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Homogeneous and heterogeneous domain decomposition methods for plate bending problems

Paola Gervasio

Department of Mathematics, University of Brescia, via Valotti, 9, 25133 Brescia, Italy

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Abstract

We consider the approximation of fourth-order problems derived, e.g., by the Kirchoff plate model and the heterogeneous coupling between a fourth-order problem and a reduced second-order one, describing, e.g., a plate-membrane model. This paper is devoted to the analysis of an iteration by subdomain method, the so-called Dirichlet/Neumann method, which is used to solve both homogeneous and heterogeneous couplings. Numerical results obtained by the spectral element method are shown.

Keywords: Domain decomposition methods; Heterogeneous coupling; Fourth order problems; Spectral element methods

1. Introduction

Fourth-order, linear elliptic differential equations in bounded domains may arise in solid mechanics to model the transversal displacements of elastic plates, or in fluid mechanics to model the stream-functions for the Navier–Stokes equations of incompressible fluids.

When coupled with second-order equations, they give rise to a heterogeneous fourth-order/second-order model that can describe, e.g., the transversal displacement of a composite elastic structure which is made of two different components, one behaving like a bending plate, the other like a membrane.

The convergence analysis of a domain decomposition approach to solve both fourth-order problem and heterogeneous coupling represents the main goal of this work. Domain decomposition methods are today largely used to reduce the computational complexity of numerical models arising from the modelling of several problems of physics and engineering (see [16,15]). They also constitute a very interesting approach to

E-mail address: gervasio@ing.unibs.it

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solve numerically heterogeneous problems which reflect realistic situations in several applied sciences (see, e.g., [15,6,17,5]).

In previous works [8,7], the domain decomposition approach to the fourth-order problem has been reformulated as a Virtual Control Approach, for which the numerical solution is reached through the minimization of a suitable cost functional and the successive resolution of local differential subproblems with Dirichlet conditions on the interface of the decomposition is required.

In this paper we propose and analyze the convergence of the so-called Dirichlet/Neumann method [13,15] applied to both homogeneous fourth-order and heterogeneous fourth-order second-order couplings. Through this method the solution of the primal problem is reduced to the successive solution of local subproblems with Dirichlet or Neumann conditions on the interface.

The numerical assessment of our theoretical results is carried out in this paper for both homogeneous and heterogeneous coupling. The fourth-order equation has been rewritten in mixed form (see [3]) to solve it numerically. A system of two second-order equations has to be solved instead of a fourth-order equation, so that conformal spectral elements (only continuous and not C^1) can be used for the approximation step. Finally, a comparison with the Virtual Control Method is done for the heterogeneous coupling, in terms of both computational efficiency and accuracy of the solution.

An outline of the paper is as follows: in Section 2 we describe the model problem and report the basic theoretical results about the fourth-order problem. In Section 3 we introduce the multidomain formulation for the fourth-order problem and the Dirichlet/Neumann iterative method, for which we prove the convergence. In Section 4 we report the numerical results obtained with spectral element approximation.

Sections 5 and 6 are devoted to the multidomain formulation and to the numerical results for the heterogeneous coupling.

2. The model problem

We consider a 3D homogeneous, isotropic thin plate of uniform thickness h and whose middle surface at equilibrium occupies a region Ω contained in the plane $x_3 = 0$. Assume that the plate is subject to a volume distribution of forces $(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ whose resultants are $f_i(x_1, x_2) = \int_{-h/2}^{h/2} \tilde{f}_i \,\mathrm{d}x_3$, for $i = 1, 2, 3$, and that the mass density per unit volume ρ is constant. In the classical thin plate theory (Kirchoff model) the transverse shear effects are neglected and this assumption leads, in small displacement theory, to the following boundary value problem for the third component u of the displacement vector $\mathbf{u}(x_1, x_2) = [v(x_1, x_2), w(x_1, x_2), u(x_1, x_2)]$:

$$\begin{cases} \rho h u_{tt} - \dfrac{\rho h^3}{12} \Delta u_{tt} + \tilde{\sigma}^2 \Delta^2 u = f_3 & \text{in } \Omega \times (0, T), \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{1}$$

where $\tilde{\sigma}^2 = Eh^3/(12(1-\mu^2))$ is the modulus of the flexural rigidity, E is the Young's modulus, $\mu \in (0, 0.5)$ is the Poisson's ratio and $\partial\cdot/\partial n$ denotes the normal derivative on the boundary.

We shall refer to (1) as the Kirchoff plate model (see [11]). Boundary conditions $u = \frac{\partial u}{\partial n} = 0$ correspond to consider a clamped plate. Approximating the time derivatives, e.g. by a classical implicit finite difference scheme with time-step Δt, one obtains the following fourth-order boundary value problem:

$$\sigma^2 \Delta^2 u - \Delta u + \alpha u = f \quad \text{in } \Omega, \tag{2}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where Ω is a Lipschitz, bounded open set in $\mathbb{R}^2$ with boundary $\partial\Omega$, $\sigma^2 = \alpha_1 \Delta t^2 E/(\rho(1-\mu^2))$, $\alpha_1 \in \mathbb{R}$ and $\alpha = 12/h^2$. Function f takes into account the resultant f_3 and other known terms.

In addition to (2) and (3) (hereafter called "homogeneous" problem), we will consider the following "heterogeneous" fourth-order second-order model:

$$\begin{cases} -\Delta u_1 + \alpha u_1 = f & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \Gamma_1 \end{cases} \begin{cases} \sigma^2 \Delta^2 u_2 - \Delta u_2 + \alpha u_2 = f & \text{in } \Omega_2, \\ u_2 = \partial u_2/\partial n = 0 & \text{on } \Gamma_2, \end{cases} \tag{4}$$

where Ω_1 and Ω_2 are two disjoint subdomains of Ω such that $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$, $S := \partial\Omega_1 \cap \partial\Omega_2$ is the interface of the decomposition and, for $i = 1, 2$, $u_i := u_{|\Omega_i}$ and $\Gamma_i := \partial\Omega \cap \partial\Omega_i$ (see Fig. 1). Model (4), which needs to be supplemented by suitable transmission conditions on S, could, for instance, describe the transversal displacement of a composite elastic structure which is made of two different components, one (corresponding to Ω_1) behaving like a membrane, the other (corresponding to Ω_2) like a bending plate.

Our aim is to solve both problem (2) and (3) and problem (4) through domain decomposition methods and, in particular, by the Dirichlet/Neumann iterations.

2.1. Weak formulation of the fourth-order problem and a priori estimates

A weak formulation of (2) and (3) reads as follows. Given $f \in L^2(\Omega)$, find $u \in H^2_0(\Omega)$ such that

$$\sigma^2(\Delta u, \Delta v)_\Omega + (\nabla u, \nabla v)_\Omega + \alpha(u, v)_\Omega = (f, v)_\Omega \quad \forall v \in H^2_0(\Omega), \tag{5}$$

where $(\cdot,\cdot)_\Omega$ denotes the L_2 inner product in Ω and

$$H^2_0(\Omega) := \left\{ v \in H^2(\Omega) : v_{|\partial\Omega} = \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0 \right\}.$$

Thanks to Lax–Milgram lemma, problem (5) has a unique solution. Moreover, if Ω is a convex polygon, then $u \in H^3(\Omega)$ (see [10, Cor. 7.3.2.5]) and there exists a positive constant C_1 such that

$$\|u\|_{H^3(\Omega)} \leqslant C_1 \|f\|_{L^2(\Omega)}. \tag{6}$$

From now on, we will consider Ω rectangle.

The approximation of problem (5) by variational numerical methods, such as finite elements or spectral elements, should require C^1-continuity across the interfaces between the elements, thus the use of Hermite's elements, which are cumbersome to implement.

A classical alternative consists of using a mixed formulation for problem (5), by which the fourth-order equation (2) is reformulated as a pair of second-order equations: find u, w such that, for $\sigma > 0$

$$\begin{cases} -\sigma\Delta u = w, \\ -\sigma\Delta w - \Delta u + \alpha u = f, \end{cases} \tag{7}$$

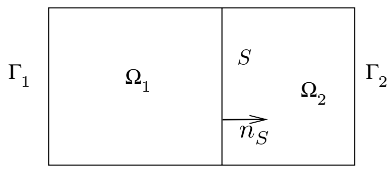

Fig. 1. A partition of Ω in two disjoint subdomains.

the equalities holding in $L^2(\Omega)$. We introduce the space:

$$\mathscr{H} = \{v \in L^2(\Omega) : \Delta v \in H^{-1}(\Omega)\}, \tag{8}$$

which is a Hilbert space for the norm

$$\|v\|_{\mathscr{H}} = \left(\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{H^{-1}(\Omega)}\right)^{1/2}. \tag{9}$$

A possible weak formulation of (7) reads as follows: given $f \in L^2(\Omega)$, find $u \in H^1_0(\Omega)$ and $w \in \mathscr{H}$ such that

$$\begin{cases} (w,v)_\Omega + \sigma\langle \Delta v, u\rangle = 0 & \forall v \in \mathscr{H}, \\ \sigma\langle \Delta w, z\rangle - (\nabla u, \nabla z)_\Omega - \alpha(u,z)_\Omega = -(f,z)_\Omega & \forall z \in H^1_0(\Omega), \end{cases} \tag{10}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. The choice of space $\mathscr{H}$ is used, for example, in [2] for the analysis of Navier–Stokes equations in vorticity-stream function formulation.

In order to prove existence and uniqueness for the solution of problem (10) we use the theory developed by Brezzi for saddle-point problems [3].

To this aim we introduce some notations and preliminary results.

Let us take

$$V = \mathscr{H}, \quad Q = H^1_0(\Omega),$$

$$a(w,v) = (w,v)_\Omega \quad \forall v,w \in V,$$

$$b(v,z) = \sigma\langle \Delta v, z\rangle \quad \forall v \in V, \ \forall z \in Q,$$

$$c(u,z) = (\nabla u, \nabla z)_\Omega + \alpha(u,z)_\Omega \quad \forall u,z \in Q.$$

Then, let $B : V \to Q'$ and $B^t : Q \to V'$ be the linear operators defined by

$$_{Q'}\langle Bv, z\rangle_Q = {}_V\langle v, B^t z\rangle_{V'} = b(v,z), \quad \forall v \in V, \ \forall z \in Q \tag{11}$$

and

$$\mathrm{Ker}\, B = \{v \in V : b(v,z) = 0, \ \forall z \in Q\},$$

$$\mathrm{Ker}\, B^t = \{z \in Q : b(v,z) = 0, \ \forall v \in V\}.$$

Finally, we introduce two linear functionals $\mathscr{F} \in Q' : {}_{Q'}\langle \mathscr{F}, z\rangle_Q = (-f,z)_\Omega$ and $\mathscr{G} \in V' : {}_{V'}\langle \mathscr{G}, v\rangle_V = (0,v)_\Omega$. With these notations, problem (10) reads: given $\mathscr{G} \in Q'$ and $\mathscr{F} \in V'$, find $w \in V$ and $u \in Q$ such that:

$$\begin{cases} a(w,v) + b(v,u) = {}_{V'}\langle \mathscr{G}, v\rangle_V & \forall v \in V, \\ b(w,z) - c(u,z) = {}_{Q'}\langle \mathscr{F}, z\rangle_Q & \forall z \in Q. \end{cases} \tag{12}$$

Lemma 2.1. *The following results hold.*

1. $a(\cdot,\cdot)$ *is a bilinear continuous form, positive semidefinite, symmetric and invertible on* Ker B.
2. $c(\cdot,\cdot)$ *is a bilinear continuous form, positive semidefinite and symmetric.*
3. Ker $B^t = \{0\}$, $b(\cdot,\cdot)$ *is a bilinear continuous form and there exists a positive constant* β *such that*

$$\sup_{v\in V} \frac{b(v,z)}{\|v\|_{\mathscr{H}}} \geqslant \beta\|z\|_{H^1_0(\Omega)}. \tag{13}$$

Proof

1. The form a is bilinear, symmetric, continuous and $a(w,w) = \|w\|_{L^2(\Omega)} \geqslant 0$, $\forall w \in V$. Moreover, $\|w\|_{L^2(\Omega)} = \|w\|_{\mathscr{H}}$ for any $w \in \mathrm{Ker}\, B = \{v \in \mathscr{H} : \Delta v = 0\}$, so that a is coercive on $\mathrm{Ker}\, B$ with coerciveness constant equal to one.
2. The form c is bilinear, symmetric, continuous and there exists a positive constant $C = C(\alpha)$: $c(u,u) = \|\nabla u\|^2_{L^2(\Omega)} + \alpha\|u\|^2_{L^2(\Omega)} \geqslant C(\alpha)\|u\|_{H^1(\Omega)}$.
3. By easy calculation it holds that $\mathrm{Ker}\, B^t = \{0\}$. The form b is bilinear and continuous. Moreover, for any $z \in Q$, choose $v \in V$ such that $v = -z$, then
$$b(v,z) = -b(z,z) = -\sigma\langle\Delta z, z\rangle = \sigma\|\nabla z\|^2_{L^2(\Omega)}.$$
Now,
$$\|\Delta z\|_{H^{-1}(\Omega)} = \sup_{\zeta\in H^1(\Omega)} \frac{\langle\Delta z, \zeta\rangle}{\|\zeta\|_{H^1(\Omega)}} = \sup_{\zeta\in H^1(\Omega)} \frac{(\nabla z, \nabla\zeta)_\Omega}{\|\zeta\|_{H^1(\Omega)}} \leqslant \|\nabla z\|_{L^2(\Omega)}$$

and, for $v = -z$,
$$\|v\|^2_{\mathscr{H}} = \|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{H^{-1}(\Omega)} \leqslant \|z\|^2_{L^2(\Omega)} + \|\nabla z\|^2_{L^2(\Omega)} \leqslant (C^2_\Omega + 1)\|\nabla z\|^2_{L^2(\Omega)},$$
being C_Ω the constant in the Poincaré inequality.

Therefore, $b(v,z) \geqslant \sigma(C^2_\Omega + 1)^{-1/2}\|v\|_{\mathscr{H}}\|\nabla z\|_{L^2(\Omega)} \geqslant \sigma/(C^2_\Omega + 1)\|v\|_{\mathscr{H}}\|z\|_{H^1(\Omega)}$ and the inf–sup condition (13) is satisfied with $\beta = \sigma/(C^2_\Omega + 1)$. □

The following result, whose proof is a consequence of Lemma 2.1 and Theorem II.1.2 in [3], holds true.

Theorem 2.1. *For every $f \in L^2(\Omega)$, problem (10) has a unique solution $(w,u) \in \mathscr{H} \times H^1_0(\Omega)$. Moreover there exists a positive constant C_2 such that*
$$\|u\|_{H^1(\Omega)} + \|w\|_{\mathscr{H}} \leqslant C_2\|f\|_{L^2(\Omega)}. \tag{14}$$

Remark 2.1. The solution u of (12) is also in $H^2_0(\Omega) \cap H^3(\Omega)$, $w \in H^1(\Omega)$ and problem (10) is equivalent to problem (5).

As a matter of fact, let us rewrite the first equation in (10) for any $v \in C^\infty(\overline{\Omega})$, since $u \in H^1_0(\Omega)$, we have:
$$0 = (w,v)_\Omega + \sigma\langle\Delta v, u\rangle = (w,v)_\Omega - \sigma(\nabla v, \nabla u)_\Omega.$$
By density arguments, it holds
$$\sigma(\nabla u, \nabla v)_\Omega = (w,v)_\Omega \quad \forall v \in H^1(\Omega) \tag{15}$$
and in particular (15) holds for any $v \in H^1_0(\Omega)$. Then, since Ω is a convex polygon and $w \in L^2(\Omega)$, the function $u \in H^1_0(\Omega)$, solution of the above problem, is also in the space $H^2(\Omega)$ [10].

By integrating by parts we have
$$(w,v)_\Omega = \sigma(\nabla v, \nabla u)_\Omega = -\sigma(\Delta u, v)_\Omega + \sigma\int_{\partial\Omega}\frac{\partial u}{\partial n}v\,\mathrm{d}s \quad \forall v \in H^1(\Omega),$$
which means $w = -\sigma\Delta u$ in $L^2(\Omega)$ and $\partial u/\partial n = 0$ on $\partial\Omega$, i.e. $u \in H^2_0(\Omega)$.

Now, by integrating by parts the second equation of (10), it holds
$$\sigma\langle\Delta w, z\rangle = (-f - \Delta u + \alpha u, z)_\Omega \quad \forall z \in H^1_0(\Omega) \tag{16}$$
and, by density arguments, we obtain $-\sigma\Delta w - \Delta u + \alpha u = f$ in $L^2(\Omega)$.

By taking now, (16) with $z \in C_0^\infty(\Omega)$, by integrating by parts twice, replacing w by $-\sigma\Delta u$ and finally by density results, we obtain problem (5). Then problem (10) is equivalent to problem (5), $u \in H^3(\Omega)$ and $w \in H^1(\Omega)$. $\square$

By the fact that $w \in H^1(\Omega)$, the mixed formulation (10) is equivalent to the following one: find $w \in H^1(\Omega)$, and $u \in H_0^1(\Omega)$ such that

$$\begin{cases} (w,v)_\Omega - \sigma(\nabla v, \nabla u)_\Omega = 0 & \forall v \in H^1(\Omega), \\ \sigma(\nabla w, \nabla z)_\Omega + (\nabla u, \nabla z)_\Omega + \alpha(u,z)_\Omega = (f,z)_\Omega & \forall z \in H_0^1(\Omega). \end{cases} \tag{17}$$

Our aim is now to write a multidomain formulation for problem (17). To do this, we have to generalize problem (17) by taking non homogeneous boundary data on a side of the domain.

We denote by Γ_j (for $j = 1,\dots,4$) the sides of Ω and by $V_j = \overline{\Gamma}_{j-1} \cap \overline{\Gamma}_j$ (for $j = 1,\dots,4$, where $\Gamma_0 = \Gamma_4$ for convenience) the vertexes of Ω. Moreover we denote by $\boldsymbol{\tau}_j$ the unit vector tangential to Γ_j, orthogonal to the outward unit normal vector $\mathbf{n}_j$ to Γ_j.

For any function $v \in H^s(\Omega)$ (with $s = 2,3$), we denote by $\gamma_j^2 : v \mapsto (v_{|\Gamma_j}, (\partial_{n_j} v)_{|\Gamma_j})$ the trace operator of order one on Γ_j and by $\gamma^{(2)}$ the trace operator from $H^s(\Omega)$ to the space $\mathbb{W}^{s,2}(\partial\Omega) := \prod_{j=1}^4 \prod_{k=0}^1 H^{s-1/2-k}(\Gamma_j)$ such that $\gamma^{(2)} v = \{\gamma_1^2 v, \gamma_2^2 v, \gamma_3^2 v, \gamma_4^2 v\}$.

It is well known that the image $\widetilde{\mathbb{W}}^{s,2}(\partial\Omega)$ of $H^s(\Omega)$ through the operator $\gamma^{(2)}$ is a subspace of $\mathbb{W}^{s,2}(\partial\Omega)$, characterized by suitable compatibility conditions at the vertexes V_j of the domain (see [10,1]), and that if we endow $\widetilde{\mathbb{W}}^{s,2}(\partial\Omega)$ by a suitable norm (see [1] for the details) then there exists a continuous extension operator from $\widetilde{\mathbb{W}}^{s,2}(\partial\Omega)$ to $H^s(\Omega)$.

From now on and until the end of this section, let S denotes a side of Ω, $\Gamma = \partial\Omega \setminus S$, $\mathbf{n}_S$ the outward normal vector to S and $\boldsymbol{\tau}_S$ the unit vector tangential to S. We introduce the space

$$\boldsymbol{\Lambda} := \left\{ \boldsymbol{\lambda} = (\lambda_1, \lambda_2) : \lambda_1 \in H^{3/2}(S), \lambda_2 \in H^{1/2}(S) : \lambda_{1|\partial S} = 0, \lambda_{2|\partial S} = 0, \left.\frac{\partial \lambda_1}{\partial \boldsymbol{\tau}_S}\right|_{\partial S} = 0 \right\}$$

and the norm $\|\cdot\|_{\boldsymbol{\Lambda}}$ that is the restriction to S of the norm given on $\widetilde{\mathbb{W}}^{2,2}(\partial\Omega)$. Then, we introduce the subspace $\boldsymbol{\Lambda}_0 \subset \boldsymbol{\Lambda}$:

$$\boldsymbol{\Lambda}_0 := \left\{ \boldsymbol{\lambda} = (\lambda_1, \lambda_2) : \lambda_1 \in H^{5/2}(S), \lambda_2 \in H^{3/2}(S) : \lambda_{1|\partial S} = 0, \lambda_{2|\partial S} = 0, \left.\frac{\partial \lambda_1}{\partial \boldsymbol{\tau}_S}\right|_{\partial S} = 0, \left.\frac{\partial \lambda_2}{\partial \boldsymbol{\tau}_S}\right|_{\partial S} = 0 \right\}.$$

For any $(\lambda_1, \lambda_2) \in \boldsymbol{\Lambda}$, let us set

$$g_1^j := \begin{cases} \lambda_1 & \text{if } S = \Gamma_j \\ 0 & \text{otherwise} \end{cases} \qquad g_2^j := \begin{cases} \lambda_2 & \text{if } S = \Gamma_j \\ 0 & \text{otherwise} \end{cases} \qquad j = 1,\dots,4$$

and let

$$\tilde{g} \text{ be an extension to } \Omega \text{ of } \left\{ \left(g_1^1, g_2^1\right), \dots, \left(g_1^4, g_2^4\right) \right\} : \gamma^{(2)} \tilde{g} = \left\{ \left(g_1^1, g_2^1\right), \dots, \left(g_1^4, g_2^4\right) \right\}. \tag{18}$$

As a consequence of the results given in [1], if $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ (resp. if $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0$), then $\tilde{g} \in H^2(\Omega)$ (resp. $\tilde{g} \in H^3(\Omega)$), moreover there exist two positive constants C_3 and C_4 such that

$$C_3 \|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}} \leqslant \|\tilde{g}\|_{H^2(\Omega)} \leqslant C_4 \|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}}, \quad \forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda} \tag{19}$$

and $\boldsymbol{\Lambda}$ is a Hilbert space.

For any subset Σ of $\partial\Omega$, we set $H_\Sigma^1(\Omega) := \{v \in H^1(\Omega) : v_{|\Sigma} = 0\}$.

Given $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \boldsymbol{\Lambda}_0$, we define the *fourth-order extension* (w_λ, u_λ) of $\boldsymbol{\lambda}$ to Ω the pair of functions $w_\lambda \in H^1(\Omega)$, $u_\lambda \in H_\Gamma^1(\Omega)$ with $u_\lambda = \lambda_1$ on S such that

$$\begin{cases} (w_\lambda, v)_\Omega - \sigma(\nabla v, \nabla u_\lambda)_\Omega = -\sigma \int_S \lambda_2 v \, ds & \forall v \in H^1(\Omega), \\ \sigma(\nabla w_\lambda, \nabla z)_\Omega + (\nabla u_\lambda, \nabla z)_\Omega + \alpha(u_\lambda, z)_\Omega = 0 & \forall z \in H^1_0(\Omega). \end{cases} \tag{20}$$

Proposition 2.1. *For any $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \boldsymbol{\Lambda}_0$, there exists a unique solution (w_λ, u_λ) of (20). Moreover, there exists a positive constant C_5 such that*

$$\|u_\lambda\|_{H^1(\Omega)} + \|w_\lambda\|_{\mathscr{H}} \leqslant C_5 \|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}}. \tag{21}$$

Proof. Let us consider an extension $\tilde{g} \in H^3(\Omega)$ of $\boldsymbol{\lambda}$ to Ω, and introduce the function $\tilde{u} = u_\lambda - \tilde{g}$. By replacing u_λ with $\tilde{u} + \tilde{g}$ in (20) and by integrating by parts, problem (20) reads: find $\tilde{u} \in H^1_0(\Omega)$, $w_\lambda \in H^1(\Omega)$ such that

$$\begin{cases} (w_\lambda, v)_\Omega - \sigma(\nabla v, \nabla \tilde{u})_\Omega = -\sigma(v, \Delta \tilde{g})_\Omega & \forall v \in H^1(\Omega), \\ \sigma(\nabla w_\lambda, \nabla z)_\Omega + (\nabla \tilde{u}, \nabla z)_\Omega + \alpha(\tilde{u}, z)_\Omega = (\Delta \tilde{g} - \alpha \tilde{g}, z)_\Omega & \forall z \in H^1_0(\Omega), \end{cases} \tag{22}$$

that is, find $\tilde{u} \in H^1_0(\Omega)$, $w_\lambda \in \mathscr{H}$ such that

$$\begin{cases} (w_\lambda, v)_\Omega + \sigma\langle \Delta v, \tilde{u}\rangle = -\sigma(v, \Delta \tilde{g})_\Omega & \forall v \in \mathscr{H}, \\ \sigma\langle \Delta w_\lambda, z\rangle - (\nabla \tilde{u}, \nabla z)_\Omega - \alpha(\tilde{u}, z)_\Omega = (-\Delta \tilde{g} + \alpha \tilde{g}, z)_\Omega & \forall z \in H^1_0(\Omega). \end{cases} \tag{23}$$

Problem (23) is of the same type of problem (12), with ${}_{V'}\langle \mathscr{G}, v\rangle_V = -\sigma(v, \Delta \tilde{g})_\Omega$ and ${}_{Q'}\langle \mathscr{F}, z\rangle_Q = (-\Delta \tilde{g} + \alpha \tilde{g}, z)_\Omega$. We remark that both $\mathscr{G}$ and $\mathscr{F}$ are well defined, since $\tilde{g} \in H^3(\Omega)$. Applying again Lemma 2.1, Theorem II.1.2 of [3] and (19) the thesis follows. □

Remark 2.2. By the same argument of Remark 2.1 and regularity results for elliptic problems with non-homogeneous Dirichlet data [10], we can prove that $u_\lambda \in H^2_\Gamma(\Omega) \cap H^3(\Omega)$ and $w_\lambda \in H^1(\Omega)$, where $H^2_\Gamma(\Omega) := \{v \in H^2(\Omega) : v_{|\Gamma} = \frac{\partial u}{\partial n}|_\Gamma = 0\}$.

3. Multidomain formulation for the homogeneous problem

We decompose the computational domain in two disjoint subdomains Ω_1 and Ω_2, such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. Moreover we ask that Ω_i ($i = 1, 2$) be rectangles (see Fig. 1). We define the interface $S = \partial\Omega_1 \cap \partial\Omega_2$, the external boundaries $\Gamma_i = \partial\Omega_i \backslash S$ for $i = 1, 2$ and the spaces:

$$\begin{aligned} &\Lambda^0 = \{\lambda \in H^{5/2}(S) : \lambda = v_{|S} \text{ for a suitable } v \in H^2_0(\Omega) \cap H^3(\Omega)\}, \\ &\Lambda = \{\mu \in H^{1/2}(S) : \mu = v_{|S} \text{ for a suitable } v \in H^1(\Omega)\}. \end{aligned} \tag{24}$$

For $i = 1, 2$, denote by $\mathscr{R}_i$ any possible extension operator from Λ to $H^1(\Omega_i)$ that satisfies $(\mathscr{R}_i\mu)_{|S} = \mu$ and by $\mathscr{R}^0_i$ any possible extension operator from Λ^0 to $H^2_{\Gamma_i}(\Omega_i) \cap H^3(\Omega_i)$ that satisfies $(\mathscr{R}^0_i\lambda)_{|S} = \lambda$.

The multidomain formulation of problem (17) reads for $i = 1, 2$: find $w_i \in H^1(\Omega_i)$ and $u_i \in H^1_\Gamma(\Omega_i)$ such that

$$\begin{aligned} &(w_i, v_i)_{\Omega_i} - \sigma(\nabla v_i, \nabla u_i)_{\Omega_i} = 0 \quad \forall v_i \in H^1_S(\Omega_i), \quad i = 1, 2, \\ &\sigma(\nabla w_i, \nabla z_i)_{\Omega_i} + (\nabla u_i, \nabla z_i)_{\Omega_i} + \alpha(u_i, z_i)_{\Omega_i} = (f, z_i)_{\Omega_i} \quad \forall z_i \in H^1_0(\Omega_i), \quad i = 1, 2, \\ &u_1 = u_2, w_1 = w_2 \quad \text{on } S, \\ &\sum_{i=1}^2 [(w_i, \mathscr{R}_i\mu)_{\Omega_i} - \sigma(\nabla \mathscr{R}_i\mu, \nabla u_i)_{\Omega_i}] = 0 \quad \forall \mu \in \Lambda, \\ &\sum_{i=1}^2 [\sigma(\nabla w_i, \nabla \mathscr{R}^0_i\lambda)_{\Omega_i} + (\nabla u_i, \nabla \mathscr{R}^0_i\lambda)_{\Omega_i} + \alpha(u_i, \mathscr{R}^0_i\lambda)_{\Omega_i}] = \sum_{i=1}^2 (f, \mathscr{R}^0_i\lambda)_{\Omega_i} \quad \forall \lambda \in \Lambda^0. \end{aligned} \tag{25}$$

Remark 3.1. The three last equations in (25) are the *transmission conditions* for the mixed formulation of the fourth-order problem (2) and (3).

We could write them in a formal way as follows:

$$\begin{aligned}
&u_1 = u_2 \quad \text{on } S,\\
&w_1 = w_2 \quad \text{on } S,\\
&\sigma\frac{\partial u_1}{\partial n_S} = \sigma\frac{\partial u_2}{\partial n_S} \quad \text{on } S,\\
&\frac{\partial u_1}{\partial n_S} + \sigma\frac{\partial w_1}{\partial n_S} = \frac{\partial u_2}{\partial n_S} + \sigma\frac{\partial w_2}{\partial n_S} \quad \text{on } S.
\end{aligned} \tag{26}$$

We note that, in view of the third condition in (26), the last one can be rewritten as $\sigma\frac{\partial w_1}{\partial n_S} = \sigma\frac{\partial w_2}{\partial n_S}$. Transmission conditions (26) guarantee that the multidomain problem (25) is equivalent to the monodomain one (17), as stated by Lemma 3.1.

Lemma 3.1. *Problem (17) is equivalent to problem (25) in the sense that if w and u are the solutions to (17), then* $w_i := w_{|\Omega_i}$ *and* $u_i := u_{|\Omega_i}$ *for* $i = 1,2$ *are the solutions of (25) and vice versa, if* w_i *and* u_i *(for* $i = 1,2$*) are the solutions of (25), then w and u, such that* $w_{|\Omega_i} = w_i$ *and* $u_{|\Omega_i} = u_i$*, are the solutions to (17).*

Proof. We begin by proving that if (w, u) is a solution of (17), then $w_i = w_{|\Omega_i}$ and $u_i = u_{|\Omega_i}$ (for $i = 1,2$) are the solutions of (25).

By construction, $u_i, w_i \in H^1(\Omega_i)$, $u_i = 0$ on Γ_i, $u_1 = u_2$ on S and $w_1 = w_2$ on S. If we consider in (17) the test functions $z \in H^1_0(\Omega)$ such that $z_{|\Omega_1} \in H^1_0(\Omega_1)$, $z_{|\Omega_2} = 0$ and $v \in H^1(\Omega)$ such that $v_{|\Omega_1} \in H^1_S(\Omega_1)$, $v_{|\Omega_2} = 0$, by putting $z_1 = z_{|\Omega_1}$ and $v_1 = v_{|\Omega_1}$ the first two equations of (25) hold for $i = 1$ (the proof is similar for $i = 2$).

Now, for any $\mu \in \Lambda$ and $\lambda \in \Lambda^0$, we set

$$z := \begin{cases} \mathscr{R}^0_1\lambda & \text{in } \Omega_1,\\ \mathscr{R}^0_2\lambda & \text{in } \Omega_2, \end{cases} \qquad v := \begin{cases} \mathscr{R}_1\mu & \text{in } \Omega_1,\\ \mathscr{R}_2\mu & \text{in } \Omega_2. \end{cases}$$

By definition of $\mathscr{R}^0_i$ and $\mathscr{R}_i$ (for $i = 1,2$), we have $z \in H^1_0(\Omega)$ and $v \in H^1(\Omega)$, so that the last two equations of (25) are satisfied too.

Vice versa, let be w_i and u_i (for $i = 1,2$) the solutions of (25), we set

$$u := \begin{cases} u_1 & \text{in } \Omega_1,\\ u_2 & \text{in } \Omega_2, \end{cases} \qquad w := \begin{cases} w_1 & \text{in } \Omega_1,\\ w_2 & \text{in } \Omega_2. \end{cases}$$

Since $w_i \in H^1(\Omega_i)$, for $i = 1,2$ and since they share the same trace on S, we have $w \in H^1(\Omega)$. By the same argument we have $u \in H^1_0(\Omega)$.

We take $v \in H^1(\Omega)$ and $z \in C^\infty_0(\Omega)$, we have that $\mu := v_{|S} \in \Lambda$ and $\lambda := z_{|S} \in \Lambda^0$, so that $(v_{|\Omega_i} - \mathscr{R}_i\mu) \in H^1_S(\Omega_i)$ and $(z_{|\Omega_i} - \mathscr{R}^0_i\lambda) \in H^1_0(\Omega_i)$, for $i = 1,2$. Therefore,

$$\begin{aligned}
(w, v)_\Omega - \sigma(\nabla v, \nabla u)_\Omega = \sum_{i=1}^{2} \Big[&\big(w_{|\Omega_i}, v_{|\Omega_i} - \mathscr{R}_i\mu\big)_{\Omega_i} - \sigma\big(\nabla(v_{|\Omega_i} - \mathscr{R}_i\mu), \nabla u_{|\Omega_i}\big)_{\Omega_i}\\
&+ \big(w_{|\Omega_i}, \mathscr{R}_i\mu\big)_{\Omega_i} - \sigma\big(\nabla\mathscr{R}_i\mu, \nabla u_{|\Omega_i}\big)_{\Omega_i}\Big] = 0 \quad \forall v \in H^1(\Omega),
\end{aligned}$$

$$\sigma(\nabla w,\nabla z)_\Omega+(\nabla u,\nabla z)_\Omega+\alpha(u,z)_\Omega=\sum_{i=1}^{2}\Big[\sigma\big(\nabla w_{|\Omega_i},\nabla\big(z_{|\Omega_i}-\mathscr{R}_i^0\lambda\big)\big)_{\Omega_i}+\big(\nabla u_{|\Omega_i},\nabla\big(z_{|\Omega_i}-\mathscr{R}_i^0\lambda\big)\big)_{\Omega_i}$$
$$+\alpha\big(u_{|\Omega_i},z_{|\Omega_i}-\mathscr{R}_i^0\lambda\big)_{\Omega_i}+\sigma\big(\nabla w_{|\Omega_i},\nabla\mathscr{R}_i^0\lambda\big)_{\Omega_i}+\big(\nabla u_{|\Omega_i},\nabla\mathscr{R}_i^0\lambda\big)_{\Omega_i}$$
$$+\alpha\big(u_{|\Omega_i},\mathscr{R}_i^0\lambda\big)_{\Omega_i}\Big]$$
$$=\sum_{i=1}^{2}\big(f,z_{|\Omega_i}-\mathscr{R}_i^0\lambda\big)_{\Omega_i}+\sum_{i=1}^{2}\big(f,\mathscr{R}_i^0\lambda\big)_{\Omega_i}=(f,z)_\Omega\quad\forall z\in C_0^\infty(\Omega).$$

By density argument, $\sigma(\nabla w,\nabla z)_\Omega+(\nabla u,\nabla z)_\Omega+\alpha(u,z)_\Omega=(f,z)_\Omega$ holds also for any $z\in H_0^1(\Omega)$. □

3.1. Iterations by subdomains: the Dirichlet/Neumann method

In order to solve problem (25), we can use an iteration by subdomains algorithm that reads as follows. Given $\boldsymbol{\lambda}^0=(\lambda_1^0,\lambda_2^0)\in\boldsymbol{\Lambda}_0$, for $k\geqslant 1$, we look for $w_i^k\in H^1(\Omega_i)$ and $u_i^k\in H^1_{\Gamma_i}(\Omega_i)$ (for $i=1,2$) such that:

$$\begin{array}{ll}
\big(w_1^k,v_1\big)_{\Omega_1}-\sigma\big(\nabla v_1,\nabla u_1^k\big)_{\Omega_1}=0 & \forall v_1\in H^1_S(\Omega_1),\\
\sigma\big(\nabla w_1^k,\nabla z_1\big)_{\Omega_1}+\big(\nabla u_1^k,\nabla z_1\big)_{\Omega_1}+\alpha\big(u_1^k,z_1\big)_{\Omega_1}=(f,z_1)_{\Omega_1} & \forall z_1\in H^1_0(\Omega_1),\\
u_1^k=\lambda_1^{k-1} & \text{on } S,\\
\big(w_1^k,\mathscr{R}_1\mu\big)_{\Omega_1}-\sigma\big(\nabla\mathscr{R}_1\mu,\nabla u_1^k\big)_{\Omega_1}=-\sigma\int_S\lambda_2^{k-1}\mu\,\mathrm{d}s & \forall\mu\in\Lambda,\\
\big(w_2^k,v_2\big)_{\Omega_2}-\sigma\big(\nabla v_2,\nabla u_2^k\big)_{\Omega_2}=0 & \forall v_2\in H^1_S(\Omega_2),\\
\sigma\big(\nabla w_2^k,\nabla z_2\big)_{\Omega_2}+\big(\nabla u_2^k,\nabla z_2\big)_{\Omega_2}+\alpha\big(u_2^k,z_2\big)_{\Omega_2}=(f,z_2)_{\Omega_2} & \forall z_2\in H^1_0(\Omega_2),\\
w_2^k=w_1^k & \text{on } S,\\
\sum_{i=1}^{2}\Big[\sigma\big(\nabla w_i^k,\nabla\mathscr{R}_i^0\lambda\big)_{\Omega_i}+\big(\nabla u_i^k,\nabla\mathscr{R}_i^0\lambda\big)_{\Omega_i}+\alpha\big(u_i^k,\mathscr{R}_i^0\lambda\big)_{\Omega_i}\Big]=\sum_{i=1}^{2}\big(f,\mathscr{R}_i^0\lambda\big)_{\Omega_i} & \forall\lambda\in\Lambda^0
\end{array}\tag{27}$$

and

$$\lambda_1^k=(1-\theta)\lambda_1^{k-1}+\theta u^k_{2|S},\quad \lambda_2^k=(1-\theta)\lambda_2^{k-1}+\theta\left.\frac{\partial u_2^k}{\partial n_S}\right|_S,\tag{28}$$

where $\theta\in(0,1)$ is a suitable relaxation parameter.

Method (27) and (28) belongs to the family of Dirichlet/Neumann methods, since it provides a sequence of problems in Ω_1 with Dirichlet conditions on S for u_1 and problems in Ω_2 with Neumann conditions on S for u_2.

In order to prove that the Dirichlet/Neumann method yields the solution of (25), we reformulate the multidomain problem (27) and (28) as a Steklov–Poincaré equation on the interface.

3.2. The Steklov–Poincaré equation

For $i=1,2$ we denote by $w_i^*\in H^1(\Omega_i)$ and $u_i^*\in H^1_0(\Omega_i)$ the solution of the following problem:

$$\begin{cases}(w_i^*,v_i)_{\Omega_i}-\sigma(\nabla v_i,\nabla u_i^*)_{\Omega_i}=0 & \forall v_i\in H^1(\Omega_i),\\ \sigma(\nabla w_i^*,\nabla z_i)_{\Omega_i}+(\nabla u_i^*,\nabla z_i)_{\Omega_i}+\alpha(u_i^*,z_i)_{\Omega_i}=(f,z_i)_{\Omega_i} & \forall z_i\in H^1_0(\Omega_i).\end{cases}\tag{29}$$

We denote by $\mathbf{\Lambda}'$ the dual space of $\mathbf{\Lambda}$ and we formally define the local Steklov–Poincaré operators $\mathscr{S}_i : \mathbf{\Lambda} \to \mathbf{\Lambda}'$: for any $\boldsymbol{\lambda} \in \mathbf{\Lambda}_0$

$$\mathscr{S}_i\boldsymbol{\lambda} := \begin{bmatrix} \sigma \dfrac{\partial w_{\lambda,i}}{\partial n_i}\bigg|_S + \dfrac{\partial u_{\lambda,i}}{\partial n_i}\bigg|_S \\ (-1)^i \sigma w_{\lambda,i|S} \end{bmatrix}, \tag{30}$$

where $(w_{\lambda,i}, u_{\lambda,i})$ is the *fourth-order extension* of $\boldsymbol{\lambda}$ to Ω_i, namely the solution of

$$\begin{cases} (w_{\lambda,i}, v_i)_{\Omega_i} - \sigma(\nabla v_i, \nabla u_{\lambda,i})_{\Omega_i} = (-1)^i \sigma \int_S \lambda_2 v_i \, \mathrm{d}s & \forall v_i \in H^1(\Omega_i), \\ \sigma(\nabla w_{\lambda,i}, \nabla z_i)_{\Omega_i} + (\nabla u_{\lambda,i}, \nabla z_i)_{\Omega_i} + \alpha(u_{\lambda,i}, z_i)_{\Omega_i} = 0 & \forall z_i \in H^1_0(\Omega_i), \end{cases} \tag{31}$$

with $u_{\lambda,i} = \lambda_1$ on S (for $i = 1,2$) and where $\mathbf{n}_i$ ($i = 1,2$) denotes the outward unit normal vector on the interface S, with respect to Ω_i. In particular $\mathbf{n}_1 = \mathbf{n}_S = -\mathbf{n}_2$.

Denoting by ${}_{\mathbf{\Lambda}'}\langle \cdot, \cdot \rangle_{\mathbf{\Lambda}}$ the duality pairing between $\mathbf{\Lambda}'$ and $\mathbf{\Lambda}$ and by $(w_{\eta,i}, u_{\eta,i})$ the fourth-order extension of $\boldsymbol{\eta}$ to Ω_i (for $i = 1,2$), we set

$$\begin{aligned} {}_{\mathbf{\Lambda}'}\langle \mathscr{S}_i\boldsymbol{\lambda}, \boldsymbol{\eta} \rangle_{\mathbf{\Lambda}} := {} & \sigma\big(\nabla w_{\lambda,i}, \nabla \mathscr{R}^0_i \eta_1\big)_{\Omega_i} + \big(\nabla u_{\lambda,i}, \nabla \mathscr{R}^0_i \eta_1\big)_{\Omega_i} + \alpha\big(u_{\lambda,i}, \mathscr{R}^0_i \eta_1\big)_{\Omega_i} \\ & + (-1)^i \sigma \int_S w_{\lambda,i|S} \eta_2 \, \mathrm{d}s \quad \forall \boldsymbol{\eta} = (\eta_1, \eta_2) \in \mathbf{\Lambda}_0. \end{aligned} \tag{32}$$

Moreover we define $\boldsymbol{\chi}_i \in \mathbf{\Lambda}'$ as follows:

$$\begin{aligned} {}_{\mathbf{\Lambda}'}\langle \boldsymbol{\chi}_i, \boldsymbol{\eta} \rangle_{\mathbf{\Lambda}} := {} & \big(f, \mathscr{R}^0_i \eta_1\big)_{\Omega_i} - \sigma\big(\nabla w^*_i, \nabla \mathscr{R}^0_i \eta_1\big)_{\Omega_i} - \big(\nabla u^*_i, \nabla \mathscr{R}^0_i \eta_1\big)_{\Omega_i} - \alpha\big(u^*_i, \mathscr{R}^0_i \eta_1\big)_{\Omega_i} \\ & - (-1)^i \sigma \int_S w^*_{i\,|S} \eta_2 \, \mathrm{d}s, \quad \forall \boldsymbol{\eta} \in \mathbf{\Lambda}_0 \end{aligned} \tag{33}$$

and we set

$$\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2, \quad \boldsymbol{\chi} = \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2. \tag{34}$$

The following Lemma holds.

Lemma 3.2. *Let (w, u) be the solution of problem (17) and let us set $\mathbf{\Lambda} = (u_{|S}, \frac{\partial u}{\partial n}|_S)$, then $\mathbf{\Lambda} \in \mathbf{\Lambda}_0$ is the solution of the Steklov–Poincaré equation*

$${}_{\mathbf{\Lambda}'}\langle \mathscr{S}\boldsymbol{\lambda}, \boldsymbol{\eta} \rangle_{\mathbf{\Lambda}} = {}_{\mathbf{\Lambda}'}\langle \boldsymbol{\chi}, \boldsymbol{\eta} \rangle_{\mathbf{\Lambda}} \quad \forall \boldsymbol{\eta} \in \mathbf{\Lambda}_0. \tag{35}$$

Conversely, if $\boldsymbol{\lambda} \in \mathbf{\Lambda}_0$ is the solution to (35), then the solution (w, u) of (17) is such that

$$w_{|\Omega_i} = w^*_i + w_{\lambda,i}, \qquad u_{|\Omega_i} = u^*_i + u_{\lambda,i}, \quad i = 1,2. \tag{36}$$

Proof. By Remark 2.1, $u \in H^3(\Omega)$, then $\boldsymbol{\lambda}$ is well defined and it belongs to $\mathbf{\Lambda}_0$. By the equivalence between problems (17) and (25), if $\boldsymbol{\lambda}$ is the trace of order one of u on S, then the restriction of the solution of (17) to Ω_i is

$$w_{|\Omega_i} = w^*_i + w_{\lambda,i}, \qquad u_{|\Omega_i} = u^*_i + u_{\lambda,i}, \quad i = 1,2.$$

We have

$$
\begin{aligned}
{}_{\Lambda'}\langle \mathscr{S}\boldsymbol{\lambda},\boldsymbol{\eta}\rangle_{\Lambda} &= \sum_{i=1}^{2}\left[\sigma\big(\nabla w_{\lambda,i},\nabla\mathscr{R}_i^0\eta_1\big)_{\Omega_i}+\big(\nabla u_{\lambda,i},\nabla\mathscr{R}_i^0\eta_1\big)_{\Omega_i}+\alpha\big(u_{\lambda,i},\mathscr{R}_i^0\eta_1\big)_{\Omega_i}\right]+\sum_{i=1}^{2}(-1)^i\sigma\int_S w_{\lambda,i|S}\eta_2\,\mathrm{d}s\\
&=\sum_{i=1}^{2}\left[\sigma\big(\nabla w_{|\Omega_i},\nabla\mathscr{R}_i^0\eta_1\big)_{\Omega_i}+\big(\nabla u_{|\Omega_i},\nabla\mathscr{R}_i^0\eta_1\big)_{\Omega_i}+\alpha\big(u_{|\Omega_i},\mathscr{R}_i^0\eta_1\big)_{\Omega_i}\right]\\
&\quad-\sum_{i=1}^{2}\left[\sigma\big(\nabla w_i^*,\nabla\mathscr{R}_i^0\eta_1\big)_{\Omega_i}+\big(\nabla u_i^*,\nabla\mathscr{R}_i^0\eta_1\big)_{\Omega_i}+\alpha\big(u_i^*,\mathscr{R}_i^0\eta_1\big)_{\Omega_i}\right]\\
&\quad+\sum_{i=1}^{2}\left[(-1)^i\sigma\int_S w_{|\Omega_i}\eta_2\,\mathrm{d}s-(-1)^i\sigma\int_S w^*_{i\,|S}\eta_2\mathrm{d}s\right]\\
&\qquad\text{(by both third and last equations in (25))}\\
&=\sum_{i=1}^{2}\left[\big(f,\mathscr{R}_i^0\eta_1\big)_{\Omega_i}-\sigma\big(\nabla w_i^*,\nabla\mathscr{R}_i^0\eta_1\big)_{\Omega_i}-\big(\nabla u_i^*,\nabla\mathscr{R}_i^0\eta_1\big)_{\Omega_i}\right.\\
&\quad\left.-\alpha\big(u_i^*,\mathscr{R}_i^0\eta_1\big)_{\Omega_i}-(-1)^i\sigma\int_S w^*_{i\,|S}\eta_2\mathrm{d}s\right]={}_{\Lambda'}\langle\boldsymbol{\chi},\boldsymbol{\eta}\rangle_{\Lambda}.
\end{aligned}
$$

Conversely, let us take the solution $\boldsymbol{\lambda}$ to (35) and set

$$w_i = w_i^* + w_{\lambda,i},\qquad u_i = u_i^* + u_{\lambda,i}\quad \text{for } i=1,2,$$

where (w_i^*, u_i^*) and $(w_{\lambda,i}, u_{\lambda,i})$ are the solutions to (29) and (31), respectively.

The first two equations and the fourth equation of (25) follow by both (29) and (31). Moreover, from

$$
\begin{aligned}
0&={}_{\Lambda'}\langle\mathscr{S}\boldsymbol{\lambda},\boldsymbol{\eta}\rangle_{\Lambda}-{}_{\Lambda'}\langle\boldsymbol{\chi},\boldsymbol{\eta}\rangle_{\Lambda}\\
&=\sum_{i=1}^{2}\left[\sigma\big(\nabla w_i,\nabla\mathscr{R}_i^0\eta_1\big)_{\Omega_i}+\big(\nabla u_i,\nabla\mathscr{R}_i^0\eta_1\big)_{\Omega_i}+\alpha\big(u_i,\mathscr{R}_i^0\eta_1\big)_{\Omega_i}-\big(f,\mathscr{R}_i^0\eta_1\big)_{\Omega_i}\right]+\sum_{i=1}^{2}(-1)^i\sigma\int_S w_{i|S}\eta_2\,\mathrm{d}s,
\end{aligned}
$$

the last equation in (25) holds and $w_1 = w_2$ on S.

Finally, since $u_1^* = u_2^* = 0$ on S and $u_{\lambda,1} = u_{\lambda,2}$ on S, then $u_1 = u_2$ on S. By the equivalence between problem (17) and (25) the thesis follows. □

Remark 3.2. In the special case where we replace $\mathscr{R}_i^0\eta_1$ by u_η^i and $\mathscr{R}_i\mu$ by w_η^i in definitions (32) and (33) ((w_η^i, u_η^i) is the solution to (31) and μ denotes the trace of w_η^i on S), thanks to the first equation in (31) with $w_{\eta,i}$ instead of v_i, we obtain

$$ {}_{\Lambda'}\langle\mathscr{S}_i\boldsymbol{\lambda},\boldsymbol{\eta}\rangle_{\Lambda} = (w_{\lambda,i}, w_{\eta,i})_{\Omega_i} + (\nabla u_{\lambda,i}, \nabla u_{\eta,i})_{\Omega_i} + \alpha(u_{\lambda,i}, u_{\eta,i})_{\Omega_i} \tag{37}$$

and

$$ {}_{\Lambda'}\langle\boldsymbol{\chi}_i,\boldsymbol{\eta}\rangle_{\Lambda} = (f, u_{\eta,i})_{\Omega_i} - (w_i^*, w_{\eta,i})_{\Omega_i} - (\nabla u_i^*, \nabla u_{\eta,i})_{\Omega_i} - \alpha(u_i^*, u_{\eta,i})_{\Omega_i}. \tag{38}$$

In this case the operators $\mathscr{S}_i$, for $i = 1,2$, are symmetric and it can be proved that they are continuous and coercive, as stated in the following section.

3.3. Convergence analysis

Lemma 3.3. *The operators $\mathscr{S}_i$ are linear, symmetric, continuous and coercive, for $i = 1, 2$.*

Proof. The linearity and symmetry follow by both definition (30) and (37).

Continuity. By formula (37) and by applying the a priori estimate (21) we have:

$$
\begin{aligned}
|{}_{\Lambda'}\langle \mathscr{S}_i\boldsymbol{\lambda}, \boldsymbol{\eta}\rangle_{\Lambda}| &\leqslant \|w_{\lambda,i}\|_{L^2(\Omega_i)}\|w_{\eta,i}\|_{L^2(\Omega_i)} + C(\alpha)\|u_{\lambda,i}\|_{H^1(\Omega_i)}\|u_{\eta,i}\|_{H^1(\Omega_i)} \\
&\leqslant C(\alpha,\sigma)(\|w_{\lambda,i}\|_{L^2(\Omega_i)} + \|u_{\lambda,i}\|_{H^1(\Omega_i)})(\|w_{\eta,i}\|_{L^2(\Omega_i)} + \|u_{\eta,i}\|_{H^1(\Omega_i)}) \\
&\leqslant K_2^{(i)}\|\boldsymbol{\lambda}\|_{\Lambda}\|\boldsymbol{\eta}\|_{\Lambda},
\end{aligned}
$$

with $K_2^{(i)} = K_2^{(i)}(\alpha, \sigma, \Omega_i) > 0$.

Coercivity. First of all we observe that for any $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0$, with $\boldsymbol{\lambda} \neq \mathbf{0}$ it holds

$$
{}_{\Lambda'}\langle \mathscr{S}_i\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_{\Lambda} = \|w_{\lambda,i}\|^2_{L^2(\Omega_i)} + \|\nabla u_{\lambda,i}\|^2_{L^2(\Omega_i)} + \alpha\|u_{\lambda,i}\|^2_{L^2(\Omega_i)} > 0. \tag{39}
$$

Then we set

$$
w_\lambda := \begin{cases} w_{\lambda,1} & \text{in } \Omega_1, \\ w_{\lambda,2} & \text{in } \Omega_2, \end{cases} \qquad u_\lambda := \begin{cases} u_{\lambda,1} & \text{in } \Omega_1, \\ u_{\lambda,2} & \text{in } \Omega_2. \end{cases}
$$

By Remark 2.2 the functions u_{λ_i} (for $i = 1, 2$) belong to $H^2_{\Gamma_i}(\Omega_i)$ and, since $u_{\lambda,1}$ and $u_{\lambda,2}$ share the same trace of order one on S, then $u_\lambda \in H^2_0(\Omega)$. The following estimate takes sense, thanks also to the fact that $\|\Delta v\|^2_{L^2(\Omega)} + \|v\|^2_{H^1(\Omega)} = \|v\|^2_{H^2(\Omega)}$ for any $v \in H^2_0(\Omega)$ (see [4]):

$$
\begin{aligned}
{}_{\Lambda'}\langle \mathscr{S}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_{\Lambda} &= \sum_{i=1}^{2}\left[\|w_{\lambda,i}\|^2_{L^2(\Omega_i)} + \|\nabla u_{\lambda,i}\|^2_{L^2(\Omega_i)} + \alpha\|u_{\lambda,i}\|^2_{L^2(\Omega_i)}\right] \\
&\geqslant C(\alpha)\left(\|w_\lambda\|^2_{L^2(\Omega)} + \|u_\lambda\|^2_{H^1(\Omega)}\right) = C(\alpha)\left(\sigma^2\|\Delta u_\lambda\|^2_{L^2(\Omega)} + \|u_\lambda\|^2_{H^1(\Omega)}\right) \\
&\geqslant C(\alpha,\sigma)\|u_\lambda\|^2_{H^2(\Omega)} \geqslant C(\alpha,\sigma,\Omega)\|\boldsymbol{\lambda}\|^2_{\Lambda}.
\end{aligned}
$$

Since $\mathscr{S}_1$ and $\mathscr{S}_2$ are two operator of the same nature and they are positive by (39), then the previous inequality implies that they are coercive too, that is there exist two positive constants $K_1^{(i)} = K_1^{(i)}(\alpha, \sigma, \Omega_i)$ (for $i = 1, 2$) such that

$$
{}_{\Lambda'}\langle \mathscr{S}_i\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_{\Lambda} \geqslant K_1^{(i)}(\alpha,\sigma,\Omega_i)\|\boldsymbol{\lambda}\|^2_{\Lambda} \quad \forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0. \qquad \square
$$

By simple calculations, we can reformulate the Dirichlet/Neumann method (27) and (28) as a preconditioned Richardson iteration for the Steklov–Poincaré equation (35):

$$
\begin{aligned}
&\text{given } \boldsymbol{\lambda}^0 \in \boldsymbol{\Lambda}_0, \\
&\boldsymbol{\lambda}^k = (1-\theta)\boldsymbol{\lambda}^{k-1} + \theta\mathscr{S}_2^{-1}(\boldsymbol{\chi} - \mathscr{S}_1\boldsymbol{\lambda}^{k-1}) \quad k \geqslant 1.
\end{aligned} \tag{40}
$$

Remark 3.3. Note that, as a consequence of their coercivity on $\boldsymbol{\Lambda}$ and thanks to Lax–Milgram lemma, both $\mathscr{S}_1$ and $\mathscr{S}_2$ are invertible on $\mathrm{Im}(\mathscr{S}_1) = \mathrm{Im}(\mathscr{S}_2)$ and $\mathscr{S}_2^{-1}\mathscr{S}_1\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0$ for any $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0$.

We introduce the $\mathscr{S}_2$-scalar product

$$
(\boldsymbol{\lambda}, \boldsymbol{\eta})_{\mathscr{S}_2} := {}_{\Lambda'}\langle \mathscr{S}_2\boldsymbol{\lambda}, \boldsymbol{\eta}\rangle_{\Lambda} \quad \forall \boldsymbol{\lambda}, \ \boldsymbol{\eta} \in \boldsymbol{\Lambda}_0.
$$

The corresponding $\mathscr{S}_2$-norm

$$\|\boldsymbol{\lambda}\|_{\mathscr{S}_2} := \sqrt{(\boldsymbol{\lambda}, \boldsymbol{\lambda})_{\mathscr{S}_2}} \quad \forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0 \tag{41}$$

is equivalent to the norm $\|\cdot\|_{\boldsymbol{\Lambda}}$, for any function $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0$. Actually, it satisfies the two-side inequality

$$K_1^{(2)}\|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}}^2 \leqslant \|\boldsymbol{\lambda}\|_{\mathscr{S}_2}^2 \leqslant K_2^{(2)}\|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}}^2 \quad \forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0, \tag{42}$$

where $K_1^{(2)}$ and $K_2^{(2)}$ are introduced in Lemma 3.3.

Given a relaxation parameter θ, consider the following operator:

$$T_\theta : \boldsymbol{\Lambda}_0 \to \boldsymbol{\Lambda}_0, \quad \boldsymbol{\lambda} \mapsto T_\theta\boldsymbol{\lambda} := (1-\theta)\boldsymbol{\lambda} - \theta\mathscr{S}_2^{-1}\mathscr{S}_1\boldsymbol{\lambda}. \tag{43}$$

Then, (40) reads

$$\begin{aligned} &\text{given } \boldsymbol{\lambda}^0 \in \boldsymbol{\Lambda}_0,\\ &\boldsymbol{\lambda}^k = T_\theta\boldsymbol{\lambda}^{k-1} + \theta\mathscr{S}_2^{-1}\boldsymbol{\chi} \quad k \geqslant 1. \end{aligned} \tag{44}$$

In order to prove the convergence of the sequence $\boldsymbol{\lambda}^k$ to the solution of (35), it is sufficient to prove that T_θ is a contraction with respect to the $\mathscr{S}_2$-norm.

Theorem 3.1. *There exist two positive constants* $\tilde{\theta} \in (0, 1]$ *and* $K_\theta \in (0, 1)$ *such that*

$$\|T_\theta\boldsymbol{\lambda}\|_{\mathscr{S}_2} \leqslant K_\theta\|\boldsymbol{\lambda}\|_{\mathscr{S}_2} \quad \forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0, \ \forall \theta \in (0, \tilde{\theta}) \tag{45}$$

i.e. T_θ *is a contraction.*

Proof. We remark that

$$T_\theta\boldsymbol{\lambda} = (1-\theta)\boldsymbol{\lambda} - \theta\mathscr{S}_2^{-1}\mathscr{S}_1\boldsymbol{\lambda} = \boldsymbol{\lambda} - \theta\mathscr{S}_2^{-1}\mathscr{S}\boldsymbol{\lambda}.$$

By the definition (41) we obtain

$$\|T_\theta\boldsymbol{\lambda}\|_{\mathscr{S}_2}^2 = {}_{\boldsymbol{\Lambda}'}\langle\mathscr{S}_2\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_{\boldsymbol{\Lambda}} - \theta_{\boldsymbol{\Lambda}'}\langle\mathscr{S}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_{\boldsymbol{\Lambda}} - \theta_{\boldsymbol{\Lambda}'}\left\langle\mathscr{S}_2\boldsymbol{\lambda}, \mathscr{S}_2^{-1}\mathscr{S}\boldsymbol{\lambda}\right\rangle_{\boldsymbol{\Lambda}} + \theta^2{}_{\boldsymbol{\Lambda}'}\left\langle\mathscr{S}\boldsymbol{\lambda}, \mathscr{S}_2^{-1}\mathscr{S}\boldsymbol{\lambda}\right\rangle_{\boldsymbol{\Lambda}},$$

then, by setting $\boldsymbol{\mu} = \mathscr{S}_2^{-1}\mathscr{S}\boldsymbol{\lambda}$ and recalling that $\mathscr{S}_2$ is symmetric, we can write

$$\|T_\theta\boldsymbol{\lambda}\|_{\mathscr{S}_2}^2 = \|\boldsymbol{\lambda}\|_{\mathscr{S}_2}^2 - 2\theta_{\boldsymbol{\Lambda}'}\langle\mathscr{S}\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle + \theta^2{}_{\boldsymbol{\Lambda}'}\langle\mathscr{S}\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle_{\boldsymbol{\Lambda}}.$$

From Lemma (3.3) and (42) it follows that

$$\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} = \|\mathscr{S}_2^{-1}\mathscr{S}\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}} \leqslant \frac{1}{K_1^{(2)}}\|\mathscr{S}\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}} \leqslant \frac{K_2^{(1)} + K_2^{(2)}}{K_1^{(2)}}\|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}},$$

$${}_{\boldsymbol{\Lambda}'}\langle\mathscr{S}\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle_{\boldsymbol{\Lambda}} \leqslant \left(K_2^{(1)} + K_2^{(2)}\right)\|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}}\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} \leqslant \frac{\left(K_2^{(1)} + K_2^{(2)}\right)^2}{\left(K_1^{(2)}\right)^2}\|\boldsymbol{\lambda}\|_{\mathscr{S}_2}^2,$$

and

$$-2\theta_{\boldsymbol{\Lambda}'}\langle\mathscr{S}\boldsymbol{\lambda}, \boldsymbol{\lambda}, \rangle_{\boldsymbol{\Lambda}} \leqslant -2\theta\left(K_1^{(1)} + K_1^{(2)}\right)\|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}}^2 \leqslant -2\theta\frac{K_1^{(1)} + K_1^{(2)}}{K_2^{(2)}}\|\boldsymbol{\lambda}\|_{\mathscr{S}_2}^2.$$

Therefore,

$$\|T_\theta\boldsymbol{\lambda}\|_{\mathscr{S}_2} \leqslant K_\theta\|\boldsymbol{\lambda}\|_{\mathscr{S}_2} \quad \forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0,$$

with

$$K_\theta = \theta^2 \left(\frac{K_2^{(1)} + K_2^{(2)}}{K_1^{(2)}} \right)^2 - 2\theta \frac{K_1^{(1)} + K_1^{(2)}}{K_2^{(2)}} + 1$$

and the thesis follows if

$$0 < \theta < \tilde{\theta} = 2 \frac{K_1^{(1)} + K_1^{(2)}}{K_2^{(2)}} \left(\frac{K_1^{(2)}}{K_2^{(1)} + K_2^{(2)}} \right)^2. \quad \square$$

4. Numerical approximation

In order to approximate the solution of the boundary-value problems in (27) we use the conformal quadrilateral spectral element method (see e.g. [14,12,9]). This approach corresponds to a Generalized Galerkin formulation of the continuous problem.

The linear systems in Ω_i ($i = 1,2$) are solved by Bi-CGStab algorithm [18], preconditioned by an incomplete LU factorization.

In order to test the convergence of our Dirichlet/Neumann (D/N) algorithm we check that

$$\max_{i=1,2} \left[\frac{\| (w_{i\mathscr{H}}^k, u_{i\mathscr{H}}^k) - (w_{i\mathscr{H}}^{k-1}, u_{i\mathscr{H}}^{k-1}) \|_{\mathbf{H}^1(\Omega_i)}}{\| (w_{i\mathscr{H}}^k, u_{i\mathscr{H}}^k) \|_{\mathbf{H}^1(\Omega_i)}} \right] \leqslant 10^{-12}, \tag{46}$$

where k is the D/N iteration counter, $\{(w_{i\mathscr{H}}^k, u_{i\mathscr{H}}^k)\}$ (for $i = 1,2$, $k \geqslant 0$) denotes the spectral element approximation of the sequence $\{(w_i^k, u_i^k)\}$ and

$$\|(w,u)\|_{\mathbf{H}^1(\Omega)} = \left(\|w\|_{H^1(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 \right)^{1/2} \quad \forall w, \ \ u \in H^1(\Omega).$$

First of all we have analyzed the convergence of the D/N method (27) for different values of the coefficient σ. The symbols N and H stand for the spectral polynomial degree and the element diameter of the mesh, respectively.

We have taken $\Omega = (-1,1)^2$, while the right-hand side and the boundary data are constructed so that the exact solution is $u(x,y) = (x^2-1)e^y + (y^2-1)e^x$, $\alpha = 0$. Moreover we have considered $\Omega_1 = (-1,0) \times (-1,1)$ and $\Omega_2 = (0,1) \times (-1,1)$.

In Table 1 we report the number of D/N iterations to satisfy the stopping criterion (46) with $\theta = 0.5$ and various values of the coefficient σ.

Table 1
Homogeneous coupling

N	σ					H	σ				
	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}		1	10^{-1}	10^{-2}	10^{-3}	10^{-4}
4	4	4	5	4	4	1/5	5	5	5	4	4
5	5	4	4	4	5	1/10	5	5	4	4	5
6	5	5	5	5	5	1/15	5	4	4	4	5
7	5	5	5	5	5	1/20	5	4	4	4	4
8	5	5	5	5	4	1/25	5	4	4	4	4

Number of D/N iterations, with $\theta = 0.5$, needed to satisfy the stopping criterion (46). At left $H = 1/2$ has been considered, at right $N = 1$.

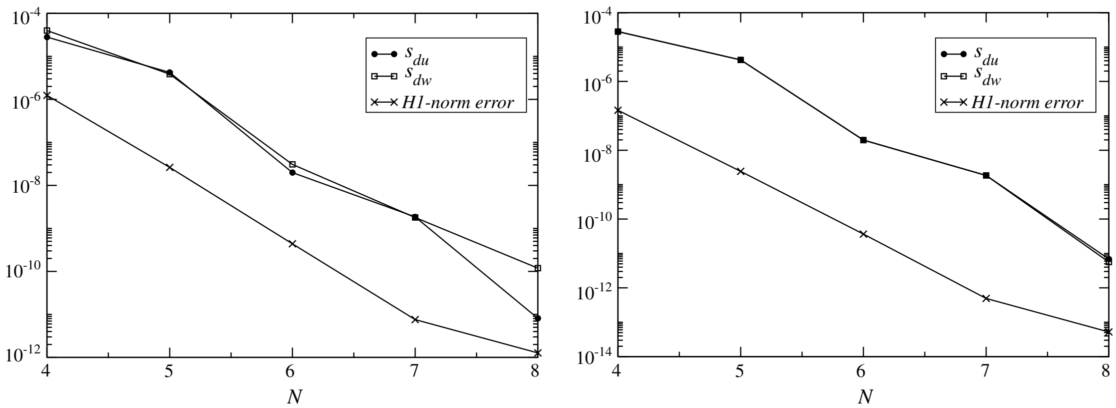

Fig. 2. Homogeneous coupling, L^{∞}-norm of the jumps on the interface and the error in $\mathbf{H}^1$-norm versus different values of the polynomial degree N, with $H = 1/2$. At left (resp. at right) the results for $\sigma = 1$ (resp. $\sigma = 10^{-3}$) are shown.

In Fig. 2 we show the L^{∞}-norm of the jumps on the interface of the normal derivative of discrete solution, that is

$$[\partial u_{\mathscr{H}}/\partial n_S]_S := (\partial u_{1\mathscr{H}}/\partial n_S - \partial u_{2\mathscr{H}}/\partial n_S)|_S \quad s_{\mathrm{d}u} := \|[\partial u_{\mathscr{H}}/\partial n_S]_S\|_{L^{\infty}(S)},$$

$$[\partial w_{\mathscr{H}}/\partial n_S]_S := (\partial w_{1\mathscr{H}}/\partial n_S - \partial w_{2\mathscr{H}}/\partial n_S)|_S \quad s_{\mathrm{d}w} := \|[\partial w_{\mathscr{H}}/\partial n_S]_S\|_{L^{\infty}(S)}$$

and the relative errors between the numerical solution and the exact one in the $\mathbf{H}^1$-norm, for two different values of σ. The L^{∞}-norms of the jumps $[u_{\mathscr{H}}]_S := (u_{1\mathscr{H}} - u_{2\mathscr{H}})|_S$, $[w_{\mathscr{H}}]_S := (w_{1\mathscr{H}} - w_{2\mathscr{H}})|_S$ are not shown, being less than 10^{-13} for all values of N, H and σ considered.

We verify that the convergence rate of the D/N method is independent of N, H and σ and that the convergence of the spectral element solution to the exact one is of exponential type.

5. The heterogeneous coupling

We consider now the heterogeneous problem (4). In this section we will give a weak formulation of it and will formulate an iteration by subdomains algorithm to find its solution.

Using the same notations introduced for both definition of the trace operator $\gamma^{(2)}$ and space $\mathbf{\Lambda}$ in Section 2.1, we introduce here the trace operator of order zero: $\gamma^{(1)}$ from $H^s(\Omega)$ (with $s = 2, 3$) to $\mathbb{W}^{s,1}(\partial\Omega) := \prod_{j=1}^{4} H^{s-1/2}(\Gamma_j)$, the space

$$\Lambda = \left\{\lambda \in H^{3/2}(S) : \lambda_{|\partial S} = 0\right\}$$

and we denote by $\|\cdot\|_{\Lambda}$ the restriction to S of the norm given on $\widetilde{\mathbb{W}}^{2,1}(\partial\Omega)$ in [1]. Moreover we introduce the subspace Λ_0 of Λ:

$$\Lambda_0 = \left\{\lambda \in H^{5/2}(S) : \lambda_{|\partial S} = 0\right\}.$$

For any $\lambda \in \Lambda$ we denote by $u_{\lambda,1} \in H^1_{\Gamma_1}(\Omega_1)$ the *second-order extension* of λ to Ω_1, namely the solution of

$$\begin{cases} (\nabla u_{\lambda,1}, \nabla z_1)_{\Omega_1} + \alpha(u_{\lambda,1}, z_1)_{\Omega_1} = 0 & \forall z_1 \in H^1_0(\Omega_1), \\ u_{\lambda,1} = \lambda & \text{on } S, \end{cases} \tag{47}$$

while for any $\lambda \in \Lambda_0$ we denote by $(w_{\lambda,2}, u_{\lambda,2}) \in H^1(\Omega_2) \times H^1_{\Gamma_2}(\Omega_2)$ the fourth-order extension of $\boldsymbol{\lambda} = (\lambda,0) \in \boldsymbol{\Lambda}_0$ (in analogy with (31)) to Ω_2:

$$\begin{cases} (w_{\lambda,2}, v_2)_{\Omega_2} + \sigma(\nabla u_{\lambda,2}, \nabla v_2)_{\Omega_2} = 0 & \forall v_2 \in H^1(\Omega_2), \\ \sigma(\nabla w_{\lambda,2}, \nabla z_2)_{\Omega_2} + (\nabla u_{\lambda,2}, \nabla z_2)_{\Omega_2} + \alpha(u_{\lambda,2}, z_2)_{\Omega_2} = 0 & \forall z_2 \in H^1_0(\Omega_2), \\ u_{\lambda,2} = \lambda & \text{on } S, \end{cases} \tag{48}$$

finally, $\mathcal{R}_1$ is any possible continuous extension operator from Λ_0 to $H^2_{\Gamma_1}(\Omega_1)$, while $\mathcal{R}_2$ is any possible continuous extension operator from Λ_0 to $H^2_{\Gamma_2}(\Omega_2) \cap H^3(\Omega_2)$.

The weak formulation of the heterogeneous problem (4) reads: find $u_1 \in H^1_{\Gamma_1}(\Omega_1)$ and $(w_2, u_2) \in H^1(\Omega_2) \times H^1_{\Gamma_2}(\Omega_2)$ such that

$$\begin{aligned} & (\nabla u_1, \nabla z_1)_{\Omega_1} + \alpha(u_1, z_1)_{\Omega_1} = (f, z_1)_{\Omega_1} && \forall z_1 \in H^1_0(\Omega_1), \\ & (w_2, v_2)_{\Omega_2} - \sigma(\nabla v_2, \nabla u_2)_{\Omega_2} = 0 && \forall v_2 \in H^1(\Omega_2), \\ & \sigma(\nabla w_2, \nabla z_2)_{\Omega_2} + (\nabla u_2, \nabla z_2)_{\Omega_2} + \alpha(u_2, z_2)_{\Omega_2} = (f, z_2)_{\Omega_2} && \forall z_2 \in H^1_0(\Omega_2), \\ & u_1 = u_2 && \text{on } S, \\ & \sigma(\nabla w_2, \nabla \mathcal{R}_2\lambda)_{\Omega_2} + \sum_{i=1}^{2} \left[(\nabla u_i, \nabla \mathcal{R}_i\lambda)_{\Omega_i} + \alpha(u_i, \mathcal{R}_i\lambda)_{\Omega_i} \right] = \sum_{i=1}^{2} (f, \mathcal{R}_i\lambda)_{\Omega_i} && \forall \lambda \in \Lambda_0. \end{aligned} \tag{49}$$

Remark 5.1. Transmission conditions of (49) can be formally written as

$$\begin{aligned} & u_1 = u_2 && \text{on } S, \\ & 0 = \sigma \frac{\partial u_2}{\partial n_S} && \text{on } S, \\ & \frac{\partial u_1}{\partial n_S} = \sigma \frac{\partial w_2}{\partial n_S} + \frac{\partial u_2}{\partial n_S} && \text{on } S \end{aligned} \tag{50}$$

and they are deduced from (26), by putting $\sigma = 0$ in Ω_1. Note that, in view of the second equation, the last one reads also $\frac{\partial u_1}{\partial n_S} = \sigma \frac{\partial w_2}{\partial n_S}$.

Remark 5.2. System (49) can be solved by an iteration by subdomains algorithm, similar to the Dirichlet/Neumann method (27) and the existence and uniqueness of solution for problem (49) will be a consequence of the convergence of such iterations.

5.1. Iterations by subdomains: the Dirichlet/Neumann method for the heterogeneous problem

Given $f \in L^2(\Omega)$ and a function $\lambda^0 \in \Lambda_0$, for $k \geqslant 1$, we look for $u_1^k \in H^1_{\Gamma_1}(\Omega_1)$, $w_2^k \in H^1(\Omega_2)$ and $u_2^k \in H^1_{\Gamma_2}(\Omega_2)$ such that:

$$\begin{aligned} & \left(\nabla u_1^k, \nabla z_1\right)_{\Omega_1} + \alpha\left(u_1^k, z_1\right)_{\Omega_1} = (f, z_1)_{\Omega_1} && \forall z_1 \in H^1_0(\Omega_1), \\ & u_1^k = \lambda^{k-1} && \text{on } S, \\ & \left(w_2^k, v_2\right)_{\Omega_2} - \sigma\left(\nabla v_2, \nabla u_2^k\right)_{\Omega_2} = 0 && \forall v_2 \in H^1(\Omega_2), \\ & \sigma\left(\nabla w_2^k, \nabla z_2\right)_{\Omega_2} + \left(\nabla u_2^k, \nabla z_2\right)_{\Omega_2} + \alpha\left(u_2^k, z_2\right)_{\Omega_2} = (f, z_2)_{\Omega_2} && \forall z_2 \in H^1_0(\Omega_2), \\ & \sigma\left(\nabla w_2^k, \nabla \mathcal{R}_2\lambda\right)_{\Omega_2} + \sum_{i=1}^{2} \left[\left(\nabla u_i^k, \nabla \mathcal{R}_i\lambda\right)_{\Omega_i} + \alpha\left(u_i^k, \mathcal{R}_i\lambda\right)_{\Omega_i} \right] = \sum_{i=1}^{2} (f, \mathcal{R}_i\lambda)_{\Omega_i} && \forall \lambda \in \Lambda_0, \end{aligned} \tag{51}$$

with

$$\lambda^k = (1-\theta)\lambda^{k-1} + \theta u^k_{2|S}, \tag{52}$$

and being $\theta \in (0,1)$ a suitable relaxation parameter.

As done for the homogeneous coupling, we reformulate the Dirichlet/Neumann method in terms of the Steklov–Poincaré operator.

We denote by $u_1^* \in H^1_0(\Omega_1)$ the solution of the problem:

$$\left(\nabla u_1^*, \nabla z_1\right)_{\Omega_1} + \alpha(u_1^*, z_1)_{\Omega_1} = (f, z_1)_{\Omega_1} \quad \forall z_1 \in H^1_0(\Omega_1), \tag{53}$$

and by $(w_2^*, u_2^*) \in H^1(\Omega_2) \times H^1_0(\Omega_2)$ the solution of problem (29) for $i = 2$.

We formally define the local Steklov–Poincaré operators $\mathscr{S}^e_i : \Lambda \to \Lambda'$: for any $\lambda \in \Lambda_0$

$$\mathscr{S}^e_1\lambda := \left.\frac{\partial u_{\lambda,1}}{\partial n_S}\right|_S, \quad \mathscr{S}^e_2\lambda := -\left(\sigma\left.\frac{\partial w_{\lambda,2}}{\partial n_S}\right|_S + \left.\frac{\partial u_{\lambda,2}}{\partial n_S}\right|_S\right). \tag{54}$$

Denoting by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality pairing between Λ' and Λ, we set

$$\begin{aligned}
&\langle\langle \mathscr{S}^e_1\lambda, \eta\rangle\rangle := (\nabla u_{\lambda,1}, \nabla \mathscr{R}_1\eta)_{\Omega_1} + \alpha(u_{\lambda,1}, \mathscr{R}_1\eta)_{\Omega_1}, \quad \forall \eta \in \Lambda_0,\\
&\langle\langle \mathscr{S}^e_2\lambda, \eta\rangle\rangle := \sigma(\nabla w_{\lambda,2}, \nabla \mathscr{R}_2\eta)_{\Omega_1} + (\nabla u_{\lambda,2}, \nabla \mathscr{R}_2\eta)_{\Omega_2} + \alpha(u_{\lambda,2}, \mathscr{R}_2\eta)_{\Omega_2}, \quad \forall \eta \in \Lambda_0
\end{aligned} \tag{55}$$

and we define the linear functionals χ_1 and $\chi_2 \in \Lambda'$: for any $\eta \in \Lambda_0$

$$\begin{aligned}
&\langle\langle \chi_1, \eta\rangle\rangle := (f, \mathscr{R}_1\eta)_{\Omega_1} - (\nabla u_1^*, \nabla \mathscr{R}_1\eta)_{\Omega_1} - \alpha(u_1^*, \mathscr{R}_1\eta)_{\Omega_1},\\
&\langle\langle \chi_2, \eta\rangle\rangle := (f, \mathscr{R}_2\eta)_{\Omega_2} - \sigma(\nabla w_2^*, \nabla \mathscr{R}_2\eta)_{\Omega_2} - (\nabla u_2^*, \nabla \mathscr{R}_2\eta)_{\Omega_2} - \alpha(u_2^*, \mathscr{R}_2\eta)_{\Omega_2}.
\end{aligned} \tag{56}$$

Finally we set

$$\mathscr{S}^e = \mathscr{S}^e_1 + \mathscr{S}^e_2, \quad \chi = \chi_1 + \chi_2. \tag{57}$$

In the following Lemma we rewrite the heterogeneous multidomain problem (49) in terms of the Steklov–Poincaré operators (55), in order to interpret the Dirichlet/Neumann algorithm as a preconditioned Richardson method and to prove the convergence of the iterations.

Lemma 5.1. *Let u_1 and (w_2, u_2) the solutions of (49) and let us set $\lambda = u_{1|S} = u_{2|S}$, then $\lambda \in \Lambda_0$ is the solution of the Steklov–Poincaré equation*

$$\langle\langle \mathscr{S}^e\lambda, \eta\rangle\rangle = \langle\langle \chi, \eta\rangle\rangle \quad \forall \eta \in \Lambda_0. \tag{58}$$

Conversely, if $\lambda \in \Lambda_0$ is the solution to (58), then the solutions u_1 and (w_2, u_2) of (49) are given by

$$u_1 = u_1^* + u_{\lambda,1},$$

$$u_2 = u_2^* + u_{\lambda,2} \quad w_2 = w_2^* + w_{\lambda,2},$$

where u_1^* is the solution of (53), $u_{\lambda,1}$ is the solution of (47), (w_2^*, u_2^*) is the solution of (29) and $(w_{\lambda,2}, u_{\lambda,2})$ is the solution of (48).

Proof. The proof follows the same steps of the proof of Lemma 3.3. □

Remark 5.3. By regularity results for the solution of second order elliptic problem in convex domains and trace inequality for polygonal domains ([1] and (19)), for any $\lambda \in \Lambda_0$ the solution $u_{\lambda,1}$ of (47) is in $H^2_{\Gamma_1}(\Omega_1)$ while the solution $(w_{\lambda,2}, u_{\lambda,2})$ of (48) is in $H^2_{\Gamma_2}(\Omega_2) \cap H^3(\Omega_2)$, then, taking into account definition (54), the interface conditions imposed in the heterogeneous problem (49) and the equivalence between (49) and the Steklov–Poincaré equation (58), it follows that $\mathrm{Im}(\mathscr{S}^e_1) = \mathrm{Im}(\mathscr{S}^e_2)$.

In the special case where we take as extension operator $\mathscr{R}_1\eta = u_{\eta,1}$ (the solution of (47)) and $\mathscr{R}_2\eta = u_{\eta,2}$ (the second component of the solution of (48)), the Steklov–Poincaré operators $\mathscr{S}_i^e$ are symmetric. As a matter of fact, we have

$$\begin{aligned}\langle\langle\mathscr{S}_1^e\lambda,\eta\rangle\rangle &= (\nabla u_{\lambda,1},\nabla u_{\eta,1})_{\Omega_1} + \alpha(u_{\lambda,1},u_{\eta,1})_{\Omega_1},\\ \langle\langle\mathscr{S}_2^e\lambda,\eta\rangle\rangle &= \sigma(\nabla w_{\lambda,2},\nabla u_{\eta,2})_{\Omega_2} + (\nabla u_{\lambda,2},\nabla u_{\eta,2})_{\Omega_2} + \alpha(u_{\lambda,2},u_{\eta,2})_{\Omega_2}\\ &\quad\text{(by the first equation in (48))}\\ &= (w_{\lambda,2},w_{\eta,2})_{\Omega_2} + (\nabla u_{\lambda,2},\nabla u_{\eta,2})_{\Omega_2} + \alpha(u_{\lambda,2},u_{\eta,2})_{\Omega_2}.\end{aligned} \tag{59}$$

In this case the following lemma holds.

Lemma 5.2. *The operator $\mathscr{S}_1^e$ is linear, symmetric, continuous and positive. The operator $\mathscr{S}_2^e$ is linear, symmetric, continuous and coercive.*

Proof. The linearity and symmetry follow by definition (54) and (59). By definition (59) and trace inequalities, there exists a positive constant $K_{2,e}^{(1)}$ such that

$$\langle\langle\mathscr{S}_1^e\lambda,\eta\rangle\rangle = (\nabla u_{\lambda,1},\nabla u_{\eta,1})_{\Omega_1} + \alpha(u_{\lambda,1},u_{\eta,1})_{\Omega_1} \leqslant C\|u_{\lambda,1}\|_{H^1(\Omega_1)}\|u_{\eta,1}\|_{H^1(\Omega_1)} \leqslant K_{2,e}^{(1)}\|\lambda\|_{\Lambda}\|\eta\|_{\Lambda}.$$

Moreover, for any $\lambda\in\Lambda$ with $\lambda\neq 0$

$$\langle\langle\mathscr{S}_1^e\lambda,\lambda\rangle\rangle = \|\nabla u_{\lambda,1}\|^2_{L^2(\Omega_1)} + \alpha\|u_{\lambda,1}\|^2_{L^2(\Omega_1)} > 0,$$

that is, $\mathscr{S}_1^e$ is positive.

The continuity of $\mathscr{S}_2^e$ can be proved following the same steps of the proof of Lemma 3.3, with $\boldsymbol{\lambda} = (\lambda, 0)$, while the coercivity of $\mathscr{S}_2^e$ on Λ is a consequence of the coercivity of $\mathscr{S}_2$ on $\boldsymbol{\Lambda}$. □

As done for the homogeneous case, the Dirichlet/Neumann method (51) and (52) can be reviewed as a preconditioned Richardson scheme for the Steklov–Poincaré equation (58):

$$\begin{aligned}&\text{given } \lambda^0\in\Lambda_0,\\ &\lambda^k = (1-\theta)\lambda^{k-1} + \theta(\mathscr{S}_2^e)^{-1}(\chi - \mathscr{S}_1^e\lambda^{k-1}) \quad k\geqslant 1.\end{aligned} \tag{60}$$

By Remark 5.3 and Lemma 5.2, for any $\lambda\in\Lambda_0$, the element $(\mathscr{S}_2^e)^{-1}\mathscr{S}_1^e\lambda$ belongs to Λ_0. Then, given a suitable relaxation parameter $\theta\in(0,1)$, we can introduce the iteration operator

$$T_\theta : \Lambda_0 \to \Lambda_0, \quad T_\theta\lambda = (1-\theta)\lambda - \theta(\mathscr{S}_2^e)^{-1}\mathscr{S}_1^e\lambda, \tag{61}$$

and the convergence of the Dirichlet/Neumann iterations is ensured by proving that T_θ is a contraction, as stated in the following theorem.

Theorem 5.1. *There exist two positive constants $\tilde\theta\in(0,1]$ and $K_\theta\in(0,1)$ such that,*

$$\|T_\theta\lambda\|_\Lambda \leqslant K_\theta\|\lambda\|_\Lambda, \quad \forall\lambda\in\Lambda_0,\ \forall\theta\in(0,\tilde\theta),$$

i.e. the iterative scheme (51) (or equivalent (60)) is convergent.

Proof. We introduce the $\mathscr{S}_2^e$-scalar product $(\lambda,\eta)_{\mathscr{S}_2^e} := \langle\langle\mathscr{S}_2^e\lambda,\eta\rangle\rangle$, for any λ, $\eta\in\Lambda_0$. By Lemma 5.1, this scalar product induces a norm equivalent to the norm $\|\cdot\|_\Lambda$.

The proof follows the same steps of proof of Theorem 3.1, by proving that T_θ is a contraction with respect to the $\mathscr{S}_2^e$-norm. Note that the coercivity of $\mathscr{S}_2^e$ and the positivity of $\mathscr{S}_1^e$ are sufficient to guarantee the coercivity of $\mathscr{S}^e$.

In particular

$$\tilde{\theta} = 2\frac{(K_1^{(2)})^3}{K_2^{(2)}(K_{2,e}^{(1)} + K_2^{(2)})^2}. \qquad \square$$

Finally, the following theorem, that ensures the well position of the heterogeneous problem (49), is a consequence of Theorem 5.1 and Lemma 5.1.

Theorem 5.2. *Given* $f \in L^2(\Omega)$, *there exist a unique solution* $u_1 \in H^1_{\Gamma_1}(\Omega_1)$ *and a unique solution* $(w_2, u_2) \in H^1(\Omega_2) \times H^1_{\Gamma_2}(\Omega_2)$ *of* (49).

6. Numerical results for the heterogeneous coupling

Test case #1

We consider the computational domain $\Omega = (-1,1)^2$, and the following data: $u = (x^2-1)e^y + (y^2-1)e^x$ on $\partial\Omega$, $\partial u/\partial n = \partial((x^2-1)e^y + (y^2-1)e^x)/\partial n$ on $\partial\Omega$ and $f = e^x((\sigma^2-1)y^2 + 3\sigma^2 - 1) + e^y((\sigma^2-1)x^2 + 3\sigma^2 - 1)$, $\alpha = 0$.

We analyze the convergence rate of the Dirichlet/Neumann method for different values of σ and for various discretization and we chose the relaxation parameter θ dynamically so as to minimize the interface error at each step.

In Table 2 the number of Dirichlet/Neumann iterations are shown for a decomposition of Ω in $\Omega_1 = (-1,0)\times(-1,1)$ and $\Omega_2 = (0,1)\times(-1,1)$. The rate of convergence is independent of the space discretization, but strongly depends on σ, as expected.

We denote by $s_\phi = \|[\phi]_S\|_{L^\infty(S)}$ the L^∞-norm of the jump on the interface S of the flux, that is

$$[\phi]_S = (\phi_1 - \phi_2)|_S = \left(\frac{\partial u_{1\mathscr{H}}}{\partial n_S} - \left(\frac{\partial u_{2\mathscr{H}}}{\partial n_S} - \sigma\frac{\partial w_{2\mathscr{H}}}{\partial n_S}\right)\right)\Bigg|_S. \tag{62}$$

In view of Remark 5.1, $[\phi]_S = 0$ if and only if the second transmission condition on S (see (50)) is satisfied. In Fig. 3 we show the behavior of s_ϕ and the relative error in H^1-norm between the numerical solution and the solution of the global fourth-order problem, versus σ and for two different values of N. Both the jump and the error tend to zero when σ vanishes. The jump of the solution u at the interface is not shown, being less than 1.e−13 in all the situations.

Moreover, when N grows, the norm of the jump s_ϕ tends to zero with spectral accuracy, with a lower bound which depends on the magnitude of σ as we can see in Fig. 4.

Table 2
Test case #1

N	σ					H	σ				
	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}		1	10^{-1}	10^{-2}	10^{-3}	10^{-4}
4	12	18	10	6	5	1/10	12	18	9	6	5
5	12	18	11	6	5	1/15	12	18	10	6	5
6	12	19	12	6	5	1/20	12	18	11	6	5
7	12	19	13	7	5	1/25	12	18	12	6	5
8	12	19	14	7	5						

Heterogeneous coupling. Number of D/N iterations needed to satisfy the stopping criterion (46). The relaxation parameter θ has been chosen dynamically. At left $H = 1/2$ has been considered, at right $N = 1$.

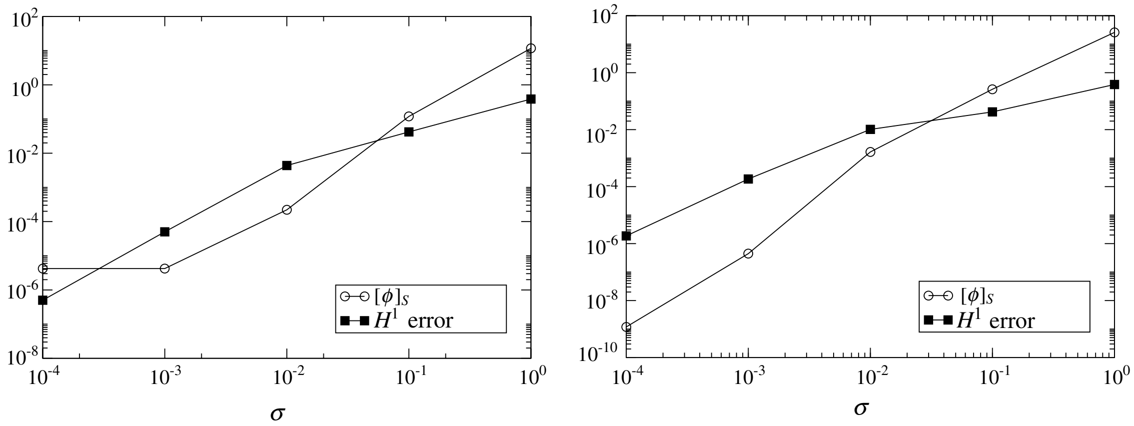

Fig. 3. Test case #1. Heterogeneous coupling, jump on the interface of the flux ϕ and the error in H^1-norm between the numerical solution and the solution of the global fourth-order problem, versus different values of σ, with $H = 1/2$. At left (resp. at right) the results for $N = 5$ (resp. $N = 8$) are shown.

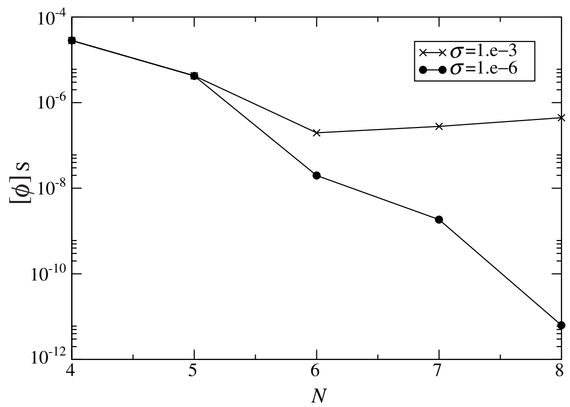

Fig. 4. Test case #1. Heterogeneous coupling, jump on the interface of the flux ϕ versus the spectral interpolation degree N.

In Table 3 we show the number of D/N iterations for different values of σ versus the position x_S of the interface S of the decomposition. In particular we have considered $\Omega_1 = (-1, x_S) \times (-1, 1)$ and $\Omega_2 = (x_S, 1) \times (-1, 1)$.

Table 3
Test case #1

x_S	σ				
	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}
−0.50	13	21	15	13	13
−0.25	11	19	14	11	11
0.0	12	18	11	6	5
0.25	11	18	10	11	11
0.5	13	17	12	13	13

Heterogeneous coupling. Number of D/N iterations for $N = 5$ and $H = 0.5$, versus the position of the interface S.

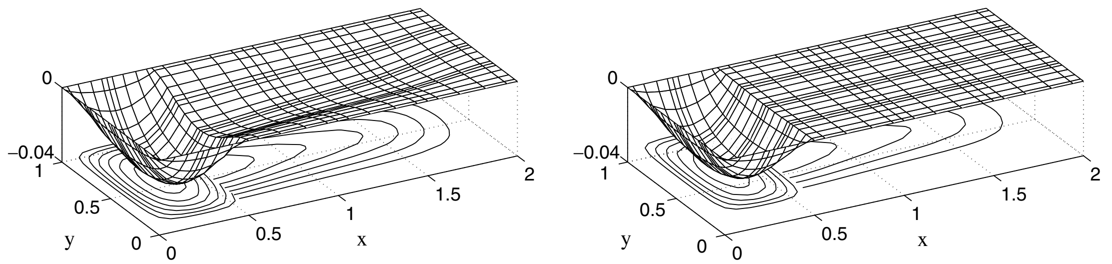

Fig. 5. Test case #2. Heterogeneous coupling: spectral element solution with $\sigma = 0.5$ (left) and $\sigma = 2$ (right). The interface is located in $x_S = 0.5$.

Test case #2

We consider now the membrane-plate heterogeneous coupling (49) with a uniform external load $f \equiv -1$ in $\Omega = (0,2) \times (0,1)$, homogeneous boundary data on $\partial\Omega$, $\alpha = 1$. The computational domain is decomposed in $\Omega_1 = (0, x_S) \times (0,1)$ and $\Omega_2 = (x_S, 2) \times (0,1)$, the spectral polynomial degree is $N = 5$.

In Fig. 5 we show the numerical solution for $x_S = 0.5$ and $\sigma = 0.5$ (at left), $\sigma = 2$ (at right), while in Table 4 we report the number of Dirichlet/Neumann iterations for various positions of the interface S and different values of σ. The discretization used for the results of this table has $N = 5$ and $H = 0.25$ in both Ω_1 and Ω_2.

6.1. Comparison with the virtual control approach

We compare now the results obtained by the Dirichlet/Neumann method on the heterogeneous coupling with those obtained by the virtual control approach (see [8,7]).

To solve problem (4) by the Virtual Control means to look for the solution of the minimization problem

$$\inf_{\lambda_1,\lambda_2} J(\lambda_1, \lambda_2), \tag{63}$$

where

$$J(\lambda_1, \lambda_2) := \frac{1}{2} \int_S \left[\left(\frac{\partial u_1}{\partial n_S} - \frac{\partial u_2}{\partial n_S} + \sigma \frac{\partial w_2}{\partial n_S} \right)^2 + \left(\sigma \frac{\partial u_2}{\partial n_S} \right)^2 \right] \mathrm{d}s \tag{64}$$

and u_1, and (w_2, u_2) are the solutions of the Dirichlet problems

$$\begin{cases} -\Delta u_1 + \alpha u_1 = f & \text{in } \Omega_1, \\ u_1 = 0 & \text{on } \partial\Omega_1 \setminus S, \\ u_1 = \lambda_1 & \text{on } S, \end{cases} \qquad \begin{cases} \sigma^2 \Delta^2 u_2 - \Delta u_2 + \alpha u_2 = f & \text{in } \Omega_2, \\ u_2 = \partial u_2/\partial n = 0 & \text{on } \partial\Omega_2 \setminus S, \\ u_2 = \lambda_1, \partial u_2/\partial n_S = \lambda_2 & \text{on } S. \end{cases}$$

We denote by u_{DN} and u_{VC} the solution of the Dirichlet/Neumann method and Virtual Control Approach, respectively and by u_{ex} the solution of the global fourth-order problem considered in the previous subsection. In Fig. 6 we compare the norm of the jump on the interface of the flux ϕ (62) and the relative errors $\|u_{\mathrm{ex}} - u_{\mathrm{DN}}\|_{1,\Omega}/\|u_{\mathrm{ex}}\|_{H^1(\Omega)}$ and $\|u_{\mathrm{ex}} - u_{\mathrm{VC}}\|_{1,\Omega}/\|u_{\mathrm{ex}}\|_{H^1(\Omega)}$. By comparing the errors with respect to the exact solution, the methods can be considered equivalent. This is not the case when we compare the computational effort. In order to solve numerically the minimum problem (63) we have used the Bi-CGStab algorithm [18] on the linear system $\nabla J = 0$. At each Bi-CGStab iteration we have to compute two matrix-vector products (that means to solve two differential subproblems in Ω_1 and two differential subproblems in Ω_2) and evaluate the gradient ∇J two times (that means to solve other two differential subproblems in Ω_1 and two differential subproblems in Ω_2). It follows that the computational effort for one iteration of

Table 4
Test case #2

x_S	σ				
	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}
0.5	9	15	12	7	5
1.0	9	15	12	6	5
1.5	9	15	12	7	5

Heterogeneous coupling. Number of D/N iterations versus the position of the interface S.

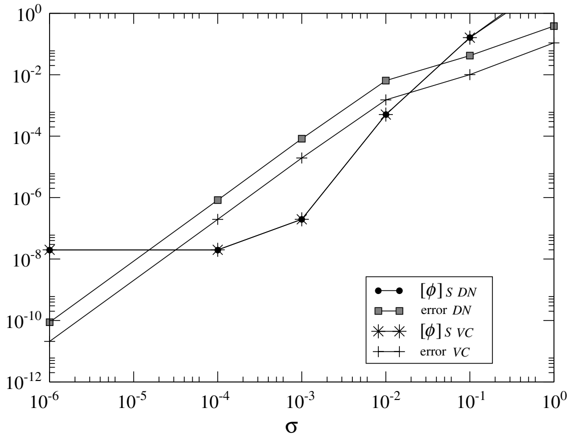

Fig. 6. Test case #1. Heterogeneous coupling: comparison between the solution of the Dirichlet/Neumann method (DN) and the solution of the Virtual Control Approach (VC): the jump of the flux on the interface and the H^1-norm error between numerical solution and global fourth-order solution.

Table 5
Test case #1

N	σ				
	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}
4	65	32	19	20	20
5	188	75	29	28	31
6	387	137	41	35	34
7	>500	272	57	44	45
8	>500	>500	74	55	54

Heterogeneous coupling. Number of iterations needed to the Bi-CGStab algorithm to converge to the solution of the minimum problem (63).

Bi-CGStab is equivalent the computational effort of four Dirichlet/Neumann iterations. In Table 5 we report the number of iterations for the Virtual Control methods needed to satisfy the stopping criterion (46). By comparing Table 5 with the left subtable in Table 2, we see that the Virtual Control method is more expensive than the Dirichlet/Neumann method. In order to reduce the computational effort of the Virtual Control method it seems mandatory to precondition the system $\nabla J = 0$.

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