Virtual Control for Fourth-Order Problems and for Heterogeneous Fourth-Order Second-Order Coupling

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Summary. We consider both approximation of fourth-order problems derived, e.g., the Kirchoff plate model, and heterogeneous coupling between a fourth-order problem and a reduced second-order problem, describing a plate-membrane model. The multidomain virtual control approach is used. This paper is devoted to the analysis and construction of the cost functional gradient in order to render the minimization procedure effective.

1 Introduction

The multidomain virtual control approach was considered in [3] coupling second-order and first-order equations with both overlapping and non-overlapping decompositions. The idea is to work with Dirichlet conditions on the interfaces of the decomposition, and the Dirichlet data (the virtual controls) are determined through the minimization of a suitable cost functional. The characterization of the cost functional is quite natural for overlapping decompositions, while it is not obvious for the non-overlapping case. These concepts were extended in [4] to fourth-order problems and to the coupling between fourth-order and second-order elliptic equations, moreover various cost functionals have been proposed for non-overlapping decompositions. Nevertheless in the previous papers the minimization algorithm used did not take advantage of the construction of the cost functional gradient in terms of adjoint problems, as is usually done in control theory.

This paper is devoted to the construction of both adjoint problems and the cost functional gradient, for the particular problem discussed here, in order to be used within a minimization method. We focus our attention on non-overlapping decompositions.
Lastly, we give numerical results which demonstrate the effectiveness of this minimization procedure versus another optimization algorithm which does not use derivatives.

We briefly recall the statement of the differential problem. We are interested in approximating the solution u of the fourth-order boundary value problem:

$$\begin{cases} \sigma^2 \Delta^2 u - \Delta u + \alpha u = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = \partial u / \partial n = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where σ and α are positive constants.

If we consider the Kirchoff plate model (see [5]) and discretize it in time by, e.g., a classic finite difference scheme, at each time step we obtain a problem like (1). In this case u denotes the vertical displacement, while σ^2 represents the modulus of the flexural rigidity and is proportional to the Young modulus of the material. The boundary conditions $u = \partial u/\partial n = 0$ correspond to considering a clamped plate.

Problem (1), restated in mixed form, formally reads: find (u, w) in Ω such that

$$\begin{cases} -\Delta u + \alpha u + \sigma \Delta w = f, \qquad -\sigma \Delta u + w = 0 & \text{in } \Omega \\ u = \partial u/\partial n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

In the classical notation for Lebesgue and Sobolev spaces, the weak form of (2) reads: given $f \in L^2(\Omega)$, find $(u, w) \in H_0^1(\Omega) \times H^1(\Omega)$ such that

$$\begin{cases} (\nabla u, \nabla z)_\Omega + \alpha(u, z)_\Omega - \sigma(\nabla w, \nabla z)_\Omega = (f, z)_\Omega & \forall z \in H_0^1(\Omega) \\ \sigma(\nabla u, \nabla v)_\Omega + (w, v)_\Omega = 0 & \forall v \in H^1(\Omega). \end{cases} \tag{3}$$

We note that the boundary condition $\partial u/\partial n = 0$ is a natural condition for the second equation of (3). Existence and uniqueness of a solution for problem (3) are proved in [2], provided that the computational domain $\Omega \subset \mathbb{R}^2$ is a convex polygon.

Further, we consider the heterogeneous fourth-order second-order model:

$$\begin{cases} -\Delta u_1 + \alpha u_1 = f & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \Gamma_1 \end{cases} \qquad \begin{cases} \sigma^2 \Delta^2 u_2 - \Delta u_2 + \alpha u_2 = f & \text{in } \Omega_2 \\ u_2 = \partial u_2/\partial n = 0 & \text{on } \Gamma_2 \end{cases} \tag{4}$$
$$+\text{transmission conditions} \quad \text{on } S,$$

where Ω_1 and Ω_2 are two disjoint subdomains of Ω such that $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$, $S := \partial\Omega_1 \cap \partial\Omega_2$ is the interface of the decomposition and, for $i = 1, 2$, $u_i := u_{|\Omega_i}$ and $\Gamma_i := \partial\Omega \cap \partial\Omega_i$ (see Figure 1). Model (4) could, for instance, describe the transversal displacement of a composite elastic structure which is made of two different components, one (corresponding to Ω_1) behaving like a membrane, the other (corresponding to Ω_2) like a bending plate.

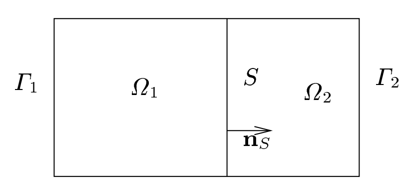

Fig. 1. A partition of Ω in two disjoint subdomains.

2 The multidomain virtual control approach

We consider a non-ovelapping decomposition of Ω into two disjoint subdomains Ω_1 and Ω_2 (see Fig. 1). For $i = 1, 2$, we define $V_i = H^1_{\Gamma_i}(\Omega_i) \times H^1(\Omega_i)$, where $H^1_{\Gamma_i}(\Omega_i) := \{v \in H^1(\Omega_i) : \ v_{|\Gamma_i} = 0\}$. Moreover we consider the space $H^{1/2}_{00}(S) := \{\varphi \in L^2(S) : \ \exists v \in H^1(\Omega), \ v_{|S} = \varphi, v_{|\partial\Omega} = 0\}$, and we set $\Lambda := H^{1/2}_{00}(S) \times H^{1/2}(S)$. Lastly, we set $((u_i, z_i))_{\Omega_i} := (\nabla u_i, \nabla z_i)_{\Omega_i} + \alpha(u_i, z_i)_{\Omega_i}$.

The multidomain formulation of (3), based on the use of virtual control, reads as follows: for $i = 1, 2$, find $(u_i, w_i) \in V_i$ such that:

$$
\begin{aligned}
&((u_1, z_1))_{\Omega_1} - \sigma(\nabla w_1, \nabla z_1)_{\Omega_1} = (f, z_1)_{\Omega_1} && \forall z_1 \in H^1_0(\Omega_1) && (5)\\
&\sigma(\nabla u_1, \nabla v_1)_{\Omega_1} + (w_1, v_1)_{\Omega_1} = \int_S \mu v_1 && \forall v_1 \in H^1(\Omega_1) && (6)\\
&((u_2, z_2))_{\Omega_2} - \sigma(\nabla w_2, \nabla z_2)_{\Omega_2} = (f, z_2)_{\Omega_2} && \forall z_2 \in H^1_0(\Omega_2) && (7)\\
&\sigma(\nabla u_2, \nabla v_2)_{\Omega_2} + (w_2, v_2)_{\Omega_2} = -\int_S \mu v_2 && \forall v_2 \in H^1(\Omega_2) && (8)\\
&u_1 = u_2 = \lambda && \text{on } S, && (9)
\end{aligned}
$$

where $\lambda \in H^{1/2}_{00}(S)$ and $\mu \in H^{1/2}(S)$ are the *virtual controls* and are determined by solving the minimization problem

$$
\inf_{(\lambda,\mu)\in\Lambda} J_1(u_1(\lambda,\mu), w_1(\lambda,\mu), u_2(\lambda,\mu), w_2(\lambda,\mu)). \tag{10}
$$

The *cost functional* J_1 is defined as

$$
J_1(\lambda,\mu) = \frac{1}{2}\int_S \left[(w_1 - w_2)^2 + \left(\left(\frac{\partial u_1}{\partial n_S} - \sigma \frac{\partial w_1}{\partial n_S} \right) - \left(\frac{\partial u_2}{\partial n_S} - \sigma \frac{\partial w_2}{\partial n_S} \right) \right)^2 \right] ds.
$$

The choice of J_1 is justified in [4], taking into account the transmission conditions for the *state equations* (5)–(9).

2.1 The minimization procedure

We solved the minimization problem (10) either by the Bi-CGStab method [9] on the linear system $\nabla J_1 = 0$, or by the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method (a quasi-Newton method with a rank-one update of the Hessian; see e.g. [7]) directly on (10). In our numerical tests we observed that the Bi-CGStab method performs well if the coefficient σ is small ($\sigma \le 10^{-2}$), if not, the acceleration parameter used in the updating step of the solution may become very small ($\le 10^{-10}$) and the convergence of the method is quite slow. The use of a preconditioner would be desirable.

Both methods require the exact evaluation of the cost functional gradient ∇J_1. It is known [6] that the gradient of the cost functional can be expressed by means of the solution of adjoint equations. The difficulty lies in finding a convenient characterization of the adjoint problem. For the sake of clarity we carry out the derivation of ∇J_1 in detail.

Without loss of generality, from now on we assume that $f \equiv 0$ (we recall that the solution of problem (5)–(9) can be written as the sum of two solutions: the first depending only on external forces and non-homogeneous boundary data (if any), and the second on the interface data (λ, μ)).

For $i = 1, 2$, we consider the state equations

$$\begin{cases} -\Delta u_i + \alpha u_i + \sigma \Delta w_i = 0 & \text{in } \Omega_i \\ -\sigma \Delta u_i + w_i = 0 & \text{in } \Omega_i \\ u_i = \partial u_i / \partial n_i = 0 & \text{on } \Gamma_i \\ u_i = \lambda, \quad \sigma \partial u_i / \partial n_S = \mu & \text{on } S; \end{cases} \tag{11}$$

the solution of (11) is denoted by $(u_i(\lambda, \mu), w_i(\lambda, \mu))$, to emphasize the dependence on the data. Accordingly, we introduce the local Steklov-Poincaré operators which appear in the first equation of (11) (see [8]):

$$\mathcal{S}_i(\lambda, \mu) := \left(\frac{\partial u_i}{\partial n_i}(\lambda, \mu) - \sigma \frac{\partial w_i}{\partial n_i}(\lambda, \mu) \right) \Big|_S . \tag{12}$$

We denote by $[w(\lambda, \mu)] = w_1(\lambda, \mu)_{|S} - w_2(\lambda, \mu)_{|S}$ the jump of the function w on the interface S, and analogous ly

$$[\mathcal{S}(\lambda, \mu)] = \left(\frac{\partial u_1}{\partial n_S}(\lambda, \mu) - \sigma \frac{\partial w_1}{\partial n_S}(\lambda, \mu) \right) |_S - \left(\frac{\partial u_2}{\partial n_S}(\lambda, \mu) - \sigma \frac{\partial w_2}{\partial n_S}(\lambda, \mu) \right) |_S .$$

We observe that $[\mathcal{S}(\lambda, \mu)] = \mathcal{S}_1(\lambda, \mu) + \mathcal{S}_2(\lambda, \mu)$.

By definition of the Fréchet derivative and by the linearity of the differential problem,for suitable regular functions η and ψ defined on S, we can formally write:

$$\begin{aligned} \langle \nabla J_1(\lambda, \mu), (\eta, \psi) \rangle = & \int_S [w(\lambda, \mu)] \, (w_1(\eta, \psi) - w_2(\eta, \psi)) \, ds \\ & + \int_S [\mathcal{S}(\lambda, \mu)] \, (\mathcal{S}_1(\eta, \psi) + \mathcal{S}_2(\eta, \psi)) \, ds. \end{aligned} \tag{13}$$

In order to express the integrals in (13) directly as functions on η and ψ, we introduce the adjoint state equations of (11): for $i = 1, 2$,

$$\begin{cases} -\Delta p_i + \alpha p_i - \sigma \Delta q_i = 0 & \text{in } \Omega_i \\ \sigma \Delta p_i + q_i = 0 & \text{in } \Omega_i \\ p_i = \partial p_i / \partial n_i = 0 & \text{on } \Gamma_i \\ p_i = [\mathcal{S}(\lambda, \mu)], \quad \sigma \partial p_i / \partial n_S = [w(\lambda, \mu)] & \text{on } S. \end{cases} \tag{14}$$

By the duality between (11) and (14) we have

$$\nabla J_1(\lambda, \mu) = \begin{bmatrix} \mathcal{S}_1^*(\lambda, \mu) + \mathcal{S}_2^*(\lambda, \mu) \\ (q_1 - q_2)|_S \end{bmatrix}, \tag{15}$$

where $\mathcal{S}_i^*(\lambda,\mu) := \left(\frac{\partial p_i}{\partial n_i} + \sigma\frac{\partial q_i}{\partial n_i}\right)\Big|_S$ (for $i = 1,2$) are the local Steklov-Poincaré operators associated to the first equation of the dual problem (14).

To prove (15) we proceed in a formal manner. We multiply the first equation in (14) by $u_i(\eta,\psi)$ and the second by $w_i(\eta,\psi)$; for $i = 1,2$, we see that

$$
\begin{aligned}
0 &= \int_{\Omega_i} (-\Delta p_i + \alpha p_i - \sigma\Delta q_i)\; u_i(\eta,\psi) d\Omega + \int_{\Omega_i} (\sigma\Delta p_i + q_i)\; w_i(\eta,\psi) d\Omega \\
&\quad \text{(by Green's formula)} \\
&= \int_{\Omega_i} (-\Delta u_i(\eta,\psi) + \alpha u_i(\eta,\psi))\; p_i d\Omega - \sigma \int_{\Omega_i} \Delta u_i(\eta,\psi)\; q_i d\Omega \\
&\quad + \int_S (p_i + \sigma q_i)\; \frac{\partial u_i}{\partial n_i}(\eta,\psi) ds - \int_S \left(\frac{\partial p_i}{\partial n_i} + \sigma\frac{\partial q_i}{\partial n_i}\right) u_i(\eta,\psi) ds \\
&\quad + \int_{\Omega_i} \sigma\Delta w_i(\eta,\psi)\; p_i d\Omega + \int_{\Omega_i} q_i\; w_i(\eta,\psi) d\Omega \\
&\quad -\sigma \int_S \frac{\partial w_i}{\partial n_i}(\eta,\psi) p_i ds + \sigma \int_S \frac{\partial p_i}{\partial n_i} w_i(\eta,\psi) ds \\
&\quad \text{(by the first two equations in (11) and the interface conditions in (14))} \\
&= \int_S [\mathcal{S}(\lambda,\mu)]\; \frac{\partial u_i}{\partial n_i}(\eta,\psi) ds + \int_S q_i\; \sigma\frac{\partial u_i}{\partial n_i}(\eta,\psi) ds - \int_S \frac{\partial p_i}{\partial n_i}\; u_i(\eta,\psi) ds \\
&\quad - \int_S \sigma\frac{\partial q_i}{\partial n_i}\; u_i(\eta,\psi) ds - \int_S [\mathcal{S}(\lambda,\mu)]\; \sigma\frac{\partial w_i}{\partial n_i}(\eta,\psi) ds \\
&\quad -(-1)^i \int_S [w(\lambda,\mu)]\; w_i(\eta,\psi) ds,
\end{aligned}
$$

and then, by the interface conditions given in (11), we obtain

$$
\int_S [\mathcal{S}(\lambda,\mu)]\; \mathcal{S}_i(\eta,\psi) ds - (-1)^i \int_S [w(\lambda,\mu)] w_i(\eta,\psi) ds = \int_S \mathcal{S}_i^*(\lambda,\mu)\eta ds + (-1)^i \int_S q_i\psi ds.
$$

It follows that

$$
\langle \nabla J_1(\lambda,\mu), (\eta,\psi)\rangle = \int_S (\mathcal{S}_1^*(\lambda,\mu) + \mathcal{S}_2^*(\lambda,\mu))\eta ds - \int_S (q_1 - q_2)_{|S}\psi ds.
$$

2.2 Numerical results

The evaluation of $J_1(\lambda,\mu)$ in a minimization algorithm (not using ∇J_1) requires the solution of a multidomain problem such as (11). In both the Bi-CGStab and BFGS algorithms, the evaluation of the cost functional and that of its gradient are done at the same time. In particular, after the evaluation of $\nabla J_1(\lambda,\mu)$, the evaluation of $J_1(\lambda,\mu)$ requires minimal additional computational effort. Each evaluation of $\nabla J_1(\lambda,\mu)$ needs the solution of two multidomain problems (that is the solution of (11) and (14)); moreover, at each iteration of the BFGS method, one evaluation of ∇J_1 is required, while, at each iteration of Bi-CGStab, two evalutations of ∇J_1 are needed.

In Table 1 we show the computational cost, in terms of the *Number of Multidomain Problems (NMP)*which we have to solve during the whole minimization algorithm, for a variant of the principal axis method (see [1]), which does not require the evaluation of the gradient, and for both the BFGS and Bi-CGStab methods with exact evaluation of ∇J_1. We also report the minimum value $\hat{J}_1$ of J_1 attained by the minimization procedure. The approximation on each Ω_i was carried out by the conformal spectral element method with interpolation degree $N = 5$ on each element (equal in both $x-$ and $y-$directions) and element diameter of the mesh $H = 0.5$. For all the methods and the values of σ, the relative error in the H^1-norm between the numerical and analytic solutions is about $6 \cdot 10^{-10}$. The advantage of using the exact evaluation of the cost functional gradient is evident, and for $\sigma \leq 10^{-2}$ the use of Bi-CGStab is preferable.

	PrAxis method		BFGS method		Bi-CGStab method	
σ	$\hat{J}_1$	NMP	$\hat{J}_1$	NMP	$\hat{J}_1$	NMP
1.	3.2980E-13	15,392	2.1627E-14	476		>1200
1.E-1	7.3888E-20	5,495	3.4173E-17	372	3.3552E-15	652
1.E-2	5.5671E-13	4,968	1.5088E-19	224	2.4491E-16	160
1.E-3	8.4217E-21	6,939	2.1297E-21	222	4.9374E-19	156

Table 1. Computational cost for Principal Axis method without the evaluation of ∇J_1, BFGS and Bi-CGStab with exact evaluation of ∇J_1. We solved problem (5)–(10) in $\Omega = (-1,1)^2$, with $\Omega_1 = (-1,0) \times (-1,1)$, $\Omega_2 = (0,1) \times (-1,1)$ and $\alpha = 1$. The right hand side f and the boundary data on $\partial\Omega$ were constructed so that the analytic solution is $u(x,y) = (x^2-1)e^y + (y^2-1)e^x$.

3 The heterogeneous coupling

We now consider problem (4). Based on the use of virtual control theory, it can be reformulated as: find $u_1 \in H^1_{\Gamma_1}(\Omega_1)$ and $(u_2, w_2) \in V_2$ such that

$$\begin{cases} ((u_1,z_1))_{\Omega_1} = (f,z_1)_{\Omega_1} & \forall z_1 \in H^1_0(\Omega_1) \\ ((u_2,z_2))_{\Omega_2} - \sigma(\nabla w_2, \nabla z_2)_{\Omega_2} = (f,z_2)_{\Omega_2} & \forall z_2 \in H^1_0(\Omega_2) \\ \sigma(\nabla u_2, v_2)_{\Omega_2} + (w_2,v_2)_{\Omega_2} = \displaystyle\int_S \mu v_2 & \forall v_2 \in H^1(\Omega_2) \\ u_1 = u_2 = \lambda & \text{on } S, \end{cases} \tag{16}$$

where the virtual controls λ, μ are determined by solving the minimization problem

$$\inf_{(\lambda,\mu)\in\Lambda} J_2(u_1(\lambda), u_2(\lambda,\mu), w_2(\lambda,\mu)) \tag{17}$$

with (see [4] for the derivation)

$$J_2(\lambda,\mu) = \frac{1}{2}\int_S \left[\left(\frac{\partial u_1}{\partial n_S} - \left(\frac{\partial u_2}{\partial n_S} - \sigma\frac{\partial w_2}{\partial n_S}\right)\right)^2 + \left(\sigma\frac{\partial u_2}{\partial n_S}\right)^2\right] ds. \tag{18}$$

3.1 The minimization procedure

In order to compute $\nabla J_2(\lambda,\mu)$ we follow the ideas of Section 2.1. The state equations for the heterogeneous coupling are given as

$$\begin{cases} -\Delta u_1 + \alpha u_1 = 0 & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \Gamma_1, \qquad u_1 = \lambda \quad \text{on } S \end{cases} \tag{19}$$

in Ω_1 and by (11) in Ω_2. The solution of (19) is denoted by $u_1(\lambda)$ and the local Steklov-Poincaré operator $\mathcal{S}_1$ which is associated to (19) is defined as $\mathcal{S}_1(\lambda) := (\partial u_1(\lambda)/\partial n_1)|_S$. Moreover, by noting that $\mu = -\sigma\partial u_2/\partial n_S$, we can rewrite J_2 as

$$J_2(\lambda,\mu) = \frac{1}{2}\int_S (\mathcal{S}_1(\lambda) + \mathcal{S}_2(\lambda,\mu))^2\, ds + \frac{1}{2}\int_S \mu^2 ds,$$

so that to minimize J_2 is equivalent to setting $\mu = 0$ and minimizing $J_S(\lambda) = \frac{1}{2}\int_S (\mathcal{S}_1(\lambda) + \mathcal{S}_2(\lambda,0))^2\, ds$.

To compute $J'_S(\lambda)$ we introduce the dual problem:

$$\begin{cases} -\Delta p_1 + \alpha p_1 = 0 \text{ in } \Omega_1 \\ p_1 = [\mathcal{S}(\lambda,0)] \quad \text{on } \Gamma_1, \qquad p_1 = 0 \text{ on } S \end{cases} \tag{20}$$

in Ω_1 and use the dual problem (14) in Ω_2, with interface conditions $p_2 = [\mathcal{S}(\lambda,0)]$, $\sigma\partial p_2/\partial n_2 = 0$, where now $[\mathcal{S}(\lambda,0)] = \mathcal{S}_1(\lambda) + \mathcal{S}_2(\lambda,0)$.

Accordingly, we denote by $\mathcal{S}_1^*(\lambda)$ the Steklov-Poincaré operator associated to the dual problem (20): $\mathcal{S}_1^*(\lambda) = (\partial p_1/\partial n_1)|_S$. By the duality between (11) and (14), (19) and (20), we obtain the following expression for $J'_S(\lambda)$:

$$J'_S(\lambda) = \mathcal{S}_1^*(\lambda) + \mathcal{S}_2^*(\lambda,0). \tag{21}$$

In fact, by proceeding in a formal way as in Section 2.1, for a suitable function η defined on S, we obtain:

$$
\begin{aligned}
0 = & \int_{\Omega_1} (-\Delta p_1 + \alpha p_1)\ u_1(\eta) d\Omega + \int_{\Omega_2} (-\Delta p_2 + \alpha p_2 - \sigma \Delta q_2)\ u_2(\eta, 0) d\Omega \\
& + \int_{\Omega_2} (\sigma \Delta p_2 + q_2)\ w_2(\eta, 0) d\Omega \\
& \text{(by Green's formula)} \\
= & \int_{\Omega_1} (-\Delta u_1(\eta) + \alpha u_1(\eta))\ p_1 d\Omega + \int_S p_1\ \frac{\partial u_1}{\partial n_1}(\eta) ds - \int_S \frac{\partial p_1}{\partial n_1}\ u_1(\eta) ds \\
& + \int_{\Omega_2} (-\Delta u_2(\eta, 0) + \alpha u_2(\eta, 0))\ p_2 d\Omega - \sigma \int_{\Omega_2} \Delta u_2(\eta, 0)\ q_2 d\Omega \\
& + \int_S (p_2 + \sigma q_2)\ \frac{\partial u_2}{\partial n_2}(\eta, 0) ds - \int_S \left(\frac{\partial p_2}{\partial n_2} + \sigma \frac{\partial q_2}{\partial n_2} \right) u_2(\eta, 0) ds \\
& + \int_{\Omega_2} \sigma \Delta w_2(\eta, 0)\ p_2 d\Omega + \int_{\Omega_2} q_2\ w_2(\eta, 0) d\Omega \\
& - \sigma \int_S \frac{\partial w_2}{\partial n_2}(\eta, 0) p_2 ds + \sigma \int_S \frac{\partial p_2}{\partial n_2} w_2(\eta, 0) ds \\
& \text{(by the first equation in (19) and the first and second equations in (11)} \\
& \text{and by the boundary conditions on } S \text{ in (20) and (14))} \\
= & \int_S [\mathcal{S}(\lambda, 0)] \left(\frac{\partial u_1}{\partial n_1}(\eta) + \left(\frac{\partial u_2}{\partial n_2}(\eta, 0) - \sigma \frac{\partial w_2}{\partial n_2}(\eta, 0) \right) \right) ds \\
& - \int_S \frac{\partial p_1}{\partial n_1} u_1(\eta) + \sigma \int_S q_2 \frac{\partial u_2}{\partial n_2}(\eta, 0) ds - \int_S \left(\frac{\partial p_2}{\partial n_2} + \sigma \frac{\partial q_2}{\partial n_2} \right) u_2(\eta, 0) ds.
\end{aligned}
$$

Then, since $u_1(\eta)|_S = u_2(\eta, 0)|_S = \eta$ and $(\partial u_2(\eta, 0)/\partial n_2)|_S = 0$ (by (11) and (19)), we see that

$$
\begin{aligned}
\langle \nabla J_S(\lambda), \eta \rangle &= \int_S [\mathcal{S}(\lambda, 0)]\ (\mathcal{S}_1(\eta) + \mathcal{S}_2(\eta, 0)) ds \\
&= \int_S (\mathcal{S}_1^*(\lambda) + \mathcal{S}_2^*(\lambda, 0))\ \eta\ ds.
\end{aligned}
$$

3.2 Numerical results for the heterogeneous coupling

The use of a single virtual control λ, instead of two, reduces the dimension of the minimization problem. In order to evaluate $J_S'(\lambda)$ we have to solve both a primal multidomain problem ((19) in Ω_1 and (11) in Ω_2) and a dual multidomain problem ((20) in Ω_1 and (14) in Ω_2). As in the homogeneous case we denote by NMP the total number of multidomain problems to be solved during the minimization algorithm.

In Table 2 we report NMP for the principal axis, BFGS and Bi-CGStab methods. Moreover we show the minimum value $\hat{J}_S$ attained by J_S for all the methods, while the $H^1(\Omega)$- norm error between the numerical solution and the global fourth-order solution $u(x,y) = (x^2-1)e^y + (y^2-1)e^x$ is shown once.

Also for the heterogeneous coupling the exact evaluation of $J'_S(\lambda)$ substantially improves the efficiency of the minimization procedure. For the heterogeneous coupling the BFGS method is more efficient than Bi-CGStab for all values of the costant σ, above all when $\sigma > 10^{-2}$.

		PrAxis method		BFGS method		Bi-CGStab method	
σ	$H^1 - err$	$\hat{J}_S$	NMP	$\hat{J}_S$	NMP	$\hat{J}_S$	NMP
1.	1.0866E-1	1.4340E-15	8,636	1.5895E-15	284	6.1703E-15	656
1.E-1	1.0100E-2	1.1719E-14	6,223	3.1870E-20	138	1.3818E-15	296
1.E-2	1.0316E-3	1.0821E-17	3,442	4.1756E-20	100	4.5086E-15	112
1.E-3	1.1834E-5	3.7528E-19	10,336	2.5455E-23	102	6.1636E-16	108

Table 2. Heterogeneous coupling. Comparison between Principal Axis method without the evaluation of J'_S, BFGS and Bi-CGStab methods with exact evaluation of J'_S, in order to solve problem (16)–(18). All data are those specified in the caption of Table 1.

In Table 3 we show NMP for two given values of σ, for different positions of the interface S of the decomposition. We observe that NMP does not substantially depend on the interface position for both the BFGS and Bi-CGStab methods.

$\sigma = 1$		
x_S	BFGS	Bi-CGStab
-0.5	300	716
-0.25	300	660
0.0	284	656
0.25	308	708
0.5	306	516

$\sigma = 0.01$		
x_S	BFGS	Bi-CGStab
-0.5	104	112
-0.25	106	108
0.0	100	112
0.25	102	112
0.5	100	104

Table 3. Heterogeneous coupling. x_S is the position of the interface S between Ω_1 and Ω_2. The *Number of Multidomain Problems* needed to solve problem (16)–(18) with BFGS and Bi-CGStab are shown. Other problem data, not specified here, are equal to those used in Table 1.

Lastly, in Figure 2 we show the numerical solution obtained for the membrane-plate heterogeneous coupling (16) with a uniform external load $f \equiv -1$, homogeneous boundary data on $\partial\Omega$, $\sigma = 1$, $\alpha = 1$. The computational domain is $\Omega = (-1,1)^2$, while $\Omega_1 = (-1,-0.25) \times (-1,1)$ and $\Omega_2 = (-0.25,1) \times (-1,1)$. Convergence to $\hat{J}_2 = 4.3397E-19$ was obtained by the BFGS method with 298 NMP.

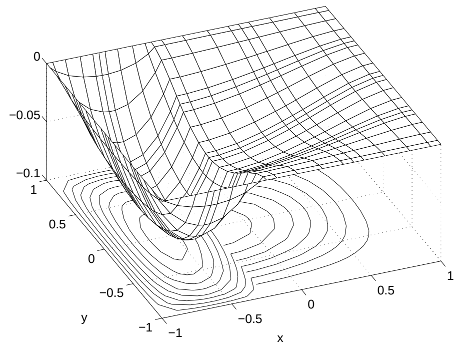

Fig. 2. Solution for the heterogeneous problem (16) in $\Omega = (-1,1)^2$ with $\Omega_1 = (-1,-0.25) \times (-1,1)$ and $\Omega_2 = (-0.25,1) \times (-1,1)$, with a uniform external load $f \equiv -1$, homogeneous boundary data on $\partial\Omega$, $\sigma = 1$, $\alpha = 1$.

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