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CONVERGENCE ANALYSIS OF HIGH ORDER ALGEBRAIC FRACTIONAL STEP SCHEMES FOR TIME-DEPENDENT STOKES EQUATIONS*

PAOLA GERVASIO†

Dedicated to Fausto Saleri

Abstract. In this paper we analyze the family of Yosida algebraic fractional step schemes proposed in [A. Quarteroni, F. Saleri, and A. Veneziani, *Comput. Methods Appl. Mech. Engrg.*, 188 (2000), pp. 505–526], [F. Saleri and A. Veneziani, *SIAM J. Numer. Anal.*, 43 (2005), pp. 174–194], and [P. Gervasio, F. Saleri, and A. Veneziani, *J. Comput. Phys.*, 214 (2006), pp. 347–365] when applied to time-dependent Stokes equations. Under suitable regularity assumptions on the data, splitting error estimates both for velocity and pressure are established. In particular we analyze the first three methods of this family, providing, respectively, convergence (of the fractional step solution towards the numerical solution achieved without any operator splitting) of orders 3/2, 5/2, 7/2 for the velocity and 1, 2, 3 for the pressure. Moreover a general way to set up higher-order schemes is proposed. The present analysis is carried out when spectral element methods are employed for space discretization.

Key words. algebraic fractional step methods, time-dependent Stokes equations, spectral element methods

AMS subject classifications. 65M12, 76D07, 76M22

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1. Introduction.

The aim of this paper is to provide convergence estimates for the class of *Yosida* methods proposed in [12, 13, 15, 8] for the numerical solution of time-dependent Stokes equations in the primitive variables velocity and pressure.

Yosida schemes are algebraic fractional step schemes in which the incompressibility constraint is relaxed in order to earn computational efficiency, and they are based on an inexact LU factorization of the matrix $\mathcal{A}$ arising from the full discretization (in space and time) of Stokes equations. The basic Yosida scheme was introduced in [13]. The accuracy analysis, carried out in [12], shows that when the time discretization is based on a backward Euler method, Yosida splitting still maintains the first order accuracy in time both for velocity and pressure (with respect to appropriate norms).

Successively, in [15, 8] two improved versions of the basic Yosida method (named *Yosida*-3 and *Yosida*-4 in the present paper) have been proposed. A pressure correction step is introduced in these schemes, with the aim of increasing the accuracy in time for both velocity and pressure. A preliminary analysis of the consistency error has been carried out for *Yosida*-3 (resp., *Yosida*-4) in [15] (resp., [8]) proving that the perturbation on the discretization matrix $\mathcal{A}$ is of third (resp., fourth) order in time with respect to the time-step. However, a complete splitting error analysis, i.e., the convergence analysis of the fractional step solution towards the numerical solution achieved without any operator splitting, is still missing. The aim of the present paper is essentially to perform such analysis when the space discretization is based on spectral element methods of $\mathbb{Q}_N - \mathbb{Q}_{N-2}$ type [11, 1, 3, 4], but we also emphasize the derivation of Yosida methods from a more general point of view than that given in

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†Department of Mathematics, University of Brescia, 25133 Brescia, Italy (gervasio@ing.unibs.it).

[15, 8], suggesting the formulation of higher-order schemes. The main result of the paper is Theorem 3.7, in which we prove that the splitting error induced by Yosida schemes (when they are coupled to BDFq (with $q = 1,\ 2$) for time discretization of the time-dependent Stokes equations) behaves like $\Delta t^{p+3/2}$ (resp., Δt^{p+1}) for the velocity (resp., for the pressure), where $p = 0$ for basic *Yosida*, $p = 1$ for *Yosida*-3, and $p = 2$ for *Yosida*-4. The thesis of Theorem 3.7 when $q = 3, 4$ has not been proved yet. Nevertheless in this paper we consider the more general contest with $q = 1, \dots, 4$ since numerical results relative to the choice $q = p + 2$ are encouraging and provide global approximation errors on the velocity of order q with respect to Δt. In the present paper no evidence is given to computational aspects concerned with Yosida methods, since the topic has been extensively discussed in [8].

This paper is organized as follows. In section 2 we recall notation and settings about Stokes equations, spectral elements discretization, and some useful properties on symmetric positive definite matrices. In section 3 we present the derivation of Yosida methods for the unsteady Stokes equations, perform the splitting error analysis when spectral element methods are used for space discretization, and give some perspective for extending the analysis to unsteady Navier–Stokes equations. In section 4 we present some numerical results corroborating the convergence analysis developed in section 3. Even if the theory presented in this paper holds in $\mathbb{R}^d$, with $d = 2, 3$, only numerical results for the case $d = 2$ are reported here.

2. Problem statement and settings. We consider time-dependent Stokes equations for Newtonian incompressible fluids in the velocity-pressure formulation. For any open bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a Lipschitz boundary $\partial\Omega$, and a positive T fixed, given a solenoidal datum $\mathbf{u}_0 \in [H^1(\Omega)]^d$, an external field $\mathbf{f} \in [L^2(0, T; H^{-1}(\Omega))]^d$, and a boundary datum $\mathbf{g} \in [L^2(0, T; H^{1/2}(\partial\Omega))]^d$, we look for the velocity field $\mathbf{u} \in [L^2(0, T; H^1(\Omega))]^d$ and the pressure field $p \in L^2(0, T; L^2_0(\Omega))$ solutions of

$$(2.1) \qquad \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \times (0, T), \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times (0, T), \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega \times (0, T), \\ \mathbf{u} = \mathbf{u}_0 & \text{in } \Omega \times \{0\}, \end{cases}$$

where $\nu > 0$ is the kinematic viscosity. It is well known that problem (2.1) admits a unique solution (see, e.g., [16]).

We approximate the time derivative by a *backward differentiation formula* (BDF) of order q. Given $\Delta t \in (0, T)$, we set $t^0 = 0$, $t^n = t^0 + n\Delta t$ (for any $n \geq 1$) and $N_T = \left[\frac{T}{\Delta t}\right]$; therefore, for any integer $n = n_0(= q - 1), \dots, N_T - 1$, we look for the solution $(\mathbf{u}^{n+1}, p^{n+1})$ of the system

$$(2.2) \qquad \begin{cases} \dfrac{\beta_{-1}}{\Delta t}\mathbf{u}^{n+1} - \nu\Delta\mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1} + \displaystyle\sum_{j=0}^{q-1} \frac{\beta_j}{\Delta t}\mathbf{u}^{n-j} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u}^{n+1} = 0 & \text{in } \Omega, \\ \mathbf{u}^{n+1} = \mathbf{g}^{n+1} & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{u}^0 = \mathbf{u}_0$ and β_j (for $j = -1, \dots, q$) are the coefficients of BDF of order q. When BDF with order greater than one are used, initial data could be provided by suitable explicit schemes (e.g., Runge–Kutta) of the same order as the BDF used.

About the space discretization, we choose quadrilateral conforming spectral elements [11]. In order to overcome instabilities due to the mixed formulation of Stokes equations, we make use of the $\mathbb{Q}_N - \mathbb{Q}_{N-2}$ scheme with staggered grids [1], according to which local polynomials of degree N in each variable are used to approximate every component of the velocity field, and local polynomials of degree $N-2$ in each variable are used to approximate the pressure ($\mathbb{Q}_N$ denotes the space of polynomials of degree less than or equal to N in each variable). By this choice, the inf-sup condition is satisfied with a constant β which is proportional to $N^{(1-d)/2}$ [1].

We introduce a conformal, regular, and quasi-uniform (see, e.g., [14]) partition $\mathcal{T}_h$ of Ω in N_e quadrilaterals T_k such that

$$\overline{\Omega} = \bigcup_{k=1}^{N_e} \overline{T}_k, \quad \text{with} \quad h = \max_{T_k \in \mathcal{T}_h} h_k, \qquad h_k = \text{diam}(T_k), \quad k = 1, \dots, N_e.$$

Let us assume that every $T_k \in \mathcal{T}_h$ is the image of the reference square $\hat{T} = (-1,1)^2$ through a smooth invertible mapping $\mathbf{F}_k : \hat{T} \mapsto T_k$ with Jacobian J_{F_k} satisfying $\det J_{F_k}(\hat{\mathbf{x}}) > 0 \ \forall \hat{\mathbf{x}} \in \hat{T}$. We set

$$\mathbb{Q}_{\mathcal{H}}(\Omega) = \{v_{\mathcal{H}} \in \mathcal{C}^0(\overline{\Omega}) : \ v_{\mathcal{H}|T_k} \circ \mathbf{F}_k \in \mathbb{Q}_N \quad \forall T_k \in \mathcal{T}_h\} \tag{2.3}$$

and $v_{N,k} := v_{\mathcal{H}|T_k} \ \forall T_k \in \mathcal{T}_h$. The subscript $\mathcal{H}$ represents the discretization level and it stands for the couple (h, N). Given $u_{\mathcal{H}}, \ v_{\mathcal{H}} \in \mathbb{Q}_{\mathcal{H}}(\Omega)$, we set

$$(u_{\mathcal{H}}, v_{\mathcal{H}})_{\mathcal{H},\Omega} = \sum_{k=1}^{N_e} (u_{N,k}, v_{N,k})_{N,T_k}, \tag{2.4}$$

where $(\cdot,\cdot)_{N,T_k}$ denotes the discrete inner product in $L^2(T_k)$, based on the Gauss–Lobatto–Legendre (GLL) quadrature formulas [3]. In each element T_k of the partition we define a local GLL grid of $(N+1)^2$ points and a local Gauss–Legendre (GL) grid of $(N-1)^2$ points. The last grid is staggered with respect to the former one and is internal to T_k. The unknowns of the discrete problem will be both the set of the velocity values on the GLL grid and the set of the pressure values on the GL grid [1]. By setting the finite-dimensional spaces

$$\begin{aligned} &\mathbf{V}_{\mathcal{H}} := [\mathbb{Q}_{\mathcal{H}}(\Omega)]^2, \qquad \mathbf{V}^0_{\mathcal{H}} := \left[\mathbb{Q}_{\mathcal{H}}(\Omega) \cap H^1_0(\Omega)\right]^2, \\ &Q_{\mathcal{H}} := \{q_{\mathcal{H}} \in L^2_0(\Omega) : \ q_{\mathcal{H}|T_k} \circ \mathbf{F}_k \in \mathbb{Q}_{N-2} \quad \forall T_k \in \mathcal{T}_h\}, \end{aligned}$$

we reformulate problem (2.1) following the Galerkin approach and replace all exact integrals in $L^2(\Omega)$ with GLL quadrature formulas; the resulting approach is named the *spectral element method with numerical integration* (SEM-NI) [4]. For any $t_n \in (0,T)$, we denote by $\tilde{\mathbf{g}}(t_n) \in [H^1(\Omega)]^2$ the extension of $\mathbf{g}(t_n)$ to Ω by any possible continuous operator from $[H^{1/2}(\partial\Omega)]^2$ to $[H^1(\Omega)]^2$; moreover, we denote its discrete counterpart by $\tilde{\mathbf{g}}^n_{\mathcal{H}} \in \mathbf{V}_{\mathcal{H}}$. At each time-step t_{n+1} (for $n = n_0, \dots, N_T - 1$) we look for the numerical solution $(\mathbf{u}^{n+1}_{\mathcal{H}}, p^{n+1}_{\mathcal{H}}) \in \mathbf{V}_{\mathcal{H}} \times Q_{\mathcal{H}}$, satisfying $(\mathbf{u}^{n+1}_{\mathcal{H}} - \tilde{\mathbf{g}}^{n+1}_{\mathcal{H}}) \in \mathbf{V}^0_{\mathcal{H}}$ and

$$\left\{ \begin{aligned} &\frac{\beta_{-1}}{\Delta t}(\mathbf{u}^{n+1}_{\mathcal{H}}, \mathbf{v}_{\mathcal{H}})_{\mathcal{H},\Omega} + \nu(\nabla \mathbf{u}^{n+1}_{\mathcal{H}}, \nabla \mathbf{v}_{\mathcal{H}})_{\mathcal{H},\Omega} \\ &\qquad - (p^{n+1}_{\mathcal{H}}, \nabla \cdot \mathbf{v}_{\mathcal{H}})_{\mathcal{H},\Omega} = \left(\mathbf{f}^{n+1} + \sum_{j=0}^{q-1} \frac{\beta_j}{\Delta t} \mathbf{u}^{n-j}_{\mathcal{H}}, \mathbf{v}_{\mathcal{H}}\right)_{\mathcal{H},\Omega} \qquad \forall \mathbf{v}_{\mathcal{H}} \in \mathbf{V}^0_{\mathcal{H}}, \\ &(\nabla \cdot \mathbf{u}^{n+1}_{\mathcal{H}}, q_{\mathcal{H}})_{\mathcal{H},\Omega} = 0 \qquad \forall q_{\mathcal{H}} \in Q_{\mathcal{H}}. \end{aligned} \right. \tag{2.5}$$

Therefore, we set N_v (resp., N_p) the total number of GLL (resp., GL) grid points in Ω. For any $\mathbf{u}_\mathcal{H} \in \mathbf{V}_\mathcal{H}$, we denote by $\mathbf{U} \in \mathbb{R}^{2N_v}$ the vector of the expansion coefficients of $\mathbf{u}_\mathcal{H}$ with respect to the Lagrange basis $\{\boldsymbol{\varphi}_j\}_{j=1}^{2N_v}$ (defined on the GLL grid) in $\mathbf{V}_\mathcal{H}$ and by $\mathsf{V}_G \subset \mathbb{R}^{2N_v}$ (resp., $\mathsf{V}_G^0 \subset \mathbb{R}^{2N_v}$) the set of the vectors $\mathbf{U}$ corresponding to functions $\mathbf{u}_\mathcal{H} \in \mathbf{V}_\mathcal{H}$ (resp., $\mathbf{V}_\mathcal{H}^0$). In a similar way, for any $p_\mathcal{H} \in Q_\mathcal{H}$ we denote by $\mathbf{P} \in \mathbb{R}^{N_p}$ the vector of the expansion coefficients of $p_\mathcal{H}$ with respect to the Lagrange basis $\{\eta_l\}_{l=1}^{N_p}$ (defined on the GL grid) in $Q_\mathcal{H}$ and by $\mathsf{Q}_G \subset \mathbb{R}^{N_p}$ the set of the arrays $\mathbf{P}$ corresponding to functions $p_\mathcal{H} \in Q_\mathcal{H}$.

We denote by $M \in \mathbb{R}^{2N_v \times 2N_v}$ the mass matrix $M_{ij} = (\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i)_{\mathcal{H},\Omega}$, by $K \in \mathbb{R}^{2N_v \times 2N_v}$ the stiffness matrix $K_{ij} = (\nabla\boldsymbol{\varphi}_j, \nabla\boldsymbol{\varphi}_i)_{\mathcal{H},\Omega}$, and by $B \in \mathbb{R}^{2N_v \times N_p}$ the matrix related to the discretization of $-\nabla\cdot$: $B_{lj} = -(\nabla \cdot \boldsymbol{\varphi}_j, \eta_l)_{\mathcal{H},\Omega}$, and we set

$$C = \frac{\beta_{-1}}{\Delta t} M + \nu K. \tag{2.6}$$

Remark 1. Matrices M, K, and C are symmetric and positive definite. Moreover, thanks to the fact that the space discretization chosen satisfies the inf-sup condition, the matrix B is a full rank matrix and $\ker(B^T) = \{\mathbf{0}\}$. Finally, we remark that when SEM-NI is used, the mass matrix M is diagonal.

The system is reduced to the unknowns internal to Ω, and the right-hand side is modified accordingly, taking into account the contributions that Lagrange basis functions associated to Dirichlet boundary nodes give to those associated to internal nodes. This step produces a right-hand side $[\mathbf{F}_1^{n+1}, \mathbf{F}_2^{n+1}]^T$ that is nonzero also in the continuity equation.

The algebraic form of (2.5) reads as follows: for $n = n_0, \dots, N_T - 1$ solve

$$\begin{cases} C\mathbf{U}^{n+1} + B^T\mathbf{P}^{n+1} = \mathbf{F}_1^{n+1} + \dfrac{1}{\Delta t}\displaystyle\sum_{j=0}^{q-1} \beta_j M \mathbf{U}^{n-j}, \\ B\mathbf{U}^{n+1} = \mathbf{F}_2^{n+1}, \end{cases} \tag{2.7}$$

or equivalently, for $n = n_0, \dots, N_T - 1$ solve

$$\mathcal{A}\mathbf{W}^{n+1} = \mathbf{G}^{n+1}, \qquad \text{with} \qquad \mathcal{A} = \begin{bmatrix} C & B^T \\ B & 0 \end{bmatrix} \tag{2.8}$$

and

$$\mathbf{W}^{n+1} = \begin{bmatrix} \mathbf{U}^{n+1} \\ \mathbf{P}^{n+1} \end{bmatrix}, \qquad \mathbf{G}^{n+1} = \begin{bmatrix} \mathbf{G}_1^{n+1} \\ \mathbf{G}_2^{n+1} \end{bmatrix} := \begin{bmatrix} \mathbf{F}_1^{n+1} + \dfrac{1}{\Delta t}\displaystyle\sum_{j=0}^{q-1} \beta_j M \mathbf{U}^{n-j} \\ \mathbf{F}_2^{n+1} \end{bmatrix}.$$

System (2.8) could be solved by a global approach such as a preconditioned Krylov method with either algebraic or differential preconditioners. Alternatively, a block LU factorization of $\mathcal{A}$ can be performed with

$$L = \begin{bmatrix} C & 0 \\ B & -BC^{-1}B^T \end{bmatrix}, \qquad U = \begin{bmatrix} I & C^{-1}B^T \\ 0 & I \end{bmatrix}, \tag{2.9}$$

so that, for any $n = n_0, \dots, N_T - 1$, system (2.7) reads also

$$\begin{array}{ll} L\text{-step}: \text{ find } \mathbf{U}^{n+1/2},\ \mathbf{P}^{n+1/2}: & \begin{cases} C\mathbf{U}^{n+1/2} = \mathbf{G}_1^{n+1}, \\ \Sigma\mathbf{P}^{n+1/2} = \mathbf{G}_2^{n+1} - B\mathbf{U}^{n+1/2}, \end{cases} \\ U\text{-step}: \text{ find } \mathbf{U}^{n+1},\ \mathbf{P}^{n+1}: & \begin{cases} \mathbf{P}^{n+1} = \mathbf{P}^{n+1/2}, \\ C(\mathbf{U}^{n+1/2} - \mathbf{U}^{n+1}) = B^T\mathbf{P}^{n+1}. \end{cases} \end{array} \tag{2.10}$$

The matrix

$$\Sigma := -BC^{-1}B^T \in \mathbb{R}^{N_p \times N_p} \tag{2.11}$$

is the so-called *pressure Schur complement* matrix and $-\Sigma > 0$. Solving system (2.8) by a block *LU* factorization offers the advantage of splitting the original problem into subproblems of smaller size, even if the assembling of Σ is, however, quite expensive. The idea of Yosida schemes consists in replacing the block *LU* factorization (2.9) with an inexact one or, equivalently, in replacing algorithm (2.10) with a more efficient one from the computational point of view. The next section will be devoted to present the derivation of Yosida schemes, starting from (2.8).

We are now going to introduce some notation and report some properties we will use in the next sections, referring to [17] and [9] for an exhaustive treatment of these subjects.

D1. Given a real square matrix $\mathcal{B}$, we write $\mathcal{B} > 0$ if it is symmetric positive definite (s.p.d.), while we write $\mathcal{B} \geq 0$ if it is symmetric semipositive definite. Moreover, if $\mathcal{B}, \mathcal{C} \in \mathbb{R}^{n\times n}$ are symmetric, we write $\mathcal{B} > \mathcal{C}$ if $\mathcal{B} - \mathcal{C} > 0$ and $\mathcal{B} \geq \mathcal{C}$ if $\mathcal{B} - \mathcal{C} \geq 0$.

D2. The eigenvalues of a matrix $\mathcal{B} \in \mathbb{R}^{n\times n}$ are denoted by $\lambda_i(\mathcal{B})$ and the *spectral radius* of $\mathcal{B}$ is defined as $\rho(\mathcal{B}) = \max\{|\lambda_i(\mathcal{B})|,\ i = 1, \dots, n\}$.

D3. For any square matrix $\mathcal{M}$ of the form

$$\mathcal{M} = \begin{bmatrix} \mathcal{C} & \mathcal{B}^T \\ \mathcal{B} & \mathcal{D} \end{bmatrix}$$

the Schur complement of the nonsingular block $\mathcal{C}$ in $\mathcal{M}$ is defined as $\mathcal{M}/\mathcal{C} := \mathcal{D} - \mathcal{B}\mathcal{C}^{-1}\mathcal{B}^T$. Moreover, $\det(\mathcal{M}) = \det(\mathcal{C})\det(\mathcal{M}/\mathcal{C})$ (*Schur's formula* [17, Thm. 2.2]).

D4. The *inertia* $In(\mathcal{B})$ of a square matrix $\mathcal{B}$ is defined to be the ordered triplet (i_+, i_-, i_0), where i_+, i_-, and i_0 are the numbers of positive, negative, and zero real parts of the eigenvalues of $\mathcal{B}$.

D5. For any matrix $\mathcal{B} \in \mathbb{R}^{n\times n}$, whose coefficients depend on Δt, we say that

$$\mathcal{B} = \mathcal{O}(\Delta t^k) \qquad \text{if} \qquad |||\mathcal{B}||| = \mathcal{O}(\Delta t^k) \qquad \text{for } \Delta t \to 0,$$

where $|||\cdot|||$ denotes the 2-norm for matrices.

P1. For any invertible matrices $\mathcal{B},\ \mathcal{C} \in \mathbb{R}^{n\times n}$ it holds that

$$(\mathcal{B} + \mathcal{C})^{-1} = \mathcal{B}^{-1}[\mathcal{B} - (\mathcal{C}^{-1} + \mathcal{B}^{-1})^{-1}]\mathcal{B}^{-1}, \tag{2.12}$$

$$\mathcal{B} - \mathcal{C} = -\mathcal{B}(\mathcal{B}^{-1} - \mathcal{C}^{-1})\mathcal{B}, \tag{2.13}$$

$$\mathcal{B}(\mathcal{B} + \mathcal{C})^{-1}\mathcal{B} = \mathcal{B} - \mathcal{C} + \mathcal{C}(\mathcal{B} + \mathcal{C})^{-1}\mathcal{C}. \tag{2.14}$$

P2. As a consequence of the Courant–Fisher minimax theorem [17], for any real symmetric matrices $\mathcal{B},\ \mathcal{C} \in \mathbb{R}^{n\times n}$ it holds that $\lambda_i(\mathcal{B}) + \lambda_{min}(\mathcal{C}) \leq \lambda_i(\mathcal{B} + \mathcal{C}) \leq \lambda_i(\mathcal{B}) + \lambda_{max}(\mathcal{C})$, $i = 1, \dots, n$.

P3. If $\mathcal{B}, \mathcal{C} \in \mathbb{R}^{n\times n}$ are s.p.d. and $\mathcal{B} - \mathcal{C} > 0$, then $\mathcal{C}^{-1} - \mathcal{B}^{-1} > 0$.

P4. If $\mathcal{B}, \mathcal{C} \in \mathbb{R}^{n\times n}$ are symmetric and $\mathcal{C} > 0$, then the eigenvalues of $\mathcal{BC}$ are all real and $In(\mathcal{BC}) = In(\mathcal{B})$ (see, e.g., [5]).

P5. If a matrix is the product of two s.p.d. matrices, then it is similar to an s.p.d. matrix (this is a consequence of P4).

P6. For any s.p.d. matrix $\mathcal{C} \in \mathbb{R}^{n\times n}$ and $\mathcal{X} \in \mathbb{R}^{m\times n}$ with $\ker(\mathcal{X}^T) = \{\mathbf{0}\}$, it holds [18] that

$$\mathcal{C}^{-1} - \mathcal{X}^T(\mathcal{X}\mathcal{C}\mathcal{X}^T)^{-1}\mathcal{X} \geq 0. \tag{2.15}$$

3. Yosida schemes. Yosida schemes come basically from an inexact block LU factorization of matrix $\mathcal{A}$. The first method of this family was introduced in [12, 13] and consists in replacing the Schur complement Σ by a suitable matrix S.

In [15] and [8] two improved versions of the *Yosida* method (here named *Yosida*-3 and *Yosida*-4) were proposed. Such improved Yosida schemes introduce a correction step on the pressure with the aim of increasing the accuracy in time for both velocity and pressure. In particular, these improved Yosida schemes are defined by a new matrix Q, inside the U-step of (2.10) and acting on the pressure, which corrects the approximation of the Schur complement Σ. All is done at the algebraic level, without a compulsory differential interpretation. This feature allows us to neglect the setting up of boundary conditions for the substeps. All methods and the analysis developed hereafter can be applied to different kinds of boundary conditions as well.

The idea of Yosida schemes is as follows: at each time-step, instead of solving system (2.8) by the block LU factorization set in (2.9), we are interested in solving a new system

$$\hat{\mathcal{A}}\widehat{\mathbf{W}}^{n+1} = \widehat{\mathbf{G}}^{n+1}, \tag{3.1}$$

where $\hat{\mathcal{A}} := \hat{L}\hat{U}$ is generated by a suitable $\hat{L}\hat{U}$ inexact factorization of $\mathcal{A}$, $\widehat{\mathbf{W}}^{n+1}$ is an approximation of $\mathbf{W}^{n+1}$ defined as

$$\widehat{\mathbf{W}}^{n+1} = \begin{bmatrix} \hat{\mathbf{U}}^{n+1} \\ \hat{\mathbf{P}}^{n+1} \end{bmatrix}, \quad \text{while} \quad \widehat{\mathbf{G}}^{n+1} = \begin{bmatrix} \mathbf{F}_1^{n+1} + \dfrac{1}{\Delta t}\displaystyle\sum_{j=0}^{q-1}\beta_j M\widehat{\mathbf{U}}^{n-j} \\ \mathbf{F}_2^{n+1} \end{bmatrix}. \tag{3.2}$$

Let $H = (\Delta t/\beta_{-1})M^{-1}$; the inexact factors $\hat{L}$ and $\hat{U}$ of $\mathcal{A}$ are chosen as follows:

$$\hat{L} = \begin{bmatrix} C & 0 \\ B & -BHB^T \end{bmatrix}, \quad \hat{U} = \begin{bmatrix} I & C^{-1}B^T \\ 0 & Q \end{bmatrix}, \tag{3.3}$$

where the nonsingular matrix Q may be set up by following different strategies (see section 3.1), and its choice leads the accuracy of the whole scheme. By setting

$$S := -BHB^T, \tag{3.4}$$

matrix $\hat{\mathcal{A}}$ reads

$$\hat{\mathcal{A}} = \begin{bmatrix} C & B^T \\ B & SQ - \Sigma \end{bmatrix}. \tag{3.5}$$

Matrix S is an approximation of the pressure Schur complement Σ of $\mathcal{A}$ and $-S > 0$.

For any $n = n_0, \dots, N_T - 1$, system (3.1) reads also

$$\left\{\begin{array}{r}C\widehat{\mathbf{U}}^{n+1} + B^T\widehat{\mathbf{P}}^{n+1} = \mathbf{F}_1^{n+1} + \dfrac{1}{\Delta t}\displaystyle\sum_{j=0}^{q-1}\beta_j M\widehat{\mathbf{U}}^{n-j},\\ B\widehat{\mathbf{U}}^{n+1} - (\Sigma - SQ)\widehat{\mathbf{P}}^{n+1} = \mathbf{F}_2^{n+1}\end{array}\right. \tag{3.6}$$

(this is the Yosida counterpart of (2.7)) or again

$$\begin{array}{ll}\hat{L}\text{-step}:\ \text{find } \widehat{\mathbf{U}}^{n+1/2},\ \widehat{\mathbf{P}}^{n+1/2}: & \left\{\begin{array}{l}C\widehat{\mathbf{U}}^{n+1/2} = \widehat{\mathbf{G}}_1^{n+1},\\ S\widehat{\mathbf{P}}^{n+1/2} = \widehat{\mathbf{G}}_2^{n+1} - B\widehat{\mathbf{U}}^{n+1/2},\end{array}\right.\\ \hat{U}\text{-step}:\ \text{find } \widehat{\mathbf{U}}^{n+1},\ \widehat{\mathbf{P}}^{n+1}: & \left\{\begin{array}{l}Q\widehat{\mathbf{P}}^{n+1} = \widehat{\mathbf{P}}^{n+1/2},\\ C(\widehat{\mathbf{U}}^{n+1/2} - \widehat{\mathbf{U}}^{n+1}) = B^T\widehat{\mathbf{P}}^{n+1}\end{array}\right.\end{array} \tag{3.7}$$

(this is the Yosida counterpart of (2.10)).

For any t^n, system (3.1) is well posed iff $\hat{\mathcal{A}}$ is invertible, i.e., iff Q is invertible. As a matter of fact, recalling the definition of $\hat{\mathcal{A}}$ and thanks to Schur's formula, we can write $\det(\hat{\mathcal{A}}) = \det(C)\det(SQ)$.

In the following section we will present some possible choices for nonsingular matrices Q and provide algebraic fractional step schemes supplying different accuracy properties.

3.1. How to set up matrix Q. System (2.7) is equivalent to (3.6) if $\hat{\mathcal{A}} = \mathcal{A}$, i.e., if either $Q = S^{-1}\Sigma$ or $Q^{-1} = \Sigma^{-1}S$, but the exact computation of Q is infeasible, since it would imply computing Σ directly, which is what we want to avoid. Our aim is to set suitable sequences of matrices $\{\tilde{Q}_p\}_{p\geq 0}$ approximating Q such that

$$\exists C > 0 \qquad |||\Sigma - S\tilde{Q}_p||| \leq C\Delta t^{p+2} \qquad \text{as} \qquad \Delta t \to 0, \tag{3.8}$$

where $|||\cdot|||$ denotes the 2-norm for matrices. The quantity $|||\Sigma - S\tilde{Q}_p|||$ is named the *consistency error* and is due to the approximation of $\mathcal{A}$ by $\hat{\mathcal{A}}$.

The direct definition of $Q = S^{-1}\Sigma$ and its definition by the inverse $Q^{-1} = \Sigma^{-1}S$ suggest two different strategies for deriving matrices $\tilde{Q}_p$ invoked in (3.8). The first strategy exploits the expansion in series of Σ (through the expansion of C^{-1}) and the matrices so derived will be denoted by $\{Q_p\}_{p\geq 0}$. The second strategy takes advantage of the expansion of Σ^{-1} as well and the resulting matrices will be denoted by $\{\hat{Q}_p\}_{p\geq 0}$.

Let us begin to set up matrices Q_p. If $\rho(\nu HK) < 1$, we can expand Σ as follows:

$$\Sigma = -BC^{-1}B^T = -B(I + \nu HK)^{-1}HB^T = -\sum_{k\geq 0} B(-\nu HK)^k HB^T. \tag{3.9}$$

By setting

$$D_k := B(-\nu HK)^k HB^T = \mathcal{O}(\Delta t^{k+1}) \quad \text{for } k = 0, 1, 2, \dots \tag{3.10}$$

we have $\Sigma = -\sum_{k\geq 0} D_k$ and $Q = S^{-1}\Sigma = -S^{-1}\sum_{k\geq 0} D_k$. Therefore, we define

$$Q_p := -S^{-1}\sum_{k=0}^{p} D_k \quad \text{for} \ \ p = 0,\ 1,\ 2, \dots. \tag{3.11}$$

The explicit forms of the first three items of $\{Q_p\}$ are

$$Q_0 = I, \qquad Q_1 = I - S^{-1}D_1, \qquad Q_2 = I - S^{-1}D_1 - S^{-1}D_2. \tag{3.12}$$

Remark 2. In the case of SEM-NI discretization, the assumption $\rho(\nu HK) < 1$ turns into

$$\nu\frac{\Delta t}{\beta_{-1}}\rho(M^{-1}K) = c\nu\frac{\Delta t}{\beta_{-1}}N^4h^{-2} < 1,$$

where c denotes a generic positive constant independent of Δt, N, and h [3, 4]. It follows that the convergence of the series $\sum_{k\geq 0}(-\nu HK)^k$ to $(I+\nu HK)^{-1}$ is guaranteed if

$$\Delta t < c\frac{\beta_{-1}h^2}{\nu\ N^4}. \tag{3.13}$$

It is worth remarking that condition (3.13) implies the nonsingularity of matrices Q_p as well.

To derive the second sequence of matrices $\{\hat{Q}_p\}$ we note that $S = -D_0$ and we write

$$Q^{-1} = \Sigma^{-1}S = \left(-\sum_{k\geq 0} D_k\right)^{-1} \cdot S = \left(S - \sum_{k\geq 1} D_k\right)^{-1} \cdot S = \left(I - S^{-1}\sum_{k\geq 1} D_k\right)^{-1}.$$

By putting $R := S^{-1}(\sum_{k\geq 1} D_k)$ it holds that $Q^{-1} = (I-R)^{-1}$ and, by assuming that $\rho(R) < 1$, it holds that $Q^{-1} = \sum_{k\geq 0} R^k$ as well. Then we rewrite Q^{-1} as a sum of matrices $\tilde{R}_k$ s.t. $\tilde{R}_k = \mathcal{O}(\Delta t^k)$ in order to highlight the dependence on powers of Δt. We have $Q^{-1} = \sum_{k\geq 0}\tilde{R}_k$ and then we set

$$\hat{Q}_p := \left(\sum_{k=0}^{p}\tilde{R}_k\right)^{-1} \qquad \text{for } p = 0,\ 1,\ 2\dots. \tag{3.14}$$

To give the explicit form of $\hat{Q}_p$ (for $p = 0, 1, 2$) we compute the powers of R up to degree 3:

$$\begin{aligned}
R &= S^{-1}D_1 + S^{-1}D_2 + S^{-1}D_3 + \mathcal{O}(\Delta t^4),\\
R^2 &= (S^{-1}D_1)^2 + S^{-1}D_1S^{-1}D_2 + S^{-1}D_2S^{-1}D_1 + \mathcal{O}(\Delta t^4),\\
R^3 &= (S^{-1}D_1)^3 + \mathcal{O}(\Delta t^4),
\end{aligned}$$

and recalling that $D_k = \mathcal{O}(\Delta t^{k+1})$ we have

$$\begin{aligned}
\tilde{R}_0 &= I,\\
\tilde{R}_1 &= S^{-1}D_1,\\
\tilde{R}_2 &= S^{-1}D_2 + (S^{-1}D_1)^2,\\
\tilde{R}_3 &= (S^{-1}D_1)^3 + S^{-1}D_1S^{-1}D_2 + S^{-1}D_2S^{-1}D_1 + S^{-1}D_3.
\end{aligned} \tag{3.15}$$

From definition (3.14) it follows that

$$\begin{aligned}
\hat{Q}_0 &= I,\\
\hat{Q}_1 &= (I + S^{-1}D_1)^{-1} = -D^{-1}S,\\
\hat{Q}_2 &= (I + S^{-1}D_1 + S^{-1}D_2 + (S^{-1}D_1)^2)^{-1}\\
&= (-D + DS^{-1}D + B(HC)^2HB^T)^{-1}S,
\end{aligned} \tag{3.16}$$

where we have set $D = BHCHB^T$ and employed the fact that $C = H^{-1} + \nu K$.

Remark 3. We analyze the bound $\rho(R) < 1$ when SEM-NI are used. By construction, the leading part of R, for $\Delta t \to 0$, is $S^{-1}D_1$, and we can infer that, for sufficiently small Δt,

$$\rho(R) \simeq \rho(S^{-1}D_1) = \frac{\nu\Delta t}{\beta_{-1}}\rho((BM^{-1}B^T)^{-1}BM^{-1}KM^{-1}B^T).$$

It is possible to prove that the condition $\rho(R) < 1$ is satisfied under assumption (3.13). As a matter of fact we set $E_0 = BM^{-1}B^T$, $E_1 = BM^{-1}KM^{-1}B^T$, $E = E_0^{-1}E_1$, and we denote by $(\lambda_i, \mathbf{p}_i)$ an eigenvalue-eigenvector pair of E or, equivalently, an eigenvalue-eigenvector pair of the generalized eigenvalue problem $E_1\mathbf{p} = \lambda E_0\mathbf{p}$. Since $E_0 > 0$, the eigenvectors $\mathbf{p}_i$ can be chosen E_0-orthogonal to one other, i.e., $\mathbf{p}_i^T E_0 \mathbf{p}_j = 0$ if $i \neq j$. Then, for any eigenvector $\mathbf{p}_i$ $(i = 1, \ldots, N_p)$, we set $\mathbf{v}_i := M^{-1/2}B^T\mathbf{p}_i$ and denote by V_E the subspace of $\mathbb{R}^{2N_v}$ spanned by $\{\mathbf{v}_i\}_{i=1}^{N_p}$. It is easy to see that $M^{-1/2}KM^{-1/2}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ holds for any $i = 1, \ldots, N_p$, that is, the eigenvalues λ_i of E are bounded by the eigenvalues of $M^{-1}K$.

PROPOSITION 3.1. *Matrices $\hat{Q}_p$ (for $p = 0, 1, 2$) are nonsingular.*

Proof. $\hat{Q}_0$ is the identity matrix, and $\hat{Q}_1 = -D^{-1}S$ is nonsingular since both D and S are nonsingular matrices. Proving $\hat{Q}_2$ is nonsingular is equivalent to proving $\hat{Q}_2^{-1} = \tilde{R}_0 + \tilde{R}_1 + \tilde{R}_2$ is nonsingular. By definition

$$\begin{aligned}\hat{Q}_2^{-1} &= -S^{-1}D + (S^{-1}D)^2 + S^{-1}B(HC)^2HB^T \\ &= S^{-1}BHC(-C^{-1} - HB^T(BHB^T)^{-1}BH + H)CHB^T\end{aligned}$$

and, since B is a full rank matrix, $\hat{Q}_2^{-1}$ is nonsingular iff the matrix

$$W = H - C^{-1} - HB^T(BHB^T)^{-1}BH \tag{3.17}$$

is nonsingular. By recalling that $H = \frac{\beta_{-1}}{\Delta t}M^{-1} > 0$, $C = H^{-1} + \nu K > 0$, and $K > 0$, we have $C - H^{-1} > 0$, and then $H - C^{-1} > 0$ (see P3 in section 2), which implies that $(H - C^{-1})$ is nonsingular. Since $\ker(B^T) \neq \{\mathbf{0}\}$, also BHB^T is nonsingular. Therefore, we set

$$\mathcal{M} = \begin{bmatrix} BHB^T & BH \\ HB^T & H - C^{-1} \end{bmatrix}, \tag{3.18}$$

and we note that matrix W is nothing more than the Schur complement of BHB^T in $\mathcal{M}$, i.e., $\mathcal{M}/(BHB^T)$.

By applying Schur's formula (D3, section 2), $\det(\mathcal{M}) = \det(BHB^T)\det(W)$ holds. On the other hand we can write $\det(\mathcal{M}) = \det(H - C^{-1})\det(\mathcal{M}/(H - C^{-1}))$ as well. Therefore

$$\det(W) = \det(\mathcal{M}/(BHB^T)) = \frac{\det(H - C^{-1})}{\det(BHB^T)}\det(\mathcal{M}/(H - C^{-1}))$$

and $\det(W) \neq 0$ iff $\det(\mathcal{M}/(H - C^{-1})) \neq 0$.

Recalling (2.12) we have $(H - C^{-1})^{-1} = H^{-1}[H - (-C + H^{-1})^{-1}]H^{-1}$, and since $-C + H^{-1} = -\nu K$, it holds that

$$\begin{aligned}\mathcal{M}/(H - C^{-1}) &= BHB^T - BH(H - C^{-1})^{-1}HB^T \\ &= BHB^T - BHH^{-1}[H + (\nu K)^{-1}]H^{-1}HB^T = -B(\nu K)^{-1}B^T.\end{aligned}$$

The thesis follows since K is nonsingular. $\square$

As stated by the following lemma, both sequences of matrices Q_p and $\hat{Q}_p$ yield a consistency error satisfying estimate (3.8). This lemma generalizes the results given in [15, 8].

LEMMA 3.2. *There exist* $\overline{\Delta t} = \overline{\Delta t}(\nu, N, h) > 0$ *and positive constants* $c_p = c_p(\nu, N, h)$ *and* $\hat{c}_p = \hat{c}_p(\nu, N, h)$ *such that for any* $\Delta t \in (0, \overline{\Delta t})$ *and* $p = 0, 1, 2$ *the following estimates hold:*

$$|||\Sigma - SQ_p||| = c_p \Delta t^{p+2} + o(\Delta t^{p+2}), \tag{3.19}$$

$$|||\Sigma - S\hat{Q}_p||| = \hat{c}_p \Delta t^{p+2} + o(\Delta t^{p+2}). \tag{3.20}$$

Proof. By definition (3.12) we have $\Sigma - SQ_p = -\sum_{k \geq p+1} D_k$.

Estimate (3.19), with $c_p = \frac{\nu^{p+1}}{(\beta_{-1})^{p+2}} \rho(B(M^{-1}K)^{p+1} M^{-1} B^T)$, immediately follows by recalling that $D_k = \mathcal{O}(\Delta t^{k+1})$ (for any integer $k \geq 0$).

In order to prove (3.20) we begin to consider the matrix $\Sigma^{-1} - \hat{Q}_p^{-1} S^{-1}$. We set $Z = \sum_{k \geq p+1} \tilde{R}_k$, so that

$$\Sigma^{-1} - \hat{Q}_p^{-1} S^{-1} = \left(\sum_{k \geq 0} \tilde{R}_k - \sum_{k=0}^{p} \tilde{R}_k \right) S^{-1} = Z S^{-1}.$$

By using (2.13) we have

$$\Sigma - S\hat{Q}_p = -\Sigma(\Sigma^{-1} - \hat{Q}_p^{-1} S^{-1}) S\hat{Q}_p = -\Sigma Z \hat{Q}_p. \tag{3.21}$$

Estimate (3.20) proceeds by recalling that $\Sigma = -\sum_{k \geq 0} D_k = \mathcal{O}(\Delta t)$, $Z = \sum_{k \geq p+1} \tilde{R}_k = \mathcal{O}(\Delta t^{p+1})$ and by noting that $\hat{Q}_p = \left(\sum_{k=0}^{p} \tilde{R}_k \right)^{-1} = \mathcal{O}(1)$. $\square$

Even though both matrices Q_p and $\hat{Q}_p$ produce consistency errors of the same order with respect to Δt, we are inclined to choose matrices $\hat{Q}_p$ instead of Q_p to set up Yosida schemes, when $p > 0$ (recall that $Q_0 = \hat{Q}_0 = I$). As a matter of fact we can easily prove that

$$\hat{c}_1 < c_1, \tag{3.22}$$

which implies $|||\Sigma - S\hat{Q}_1||| < |||\Sigma - SQ_1|||$ for sufficiently small Δt.

By exploiting the definition of $\hat{Q}_1$ and by using (2.14), with $\mathcal{B} = S = -D_0$ and $\mathcal{C} = D_1$, we write the difference $\Sigma - S\hat{Q}_1$ as follows:

$$\begin{aligned} \Sigma - S\hat{Q}_1 &= -\sum_{k \geq 0} D_k - S(S + D_1)^{-1} S = -\sum_{k \geq 0} D_k + D_0 + D_1 - D_1(D_1 - D_0)^{-1} D_1 \\ &= -D_2 - D_1(D_1 - D_0)^{-1} D_1 + o(\Delta t^3). \end{aligned}$$

On the other hand, $\Sigma - SQ_1 = -D_2 + o(\Delta t^3)$. It is immediate to verify that both D_2 and $D_1(D_1 - D_0)^{-1} D_1$ are symmetric matrices, and then the 2-norm $|||\cdot|||$ coincides with the spectral radius $\rho(\cdot)$; moreover it holds that $D_2 > 0$ and $D_1(D_1 - D_0)^{-1} D_1 < 0$. If we apply P2 of section 2, we can conclude that

$$\lambda_{max}(D_2 + D_1(D_1 - D_0)^{-1} D_1) \leq \lambda_{max}(D_2) + \lambda_{max}(D_1(D_1 - D_0)^{-1} D_1) < \lambda_{max}(D_2),$$

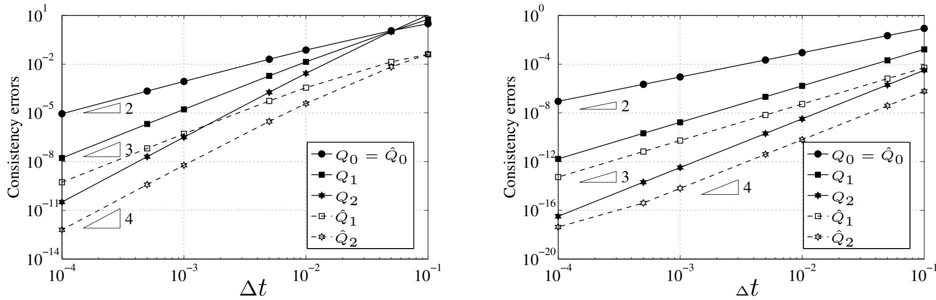

FIG. 3.1. *The consistency errors* $|||\Sigma - SQ_p|||$ *and* $|||\Sigma - S\hat{Q}_p|||$ *for* $p = 0, 1, 2$. *The viscosity is* $\nu = 10^{-2}$ *at left and* $\nu = 10^{-4}$ *at right.* 4×4 *equal square spectral elements with polynomial degree* $N = 5$ *are used.*

that is,

$$\hat{c}_1 \Delta t^3 = \rho(-D_2 - D_1(D_1 - D_0)^{-1} D_1) < \rho(-D_2) = c_1 \Delta t^3.$$

Results shown in Figure 3.1 confirm (3.22). The proof that $\hat{c}_2 < c_2$ (which implies $|||\Sigma - S\hat{Q}_2||| \leq |||\Sigma - SQ_2|||$ for small Δt) is quite long and technical, so we only report numerical results (again in Figure 3.1) confirming such assertion.

In order to compare the two approaches from a computational point of view, we note that both matrices Q_p and $\hat{Q}_p$ are involved in the solution of a linear system like $Q\hat{\mathbf{P}}^{n+1} = \hat{\mathbf{P}}^{n+1/2}$ at each time-step. When $p = 0$ it holds that $Q_0 = \hat{Q}_0 = I$. When $p \geq 1$, solving systems $Q_p\hat{\mathbf{P}}^{n+1} = \hat{\mathbf{P}}^{n+1/2}$ means performing 2 $(N_p \times 2N_v)$-matrix-vector products and solving one linear system whose matrix is $\sum_{k=0}^{p} D_k$. If we choose a direct algebraic solver, the matrix $\sum_{k=0}^{p} D_k$ may be assembled and factorized once, and the computational cost related to the solution of all systems of type $Q\hat{\mathbf{P}}^{n+1} = \hat{\mathbf{P}}^{n+1/2}$ (for any $n = n_0, \ldots, N_T - 1$) is proportional to $N_p^3 + N_p \cdot N_v \cdot N_T$. Otherwise, solving systems $\hat{Q}_p\hat{\mathbf{P}}^{n+1} = \hat{\mathbf{P}}^{n+1/2}$ means performing $2 \cdot p$ $(N_p \times 2N_v)$-matrix-vector products, $2 \cdot p$ $(N_v \times N_v)$-matrix-vector products, and solving p linear systems whose matrix is S. By recalling that matrix S is already assembled and factorized inside the algorithm, since it is used in the $\hat{L}$-substep of (3.7), the computational cost related to the solution of all $\hat{Q}\hat{\mathbf{P}}^{n+1} = \hat{\mathbf{P}}^{n+1/2}$ (for any $n = n_0, \ldots, N_T - 1$) is proportional to $(N_v + N_p) \cdot N_v \cdot N_T$. In conclusion, when we choose a direct algebraic solver to solve $Q\hat{\mathbf{P}}^{n+1} = \hat{\mathbf{P}}^{n+1/2}$, if N_T is large (i.e., $N_T \gtrsim N_v$), the choice $Q = Q_p$ will be more convenient than $Q = \hat{Q}_p$. Nevertheless, matrices $\hat{Q}_p$ (3.14) are preferable to matrices Q_p (3.11) for what concerns accuracy with respect to Δt. In this paper we have preferred accuracy properties and have chosen to approximate Q by $\hat{Q}_p$ inside the definition of $\hat{\mathcal{A}}$ (or equivalently in either (3.6) or (3.7)).

The choice $\hat{Q}_0$ yields the *Yosida* (or *Yosida*-2) method [13, 12], while the choice $\hat{Q}_1$ (resp., $\hat{Q}_2$) provides the *Yosida*-3 (resp., *Yosida*-4) scheme introduced in [15] (resp., [8]). From now on the term *Yosida*-$(p + 2)$ (for $p = 0, 1, 2$) will denote the Yosida method obtained by replacing Q with $\hat{Q}_p$.

Finally, for any $n \geq n_0$ we will denote by $(\hat{\mathbf{u}}^n_{\mathcal{H},p}, \hat{p}^n_{\mathcal{H},p}) \in \mathbf{V}_{\mathcal{H}} \times Q_{\mathcal{H}}$ the numerical solution of (3.6) obtained by replacing Q with $\hat{Q}_p$, while we will denote by $\hat{\mathbf{U}}^n_p$ (resp., $\hat{\mathbf{P}}^n_p$) the vector of the expansion coefficients of $\hat{\mathbf{u}}_{\mathcal{H},p}$ (resp., $\hat{p}_{\mathcal{H},p}$) with respect to the Lagrange basis in $V_{\mathcal{H}}$ (resp., $Q_{\mathcal{H}}$).

3.2. Convergence analysis. We define the *global errors* as the errors between the exact solution of problem (2.1) and the numerical solution $(\hat{\mathbf{u}}_{\mathcal{H},p}, \hat{p}_{\mathcal{H},p})$ obtained with any *Yosida*-$(p+2)$ method, i.e.,

$$\begin{aligned} \|\mathbf{u} - \hat{\mathbf{u}}_{\mathcal{H},p}\|_{\ell^2(H^1)} &:= \left(\Delta t \sum_{n=n_0}^{N_T-1} \|\mathbf{u}(t_n) - \hat{\mathbf{u}}^n_{\mathcal{H},p}\|^2_{H^1(\Omega)} \right)^{1/2}, \\ \|p - \hat{p}_{\mathcal{H},p}\|_{\ell^2(L^2)} &:= \left(\Delta t \sum_{n=n_0}^{N_T-1} \|p(t_n) - \hat{p}^n_{\mathcal{H},p}\|^2_{L^2(\Omega)} \right)^{1/2}. \end{aligned} \tag{3.23}$$

We can upper-bound global errors (3.23) as follows:

$$\begin{aligned} \|\mathbf{u} - \hat{\mathbf{u}}_{\mathcal{H},p}\|_{\ell^2(H^1)} &\le \|\mathbf{u} - \mathbf{u}_{\mathcal{H}}\|_{\ell^2(H^1)} + \|\mathbf{u}_{\mathcal{H}} - \hat{\mathbf{u}}_{\mathcal{H},p}\|_{\ell^2(H^1)}, \\ \|p - \hat{p}_{\mathcal{H},p}\|_{\ell^2(L^2)} &\le \|p - p_{\mathcal{H}}\|_{\ell^2(L^2)} + \|p_{\mathcal{H}} - \hat{p}_{\mathcal{H},p}\|_{\ell^2(L^2)}, \end{aligned}$$

where the first terms on the right-hand sides are the errors due to BDF, while the second ones, induced by the *Yosida*-$(p+2)$ scheme, are named *splitting errors* and are the errors between the solution $(\mathbf{u}_{\mathcal{H}}, p_{\mathcal{H}})$ of (2.10) and the solution $(\hat{\mathbf{u}}_{\mathcal{H},p}, \hat{p}_{\mathcal{H},p})$ obtained by solving (3.6) with $Q \simeq \hat{Q}_p$. Therefore, for $p = 0$, 1, 2, we are interested in analyzing the *splitting errors*:

$$\begin{aligned} \|\mathbf{u}_{\mathcal{H}} - \hat{\mathbf{u}}_{\mathcal{H},p}\|_{\ell^2(H^1)} &:= \left(\Delta t \sum_{n=n_0}^{N_T-1} \|\mathbf{u}^n_{\mathcal{H}} - \hat{\mathbf{u}}^n_{\mathcal{H},p}\|^2_{H^1(\Omega)} \right)^{1/2}, \\ \|p_{\mathcal{H}} - \hat{p}_{\mathcal{H},p}\|_{\ell^2(L^2)} &:= \left(\Delta t \sum_{n=n_0}^{N_T-1} \|p^n_{\mathcal{H}} - \hat{p}^n_{\mathcal{H},p}\|^2_{L^2(\Omega)} \right)^{1/2}. \end{aligned} \tag{3.24}$$

Remark 4. From the analysis carried out in [10] it is known that the solution of the Navier–Stokes problem can be regular for $t \to 0$ under unrealistic nonlocal compatibility conditions. On the other hand, accuracy in the usual norms $\ell^2(H^1)$ and $\ell^2(L^2)$ can be attained only if the solution is regular enough, even when using high order discretization schemes. Otherwise, high order in time can be recovered only in suitable weighted norms. In what follows, we will assume that high regularity hypotheses hold. This ensures that when using BDFq time discretization of order $q(>1)$, without the fractional step approach, optimal accuracy is q for both velocity and pressure. Without such high regularity assumptions, optimal convergence estimates given in the present paper will hold only far from $t = 0$. We denote by $|| \cdot ||_{\ell^2_w(V)}$ a suitable time-weighted norm; the scheme shown in Table 3.1 summarizes results known in literature.

From now on, by $(\cdot,\cdot)$ and $\|\cdot\| = (\cdot,\cdot)^{1/2}$ we will denote the classic Euclidean inner product and the Euclidean norm, respectively, in $\mathbb{R}^n$.

Let $\mathbf{U}$, $\mathbf{V} \in \mathsf{V}_G$ be the vectors associated to $\mathbf{u}_{\mathcal{H}}$, $\mathbf{v}_{\mathcal{H}} \in \mathbf{V}_{\mathcal{H}}$. By recalling that M and K are the mass and stiffness matrices, respectively, it holds that $(M\mathbf{U}, \mathbf{V}) = \mathbf{V}^T M \mathbf{U} = (\mathbf{u}_{\mathcal{H}}, \mathbf{v}_{\mathcal{H}})_{\mathcal{H},\Omega}$ and $(K\mathbf{U}, \mathbf{U}) = (\nabla \mathbf{u}_{\mathcal{H}}, \nabla \mathbf{v}_{\mathcal{H}})_{\mathcal{H},\Omega}$. Then we set

$$\|\mathbf{U}\|_0 := \sqrt{(M\mathbf{U}, \mathbf{U})}, \qquad \|\mathbf{U}\|_1 := \sqrt{(K\mathbf{U}, \mathbf{U})}.$$

It is worth pointing out that the norm $|| \cdot ||_1$ is the discrete counterpart to the L^2-norm of $\nabla \mathbf{u}$. Thanks to the Poincaré inequality, which states that, for any function

TABLE 3.1

	Low regularity assumptions on the exact solution (no compatibility conditions imposed)	High regularity assumptions on the exact solution
BDF1 in time	$\|\mathbf{u}-\mathbf{u}_{\mathcal{H}}\|_{\ell^2(H^1)}=\mathcal{O}(\Delta t)$ $\|p-p_{\mathcal{H}}\|_{\ell^2(L^2)}=\mathcal{O}(\Delta t^{1/2})$	$\|\mathbf{u}-\mathbf{u}_{\mathcal{H}}\|_{\ell^2(H^1)}=\mathcal{O}(\Delta t)$ $\|p-p_{\mathcal{H}}\|_{\ell^2(L^2)}=\mathcal{O}(\Delta t)$
Proof:	see [2]	guidelines of [2]
BDFq in time	$\|\mathbf{u}-\mathbf{u}_{\mathcal{H}}\|_{\ell^2(H^1)}=\mathcal{O}(\Delta t)$ $\|p-p_{\mathcal{H}}\|_{\ell^2(L^2)}=\mathcal{O}(\Delta t^{1/2})$ $\|\mathbf{u}-\mathbf{u}_{\mathcal{H}}\|_{\ell^2_w(H^1)}=\mathcal{O}(\Delta t^q)$ $\|p-p_{\mathcal{H}}\|_{\ell^2_w(L^2)}=\mathcal{O}(\Delta t^{q-1})$	$\|\mathbf{u}-\mathbf{u}_{\mathcal{H}}\|_{\ell^2(H^1)}=\mathcal{O}(\Delta t^q)$ $\|p-p_{\mathcal{H}}\|_{\ell^2(L^2)}=\mathcal{O}(\Delta t^q)$
Proof:	see [6] for $q=2$	guidelines of [6, 2]

$u\in H^1_0(\Omega)$, there exists a positive constant C_Ω such that $\|u\|_{L^2(\Omega)}\le C_\Omega\|\nabla u\|_{L^2(\Omega)}$, and thanks to the equivalence between the continuous L^2-norm and its discrete counterpart induced by the discrete inner product (2.4) [14, 7], $\|u_{\mathcal{H}}\|_{\mathcal{H},\Omega}\le\|u_{\mathcal{H}}\|_{L^2(\Omega)}\le 3\|u_{\mathcal{H}}\|_{\mathcal{H},\Omega}$, the following (discrete) Poincaré inequality holds:

$$\|\mathbf{V}\|_0\le 3C_\Omega\|\mathbf{V}\|_1 \qquad \text{for any } \mathbf{V}\in\mathsf{V}^0_G. \tag{3.25}$$

Moreover, since K is s.p.d., for any $\mathbf{U},\mathbf{V}\in\mathsf{V}_G$ it holds that

$$(K\mathbf{U},\mathbf{V})=(K^{1/2}\mathbf{U},K^{1/2}\mathbf{V})\le\|K^{1/2}\mathbf{U}\|\|K^{1/2}\mathbf{V}\|=\|\mathbf{U}\|_1\|\mathbf{V}\|_1. \tag{3.26}$$

Finally, from Lemma 3.2, we immediately deduce that, for $p=0,1,2$,

$$((\Sigma-S\hat{Q}_p)\mathbf{P},\mathbf{Q})\le\hat{c}_p\Delta t^{p+2}\|\mathbf{P}\|\|\mathbf{Q}\| \qquad \text{for any } \mathbf{P},\mathbf{Q}\in\mathsf{Q}_G. \tag{3.27}$$

Remark 5. In order to prove convergence estimates for the Yosida schemes, we need to prove that matrix $\Sigma-S\hat{Q}_p$ is definite. With Lemmas 3.3, 3.4, and 3.5 we prove that $\Sigma-S\hat{Q}_0>0$, $-(\Sigma-S\hat{Q}_1)\ge 0$ and, under suitable conditions, that $\Sigma-S\hat{Q}_2>0$. Lemma 3.3 has been proved in [13] by using the analogy with the Yosida regularization operator; we present here a shorter proof, based on s.p.d. matrices properties. Lemma 3.4 has been proved in [15] with different arguments. Here we follow an approach exploiting (2.15), yielding a shorter proof.

LEMMA 3.3. $\Sigma-S\hat{Q}_0>0$.

Proof. Inside the proof of Lemma 3.1 we have proved that $(H-C^{-1})>0$. Since $\ker(B^T)=\{\mathbf{0}\}$, it follows that $\Sigma-S\hat{Q}_0=\Sigma-S=B(H-C^{-1})B^T>0$. ∎

LEMMA 3.4. $-(\Sigma-S\hat{Q}_1)\ge 0$.

Proof. By definitions of Σ, S, and $\hat{Q}_1$ we have

$$-(\Sigma-S\hat{Q}_1)=B[C^{-1}-(BH)^T(BHCHB^T)^{-1}BH]B^T$$

and since $\ker(B^T)=\{\mathbf{0}\}$, $-(\Sigma-S\hat{Q}_1)\ge 0$ iff $C^{-1}-(BH)^T(BHCHB^T)^{-1}BH\ge 0$. The thesis follows by putting $\mathcal{X}=BH$ and $\mathcal{C}=C$ in (2.15). ∎

Inside the proof of the following lemma, we will use the matrix

$$\tilde{\tilde{B}}:=-D_3-D_1S^{-1}D_2-D_2S^{-1}D_1-D_1(S^{-1}D_1)^2. \tag{3.28}$$

Matrix $\tilde{\tilde{B}}$ is symmetric, and numerical computations have shown that $\tilde{\tilde{B}}$ has real nonnegative eigenvalues for any space discretization we have considered. Nevertheless we cannot prove that it is positive definite, even if it is worth noting that nonnegativeness of its eigenvalues does not depend on the time-step (recalling definitions of matrices D_k, all the addend in (3.28) are $\mathcal{O}(\Delta t^3)$).

LEMMA 3.5. *If $\tilde{\tilde{B}} > 0$ and if Δt is sufficiently small, then $\Sigma - S\hat{Q}_2 > 0$.*

Proof. As a particular case of (3.21) we have

$$\Sigma - S\hat{Q}_2 = -\Sigma(\Sigma^{-1} - \hat{Q}_2^{-1}S^{-1})S\hat{Q}_2, \tag{3.29}$$

and we analyze the sign of the three factors of the right-hand side of (3.29). Let us start by proving that if Δt is small enough, then $-S\hat{Q}_2 > 0$. We set

$$\tilde{B} := D - DS^{-1}D - B(HC)^2HB^T, \tag{3.30}$$

so that $-S\hat{Q}_2 = S\tilde{B}^{-1}S$. Since $\tilde{B}$ is symmetric, then $-S\hat{Q}_2$ is symmetric as well (and $\Sigma - S\hat{Q}_2$ is also symmetric); moreover we can apply Sylvester's law of inertia [9] and conclude that $-S\hat{Q}_2 > 0$ iff $\tilde{B} > 0$. In order to prove that $\tilde{B} > 0$ we apply P2 of section 2. By construction, both D and $-DS^{-1}D - B(HC)^2HB^T$ are symmetric matrices, and we can write that $\lambda_{min}(\tilde{B}) \geq \lambda_{min}(D) + \lambda_{min}(-DS^{-1}D - B(HC)^2HB^T)$. Since $D = -S - D_1 = \mathcal{O}(\Delta t)$ and $-DS^{-1}D - B(HC)^2HB^T = -D_2 - D_1S^{-1}D_1 = \mathcal{O}(\Delta t^3)$, there exist two real constants c_1 and c_2 such that $\lambda_{min}(D) = c_1\Delta t$ and $\lambda_{min}(-DS^{-1}D - B(HC)^2HB^T) = c_2\Delta t^3$; therefore $c_1 + c_2\Delta t^2 > 0$ is a sufficient condition to have $\lambda_{min}(\tilde{B}) > 0$. Since $D = BHCHB^T > 0$, then $c_1 > 0$, while we cannot give any estimate on c_2. If $c_2 > 0$, no restrictions on Δt are required; otherwise if $c_2 < 0$, we have to assume that $\Delta t < \overline{\Delta t} := (-c1/c2)^{1/2}$.

Let us consider now the matrix $-(\Sigma^{-1} - \hat{Q}_2^{-1}S^{-1}) = -\tilde{R}_3S^{-1} - \sum_{k\geq 4}\tilde{R}_kS^{-1}$, where $\tilde{R}_3S^{-1} = \mathcal{O}(\Delta t^2)$ and $\sum_{k\geq 4}\tilde{R}_kS^{-1} = \mathcal{O}(\Delta t^3)$, and we analyze only the sign of $-\tilde{R}_3S^{-1}$. By (3.15) we have $-\tilde{R}_3S^{-1} = S^{-1}\tilde{\tilde{B}}S^{-1}$, so that $-\tilde{R}_3S^{-1} > 0$ since $\tilde{\tilde{B}} > 0$. Whatever the sign of $-\sum_{k\geq 4}\tilde{R}_kS^{-1}$ may be, for sufficiently small Δt, the matrix $-(\Sigma^{-1} - \hat{Q}_2^{-1}S^{-1})$ will be s.p.d.

Finally, starting from (3.29) we can write $(\Sigma - S\hat{Q}_2)(-S\hat{Q}_2)^{-1} = \Sigma(\Sigma^{-1} - \hat{Q}_2^{-1}S^{-1})$ and the thesis follows by noting that the matrix on the right is similar to a s.p.d. matrix (P5 of section 2) and by applying P4 of section 2, with $\mathcal{B} = \Sigma - S\hat{Q}_2$ and $\mathcal{C} = (-S\hat{Q}_2)^{-1}$. □

It is worth noting that the stability of BDFq+ *Yosida*-$(p+2)$ depends on stability of both BDFq scheme and Yosida splitting. The following lemma extends stability results proved in [12, 15] and highlights the stability of Yosida splitting when the associated BDFq method is absolutely stable, i.e., $q = 1, 2$. On the contrary, when either BDF3 or BDF4 is considered, it is natural to expect that a stability condition due to BDF has to be ensured as well.

LEMMA 3.6 (stability). *BDFq+ Yosida-$(p+2)$ scheme* (3.7) *is unconditionally stable if $q = 1, 2$ and $p = 0$, while it is conditionally stable if $q = 1, 2$ and $p > 0$.*

Proof. We consider system (3.6) with $Q = \hat{Q}_p$, $\mathbf{F}_1 = \mathbf{0}$, $\mathbf{F}_2 = \mathbf{0}$. By using either the identity $2(a-b,a) = a^2 - b^2 + (a-b)^2$ (when $q = 1$) or $2(3a - 4b + c, a) = a^2 - b^2 + (2a-b)^2 - (2b-c)^2 + (a-2b+c)^2$ (when $q = 2$), by summing from $n = n_0$ up to $N_T - 1$, by Lemma 3.3, and by neglecting some positive terms, it holds that $\|\hat{\mathbf{U}}_p^{N_T}\|_0^2 \leq C$, where C is a positive constant depending only on initial data.

When $q = 1, 2$ and $p = 1$, we proceed as before. In particular, when $p = 1$, in view of Lemma 3.4, a sufficient condition to ensure stability is furnished by $\nu\|\hat{\mathbf{U}}^{n+1}\|_1^2 + ((\Sigma - S\hat{Q}_p)\hat{\mathbf{P}}^{n+1}, \hat{\mathbf{P}}^{n+1}) \geq 0$ for any $n \geq n_0$. By recalling (3.20), the previous estimate infers a bound on Δt. Finally, when $p=2$, Lemma 3.5 ensures positiveness of $\Sigma - S\hat{Q}_p$ under suitable restrictions on Δt. ∎

We set $\mathbf{E}^n_{U,p} = \mathbf{U}^n - \widehat{\mathbf{U}}^n_p$, $\mathbf{E}^n_{P,p} = \mathbf{P}^n - \widehat{\mathbf{P}}^n_p$, and $\mathbf{E}^n_p = [\mathbf{E}^n_{U,p}, \mathbf{E}^n_{P,p}]^T$, recalling that p is the subindex of $\hat{Q}_p$.

THEOREM 3.7 (convergence). *Let us consider systems* (2.7) *and* (3.6) *with either* $q=1$ *or* $q=2$*, and* (3.6) *with* $Q = \hat{Q}_p$ *for* $p = 0,1,2$. *If* Δt *is sufficiently small, then there exist two positive constants* C_p *and* $\tilde{C}_p$*, depending on space discretization, but independent of* Δt*, such that the following convergence estimates hold:*

$$\|\mathbf{E}^{N_T}_{U,p}\|_0^2 + \nu\Delta t \sum_{n=n_0}^{N_T-1} \|\mathbf{E}^{n+1}_{U,p}\|_1^2 \le C_p \Delta t^{2p+3}, \tag{3.31}$$

$$\Delta t \sum_{n=n_0}^{N_T-1} \|\mathbf{E}^{n+1}_{P,p}\|^2 \le \tilde{C}_p \Delta t^{2p+2}, \tag{3.32}$$

i.e., $\|\mathbf{u}_{\mathcal{H}} - \hat{\mathbf{u}}_{\mathcal{H},p}\|_{l^\infty(L^2)} \le C_p\Delta t^{p+3/2}$, $\|\mathbf{u}_{\mathcal{H}} - \hat{\mathbf{u}}_{\mathcal{H},p}\|_{l^2(H^1)} \le C_p\Delta t^{p+3/2}$, *and* $\|p_{\mathcal{H}} - \hat{p}_{\mathcal{H},p}\|_{l^2(L^2)} \le \tilde{C}_p \Delta t^{p+1}$.

*Proof. BDF*1. We start by considering BDF1 for time approximation of both systems (2.7) and (3.6). They read

$$\left\{\begin{array}{l} \dfrac{M}{\Delta t}(\mathbf{U}^{n+1} - \mathbf{U}^n) + \nu K\mathbf{U}^{n+1} + B^T\mathbf{P}^{n+1} = \mathbf{F}_1^{n+1}, \\ B\mathbf{U}^{n+1} = \mathbf{F}_2^{n+1} \end{array}\right. \tag{3.33}$$

and

$$\left\{\begin{array}{l} \dfrac{M}{\Delta t}(\hat{\mathbf{U}}^{n+1}_p - \hat{\mathbf{U}}^n_p) + \nu K\hat{\mathbf{U}}^{n+1}_p + B^T\hat{\mathbf{P}}^{n+1}_p = \mathbf{F}_1^{n+1}, \\ B\hat{\mathbf{U}}^{n+1}_p - (\Sigma - S\hat{Q}_p)\hat{\mathbf{P}}^{n+1}_p = \mathbf{F}_2^{n+1}. \end{array}\right. \tag{3.34}$$

By subtracting (3.34) from (3.33), we obtain

$$\left\{\begin{array}{l} \dfrac{M}{\Delta t}(\mathbf{E}^{n+1}_{U,p} - \mathbf{E}^n_{U,p}) + \nu K\mathbf{E}^{n+1}_{U,p} + B^T\mathbf{E}^{n+1}_{P,p} = \mathbf{0}, \\ B\mathbf{E}^{n+1}_{U,p} - (\Sigma - S\hat{Q}_p)\mathbf{E}^{n+1}_{P,p} = -(\Sigma - S\hat{Q}_p)\mathbf{P}^{n+1}, \end{array}\right. \tag{3.35}$$

then multiply the first and the second equations of (3.35) by $\mathbf{V}^T = 2\Delta t(\mathbf{E}^{n+1}_{U,p})^T$ and $\mathbf{Q}^T = 2\Delta t(\mathbf{E}^{n+1}_{P,p})^T$, respectively, and we subtract the second equation from the first one. We obtain

$$\begin{aligned} &(M(\mathbf{E}^{n+1}_{U,p} - \mathbf{E}^n_{U,p}), \mathbf{E}^{n+1}_{U,p}) + 2\nu\Delta t(K\mathbf{E}^{n+1}_{U,p}, \mathbf{E}^{n+1}_{U,p}) \\ &\quad + 2\Delta t((\Sigma - S\hat{Q}_p)\mathbf{E}^{n+1}_{P,p}, \mathbf{E}^{n+1}_{P,p}) = 2\Delta t((\Sigma - S\hat{Q}_p)\mathbf{P}^{n+1}, \mathbf{E}^{n+1}_{P,p}). \end{aligned} \tag{3.36}$$

Moreover, by using the identity $2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2$ and the definite positivity/negativity of $(\Sigma - S\hat{Q}_p)$ (Lemmas 3.3, 3.4, and 3.5), we have

$$\begin{aligned} &\|\mathbf{E}^{n+1}_{U,p}\|_0^2 - \|\mathbf{E}^n_{U,p}\|_0^2 + \|\mathbf{E}^{n+1}_{U,p} - \mathbf{E}^n_{U,p}\|_0^2 + 2\nu\Delta t\|\mathbf{E}^{n+1}_{U,p}\|_1^2 \\ &\quad \le 2\Delta t((\Sigma - S\hat{Q}_p)\mathbf{P}^{n+1}, \mathbf{E}^{n+1}_{P,p}) - 2\gamma\Delta t((\Sigma - S\hat{Q}_p)\mathbf{E}^{n+1}_{P,p}, \mathbf{E}^{n+1}_{P,p}), \end{aligned}$$

where $\gamma = 0$ when $p=0$ or when $p=2$ and $\tilde{\tilde{B}} > 0$; otherwise $\gamma = 1$ when $p=1$ or $p=2$ and $\tilde{\tilde{B}}$ is not positive definite.

By applying the Cauchy–Schwarz inequality, estimate (3.27), and the Young inequality it follows that

$$\begin{aligned}\|\mathbf{E}_{U,p}^{n+1}\|_0^2 - \|\mathbf{E}_{U,p}^{n}\|_0^2 + \|\mathbf{E}_{U,p}^{n+1} - \mathbf{E}_{U,p}^{n}\|_0^2 + 2\nu\Delta t\|\mathbf{E}_{U,p}^{n+1}\|_1^2 \\ \leq \frac{1}{2\varepsilon}\|(\Sigma - S\hat{Q}_p)\mathbf{P}^{n+1}\|^2 + 2\varepsilon\Delta t^2\|\mathbf{E}_{P,p}^{n+1}\|^2 + 2\gamma\hat{c}_p^2\Delta t^{p+3}\|\mathbf{E}_{P,p}^{n+1}\|^2 \\ \leq \frac{\hat{c}_p^2}{2\varepsilon}\Delta t^{2p+4}\|\mathbf{P}^{n+1}\|^2 + \left(2\varepsilon\Delta t^2 + 2\gamma\hat{c}_p^2\Delta t^{p+3}\right)\|\mathbf{E}_{P,p}^{n+1}\|^2,\end{aligned}$$

and by summing on $n = n_0, \ldots, N_T - 1$ we obtain

$$\begin{aligned}\|\mathbf{E}_{U,p}^{N_T}\|_0^2 + \sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1} - \mathbf{E}_{U,p}^{n}\|_0^2 + 2\nu\Delta t\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1}\|_1^2 \\ \leq \frac{\hat{c}_p^2}{2\varepsilon}\Delta t^{2p+3}\left(\Delta t\sum_{n=n_0}^{N_T-1}\|\mathbf{P}^{n+1}\|^2\right) \\ + \left(2\varepsilon\Delta t^2 + 2\gamma\hat{c}_p^2\Delta t^{p+3}\right)\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{P,p}^{n+1}\|^2.\end{aligned} \tag{3.37}$$

We denote by β the *inf-sup* constant which depends on the space discretization but that is independent of Δt. Thanks to the *inf-sup* condition, to the first equation in (3.35), to (3.25) and (3.26), we have

$$\begin{aligned}\|\mathbf{E}_{P,p}^{n+1}\| \leq \frac{1}{\beta}\sup_{\substack{\mathbf{V}\in\mathbf{V}_G^0\\ \mathbf{V}\neq\mathbf{0}}}\frac{|(B^T\mathbf{E}_{P,p}^{n+1}, \mathbf{V})|}{\|\mathbf{V}\|_1} = \frac{1}{\beta}\sup_{\substack{\mathbf{V}\in\mathbf{V}_G^0\\ \mathbf{V}\neq\mathbf{0}}}\frac{|(\frac{M}{\Delta t}(\mathbf{E}_{U,p}^{n+1} - \mathbf{E}_{U,p}^{n}) + \nu K\mathbf{E}_{U,p}^{n+1}, \mathbf{V})|}{\|\mathbf{V}\|_1} \\ \leq \frac{1}{\beta}\left[\frac{3C_\Omega}{\Delta t}\|\mathbf{E}_{U,p}^{n+1} - \mathbf{E}_{U,p}^{n}\|_0 + \nu\|\mathbf{E}_{U,p}^{n+1}\|_1\right]\end{aligned}$$

and then

$$\Delta t\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{P,p}^{n+1}\|^2 \leq \frac{\Delta t}{\beta^2}\left[\frac{18C_\Omega^2}{\Delta t^2}\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1} - \mathbf{E}_{U,p}^{n}\|_0^2 + 2\nu^2\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1}\|_1^2\right]. \tag{3.38}$$

Now we note that there exists a positive constant C_P such that $\Delta t(\sum_{n=n_0}^{N_T-1}\|\mathbf{P}^{n+1}\|^2) \leq C_P$. As a matter of fact, $p \in L^2(0, T; L_0^2(\Omega))$, while the discretization error on the pressure associated to the pure BDF1 scheme (3.33) satisfies the inequality $\Delta t\sum_{n=n_0}^{N_T-1} \|p(t_{n+1}) - p_{\mathcal{H}}(t_{n+1})\|^2 \leq c\Delta t$ [2].

Therefore, we replace (3.38) in (3.37) and obtain

$$\begin{aligned}\|\mathbf{E}_{U,p}^{N_T}\|_0^2 + \sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1} - \mathbf{E}_{U,p}^{n}\|_0^2 + 2\nu\Delta t\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1}\|_1^2 \\ \leq \frac{\hat{c}_p^2 C_P}{2\varepsilon}\Delta t^{2p+3} + \left(\frac{\varepsilon\Delta t}{2} + \gamma\hat{c}_p^2\Delta t^{p+2}\right)\frac{\Delta t}{\beta^2} \\ \times\left[\frac{18C_\Omega^2}{\Delta t^2}\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1} - \mathbf{E}_{U,p}^{n}\|_0^2 + 2\nu^2\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1}\|_1^2\right].\end{aligned}$$

If $\gamma = 0$, we choose $\varepsilon = \beta^2/(18C_\Omega^2)$, we move on the left the terms depending on $\mathbf{E}_U$, and, under the assumption $\Delta t \le 18C_\Omega^2/\nu$, we obtain

$$\|\mathbf{E}_{U,p}^{N_T}\|_0^2 + \frac{1}{2}\sum_{n=n_0}^{N_T-1} \|\mathbf{E}_{U,p}^{n+1} - \mathbf{E}_{U,p}^n\|_0^2 + \nu\Delta t \sum_{n=n_0}^{N_T-1} \|\mathbf{E}_{U,p}^{n+1}\|_1^2 \le \frac{9C_\Omega^2 \hat{c}_p^2 C_P}{\beta^2}\Delta t^{2p+3}. \tag{3.39}$$

Otherwise, if $\gamma = 1$, we choose $\varepsilon = \beta^2/(27C_\Omega^2)$, again we move on the left the terms depending on $\mathbf{E}_{U,p}$, and we have

$$\begin{aligned}\|\mathbf{E}_{U,p}^{N_T}\|_0^2 &+ \left(\frac{2}{3} - \Delta t^{p+1}\frac{18\hat{c}_p^2 C_\Omega^2}{\beta^2}\right)\sum_{n=n_0}^{N_T-1} \|\mathbf{E}_{U,p}^{n+1} - \mathbf{E}_{U,p}^n\|_0^2 \\ &+ \nu\Delta t\left(2 - \frac{\nu\Delta t}{27C_\Omega^2} - \frac{2\nu\hat{c}_p^2\Delta t^{2p+3}}{\beta^2}\right)\sum_{n=n_0}^{N_T-1} \|\mathbf{E}_{U,p}^{n+1}\|_1^2 \le \frac{27C_\Omega^2\hat{c}_p^2 C_P}{2\beta^2}\Delta t^{2p+3}.\end{aligned}$$

If we require that $\Delta t \le \min\left\{\left(\frac{\beta}{6\sqrt{3}\hat{c}_p C_\Omega}\right)^{(p+1)/2}, \frac{27C_\Omega^2}{2\nu}, \left(\frac{\beta}{2\hat{c}_p\sqrt{\nu}}\right)^{1/(2p+3)}\right\}$, an estimate like (3.39), but with a different numeric constant, is true as well.

In conclusion, by neglecting some positive terms the estimate (3.31) holds, that is, the *Yosida*-$(p+2)$ method is $(p+3/2)$th order accurate on the velocity with respect to both $\ell^\infty(L^2)$- and $\ell^2(H^1)$-norms, with $C_p = C_p(C_\Omega, \hat{c}_p, C_P, \beta)$.

Finally, from (3.38) and (3.31), the estimate (3.32) on the pressure holds as well, that is, the *Yosida*-$(p+2)$ is $(p+1)$th order accurate on the pressure with respect to the $\ell^2(L^2)$-norm, with $\tilde{C}_p = \tilde{C}_p(C_\Omega, \hat{c}_p, C_P, \beta)$.

*BDF*2. We take into account both systems (2.7) and (3.6) with $q = 2$ and subtract the second system from the first one, obtaining

$$\begin{cases} \dfrac{M}{2\Delta t}(3\mathbf{E}_{U,p}^{n+1} - 4\mathbf{E}_{U,p}^n + \mathbf{E}_{U,p}^{n-1}) + \nu K\mathbf{E}_{U,p}^{n+1} + B^T\mathbf{E}_{P,p}^{n+1} = \mathbf{0}, \\ B\mathbf{E}_{U,p}^{n+1} - (\Sigma - S\hat{Q}_p)\mathbf{E}_{P,p}^{n+1} = -(\Sigma - S\hat{Q}_p)\mathbf{P}^{n+1}. \end{cases} \tag{3.40}$$

Then we multiply the first and the second equations of (3.40) by $\mathbf{V}^T = 4\Delta t(\mathbf{E}_{U,p}^{n+1})^T$ and $\mathbf{Q}^T = 4\Delta t(\mathbf{E}_{P,p}^{n+1})^T$, respectively, and subtract the second equation from the first one. We obtain

$$\begin{aligned}2(M(3\mathbf{E}_{U,p}^{n+1} - 4\mathbf{E}_{U,p}^n + \mathbf{E}_{U,p}^{n-1}), \mathbf{E}_{U,p}^{n+1}) + 4\nu\Delta t(K\mathbf{E}_{U,p}^{n+1}, \mathbf{E}_{U,p}^{n+1}) \\ + 4\Delta t((\Sigma - S\hat{Q}_p)\mathbf{E}_{P,p}^{n+1}, \mathbf{E}_{P,p}^{n+1}) = 4\Delta t((\Sigma - S\hat{Q}_p)\mathbf{P}^{n+1}, \mathbf{E}_{P,p}^{n+1}).\end{aligned} \tag{3.41}$$

Moreover, by using the identity $2(3a - 4b + c, a) = |a|^2 - |b|^2 + |2a - b|^2 - |2b - c|^2 + |a - 2b + c|^2$, working as in the proof for BDF1, and by neglecting some positive terms, we obtain

$$\begin{aligned}\|\mathbf{E}_{U,p}^{N_T}\|_0^2 + \sum_{n=n_0}^{N_T-1} \|\mathbf{E}_{U,p}^{n+1} - 2\mathbf{E}_{U,p}^n + \mathbf{E}_{U,p}^{n-1}\|_0^2 + 4\nu\Delta t\sum_{n=n_0}^{N_T-1} \|\mathbf{E}_{U,p}^{n+1}\|_1^2 \\ \le \frac{\hat{c}_p^2}{2\varepsilon}C_P\Delta t^{2p+3} + 4\left(2\varepsilon\Delta t^2 + \gamma\hat{c}_p^2\Delta t^{p+3}\right)\sum_{n=n_0}^{N_T-1} \|\mathbf{E}_{P,p}^{n+1}\|^2.\end{aligned} \tag{3.42}$$

By applying the *inf-sup* condition, we have

$$\begin{aligned}\|\mathbf{E}_{P,p}^{n+1}\| &\leq \frac{1}{\beta} \sup_{\substack{\mathbf{V}\in\mathbf{V}_G^0\\ \mathbf{V}\neq\mathbf{0}}} \frac{|(B^T\mathbf{E}_{P,p}^{n+1},\mathbf{V})|}{\|\mathbf{V}\|_1} \\ &\leq \frac{1}{\beta}\left[\frac{3C_\Omega}{2\Delta t}\|3\mathbf{E}_{U,p}^{n+1}-4\mathbf{E}_{U,p}^{n}+\mathbf{E}_{U,p}^{n-1}\|_0+\nu\|\mathbf{E}_{U,p}^{n+1}\|_1\right] \\ &\leq \frac{1}{\beta}\left[\frac{3C_\Omega}{2\Delta t}\|\mathbf{E}_{U,p}^{n+1}-2\mathbf{E}_{U,p}^{n}+\mathbf{E}_{U,p}^{n-1}\|_0+2\|\mathbf{E}_{U,p}^{n+1}-\mathbf{E}_{U,p}^{n}\|_0+\nu\|\mathbf{E}_{U,p}^{n+1}\|_1\right]\end{aligned}$$

and then

$$\begin{aligned}\Delta t\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{P,p}^{n+1}\|^2 \leq \frac{3\Delta t}{\beta^2}\Bigg[\nu^2\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1}\|_1^2 \\ +\frac{9C_\Omega^2}{4\Delta t^2}\sum_{n=n_0}^{N_T-1}\left(\|\mathbf{E}_{U,p}^{n+1}-2\mathbf{E}_{U,p}^{n}+\mathbf{E}_{U,p}^{n-1}\|_0^2+4\|\mathbf{E}_{U,p}^{n+1}-\mathbf{E}_{U,p}^{n}\|_0^2\right)\Bigg].\end{aligned} \tag{3.43}$$

Now we have to estimate the term $\|\mathbf{E}_{U,p}^{n+1}-\mathbf{E}_{U,p}^{n}\|_0^2$ which is not present on the left-hand side of (3.42). To this aim we multiply the first equation of (3.40) by $4\Delta t(\mathbf{E}_{U,p}^{n+1}-\mathbf{E}_{U,p}^{n})^T$, the second equation of (3.42) at both time t_{n+1} and t_n by $4\Delta t(\mathbf{E}_{p,p}^{n+1})^T$, and we linearly combine them in order to cut the term $(B\mathbf{E}_{P,p}^{n+1},\mathbf{E}_{U,p}^{n+1}-\mathbf{E}_{U,p}^{n})$. By rewriting $3\mathbf{E}_{U,p}^{n+1}-4\mathbf{E}_{U,p}^{n}+\mathbf{E}_{U,p}^{n-1}=2(\mathbf{E}_{U,p}^{n+1}-\mathbf{E}_{U,p}^{n})+(\mathbf{E}_{U,p}^{n+1}-\mathbf{E}_{U,p}^{n})-(\mathbf{E}_{U,p}^{n}-\mathbf{E}_{U,p}^{n-1})$ by applying $(2a,a-b)=|a|^2-|b|^2+|a-b|^2$, we have

$$\begin{aligned}&5\|\mathbf{E}_{U,p}^{n+1}-\mathbf{E}_{U,p}^{n}\|_0^2-\|\mathbf{E}_{U,p}^{n}-\mathbf{E}_{U,p}^{n-1}\|_0^2+\|\mathbf{E}_{U,p}^{n+1}-2\mathbf{E}_{U,p}^{n}+\mathbf{E}_{U,p}^{n-1}\|_0^2 \\ &\quad+2\nu\Delta t\left(\|\mathbf{E}_{U,p}^{n+1}\|_1^2-\|\mathbf{E}_{U,p}^{n}\|_1^2+\|\mathbf{E}_{U,p}^{n+1}-\mathbf{E}_{U,p}^{n}\|_1^2\right) \\ &\quad=-4\Delta t((\Sigma-S\hat{Q}_p)(\mathbf{P}^{n+1}-\mathbf{P}^n),\mathbf{E}_{P,p}^{n+1})+4\Delta t((\Sigma-S\hat{Q}_p)(\mathbf{E}_{P,p}^{n+1}-\mathbf{E}_{P,p}^{n}),\mathbf{E}_{P,p}^{n+1}).\end{aligned}$$

Therefore, by applying the same techniques as before, we have

$$\begin{aligned}&4\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1}-\mathbf{E}_{U,p}^{n}\|_0^2+\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1}-2\mathbf{E}_{U,p}^{n}+\mathbf{E}_{U,p}^{n-1}\|_0^2 \\ &\quad+2\nu\Delta t\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1}-\mathbf{E}_{U,p}^{n}\|_1^2 \\ &\quad\leq\frac{4}{\varepsilon}\hat{c}_p^2C_P\Delta t^{2p+3}+\left(2\varepsilon\Delta t+3\hat{c}_p\Delta t^{p+2}\right)\left(\Delta t\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{P,p}^{n+1}\|^2\right),\end{aligned} \tag{3.44}$$

where we have used $\sum_n\|\mathbf{P}^{n+1}-\mathbf{P}^n\|^2\leq 2\sum_n\|\mathbf{P}^{n+1}\|^2$ and $\sum_n\|\mathbf{E}_{P,p}^{n+1}-\mathbf{E}_{P,p}^{n}\|^2\leq 2\sum_n\|\mathbf{E}_{P,p}^{n+1}\|^2$. By summing inequalities (3.42) and (3.44) and by neglecting some positive terms, it holds that

$$
\begin{aligned}
&\|\mathbf{E}_{U,p}^{N_T}\|_0^2 + 2\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1} - 2\mathbf{E}_{U,p}^{n} + \mathbf{E}_{U,p}^{n-1}\|_0^2 + 4\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1} - \mathbf{E}_{U,p}^{n}\|_0^2 \\
&\quad + 2\nu\Delta t\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1}\|_1^2 \\
&\quad \leq \frac{9}{2\varepsilon}\hat{c}_p^2 C_P \Delta t^{2p+3} + (10\varepsilon\Delta t + 7\hat{c}_p\Delta t^{p+2})\left(\Delta t\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{P,p}^{n+1}\|^2\right).
\end{aligned}
\tag{3.45}
$$

By using (3.43) in (3.45) and by moving the terms with $\mathbf{E}_{U,p}$ on the left, we have

$$
\begin{aligned}
&\|\mathbf{E}_{U,p}^{N_T}\|_0^2 + \left(4 - \frac{3}{\beta_2}(10\varepsilon\Delta t + 7\hat{c}_p\Delta t^{p+2})\frac{9c_\Omega^2}{4\Delta t}\right)\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1} - \mathbf{E}_{U,p}^{n}\|_0^2 \\
&\quad + \left(2 - \frac{3}{\beta_2}(10\varepsilon\Delta t + 7\hat{c}_p\Delta t^{p+2})\frac{9c_\Omega^2}{4\Delta t}\right)\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1} - 2\mathbf{E}_{U,p}^{n} + \mathbf{E}_{U,p}^{n-1}\|_0^2 \\
&\quad + \left(4\nu\Delta t - \frac{3}{\beta_2}(10\varepsilon\Delta t + 7\hat{c}_p\Delta t^{p+2})\nu^2\Delta t\right)\sum_{n=n_0}^{N_T-1}\|\mathbf{E}_{U,p}^{n+1}\|_1^2 \leq \frac{9}{2\varepsilon}\hat{c}_p^2 C_P\Delta t^{2p+3}.
\end{aligned}
$$

We choose $\varepsilon = \frac{1}{10}(\beta^2/(27C_\Omega^2) - 7\hat{c}_p\Delta t^{p+1})$ and, under the assumption $\Delta t \leq \min\{27 C_\Omega^2/\nu, (4\beta^2/(189\hat{c}_p C_\Omega^2))^{1/(p+1)}\}$, the estimate (3.31) follows. The estimate (3.32) follows by both (3.31) and (3.43). ∎

Remark 6. It is worth noting that the assumptions required on the time-step by the previous theorem are not very restrictive; note that for $\Omega \subset \mathbb{R}^d$ it holds that $C_\Omega = \frac{2meas(\Omega)}{d}$ and $\beta = \mathcal{O}(N^{(1-d)/2})$. The most restrictive condition on the time-step is that given in (3.13) ensuring the series $-\sum_{k\geq 0} D_k$ converges to Σ.

Remark 7. When either BDF3 or BDF4 are considered, under suitable stability conditions on the time-step, we expect that convergence estimates like (3.31)–(3.32) hold for the couple BDFq-*Yosida*-$(p+2)$, as well as for $q = 3, 4$, as numerical results of the next section show.

Remark 8 (extensions to the Navier–Stokes case). We consider now the semi-discretization of unsteady Navier–Stokes equations by the following semi-implicit scheme: for any $n = n_0, \ldots, N_T - 1$, look for the solution $(\mathbf{u}^{n+1}, p^{n+1})$ of the system

$$
\begin{cases}
\dfrac{\beta_{-1}}{\Delta t}\mathbf{u}^{n+1} - \nu\Delta\mathbf{u}^{n+1} + \mathcal{N}(\mathbf{u}^{n-q+1}, \ldots, \mathbf{u}^{n}; \mathbf{u}^{n+1}) & \\
\qquad\qquad + \nabla p^{n+1} = \mathbf{f}^{n+1} + \displaystyle\sum_{j=0}^{q-1}\frac{\beta_j}{\Delta t}\mathbf{u}^{n-j} & \text{in } \Omega, \\
\nabla\cdot\mathbf{u}^{n+1} = 0 & \text{in } \Omega, \\
\mathbf{u}^{n+1} = \mathbf{g}^{n+1} & \text{on } \partial\Omega,
\end{cases}
\tag{3.46}
$$

where $\mathcal{N}$ represents either a full *explicit* discretization of the convective term if

$$
\mathcal{N}(\mathbf{u}^{n-q+1}, \ldots, \mathbf{u}^{n}; \mathbf{u}^{n+1}) := \sum_{j=0}^{q-1}\alpha_j(\mathbf{u}^{n-j}\cdot\nabla)\mathbf{u}^{n-j}
\tag{3.47}
$$

or a *semi-implicit* discretization of the convective term if

$$\mathcal{N}(\mathbf{u}^{n-q+1},\dots,\mathbf{u}^n;\mathbf{u}^{n+1}) = (\mathbf{u}^* \cdot \nabla)\mathbf{u}^{n+1}, \qquad \mathbf{u}^* = \sum_{j=0}^{q-1} \alpha_j \mathbf{u}^{n-j}, \tag{3.48}$$

and where $\alpha_j \in \mathbb{R}$ (for $j = 0,\dots,q-1$) are the coefficients of an extrapolation formula of order q. When the convective term is handled by the explicit form (3.47) the Yosida method can be used to solve (3.46), (3.47) and the analysis can be easily extended provided that $\mathcal{N}(\mathbf{u}_{\mathcal{H}}^{n-q+1},\dots,\mathbf{u}_{\mathcal{H}}^n;\mathbf{u}_{\mathcal{H}}^{n+1}) - \mathcal{N}(\hat{\mathbf{u}}_{\mathcal{H}}^{n-q+1},\dots,\hat{\mathbf{u}}_{\mathcal{H}}^n;\hat{\mathbf{u}}_{\mathcal{H}}^{n+1})$ is bounded. Otherwise, when the convective term is treated by the semi-implicit form (3.48) the counterpart of (3.6) reads as follows: for $n = n_0,\dots,N_T-1$ solve

(3.49)
$$\begin{cases} \left(\dfrac{\beta_{-1}}{\Delta t} M + C_n(\{\hat{\mathbf{U}}^{n-j}\}_{j=0}^{q-1})\right)\widehat{\mathbf{U}}^{n+1} + B^T\widehat{\mathbf{P}}^{n+1} = \mathbf{F}_1^{n+1} + \dfrac{1}{\Delta t}\displaystyle\sum_{j=0}^{q-1}\beta_j M\widehat{\mathbf{U}}^{n-j}, \\ B\widehat{\mathbf{U}}^{n+1} - (-B(C_n(\{\hat{\mathbf{U}}^{n-j}\}_{j=0}^{q-1}))^{-1}B^T - SQ)\widehat{\mathbf{P}}^{n+1} = \mathbf{F}_2^{n+1}, \end{cases}$$

where $C_n(\{\mathbf{U}^{n-j}\}_{j=0}^{q-1}) := \frac{\beta_{-1}}{\Delta t}M + \nu K + N(\mathbf{U}^{n-q+1},\dots,\mathbf{U}^n)$, and $N(\mathbf{U}^{n-q+1},\dots,\mathbf{U}^n)$ is the matrix related to the discretization of the convective term. It is immediate to see that matrix Σ is now time-dependent and both the derivation and analysis of Yosida schemes cannot be easily extended starting from the Stokes case.

Nevertheless, many numerical results presented in [8, 7] show that the convergence orders proved in Theorem 3.7 for the Stokes case still hold for Navier–Stokes equations with semi-implicit treatment of the convective term.

4. Numerical results on time-dependent Stokes equations. We consider the computational domain $\Omega = (-1,1)^2$ and $t \in (0,1)$, while the right-hand side, the boundary conditions, and the initial conditions are set so that the exact solution is

$$\begin{aligned} \mathbf{u}(x,y,t) &= [(t+1)\sin(x)\sin((t+1)y), \cos(x)\cos((t+1)y)]^T, \\ p(x,y,t) &= \cos(x)\sin((t+1)y). \end{aligned} \tag{4.1}$$

In Figure 4.1 we report the splitting errors (3.24) obtained by using either BDF1 or BDF2. In Figure 4.2 we show the global errors (3.23) for $p = 0,1,2$ by using

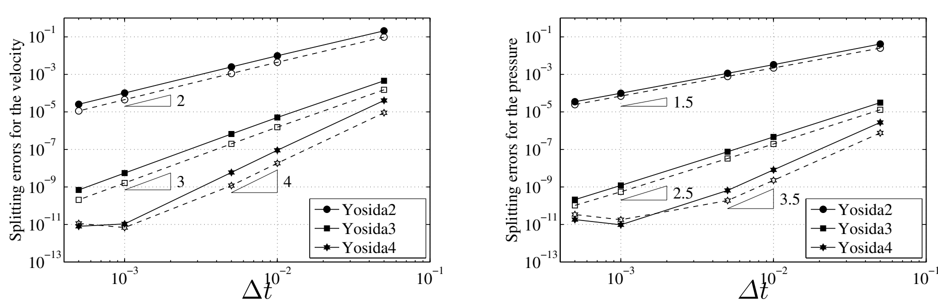

FIG. 4.1. *Splitting errors* (3.24) *for the exact solution* (4.1). *Black symbols refer to runs with* BDF1*, while empty symbols refer to runs with* BDF2*.* $\nu = 10^{-3}$*,* $N = 16$*, and* $h = 2$ *(one spectral element). The convergence history of Yosida-*4 *is stopped by the spectral element method discretization error.*

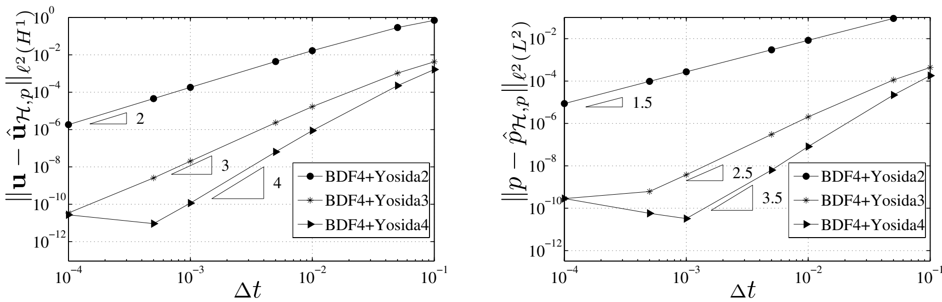

FIG. 4.2. *Global errors* (3.23) *for the exact solution* (4.1). $\nu = 10^{-3}$, $N = 16$, *and* $h = 1$ *(*2×2 *spectral elements). The convergence history of BDF*4 + *Yosida*-4 *is stopped by both spectral element method discretization and rounding errors. The splitting errors prevail over the errors due to BDF*4.

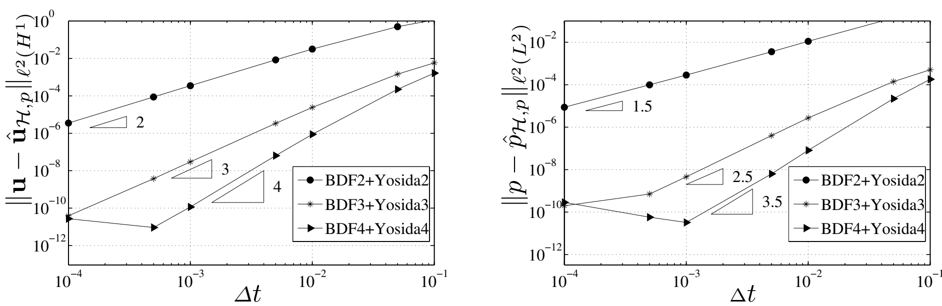

FIG. 4.3. *Global errors* (3.23) *for the exact solution* (4.1). $\nu = 10^{-3}$, $N = 16$, *and* $H = 1$ *(*2×2 *spectral elements). The convergence history of BDF*4 + *Yosida*-4 *is stopped by both spectral element method discretization and rounding errors. The splitting errors due to Yosida*-$(p+2)$ *schemes are of the same order as those produced by BDFp.*

BDF4 time approximation; finally in Figure 4.3 we show the global errors (3.23) for $p = 0, 1, 2$ by using BDFq time approximation, with $q = p + 2$.

Even if Theorem 3.7 ensures that the splitting errors (3.24) behave like $\Delta t^{p+3/2}$ for the velocity and Δt^{p+1} for the pressure, numerical results on the solution (4.1) provide that the splitting errors of *Yosida*-$(p + 2)$ schemes behave like Δt^{p+2} for the velocity and like $\Delta t^{p+3/2}$ for the pressure for both BDF1 and BDF2. Global errors depend on convergence order of both BDF and *Yosida*-$(p + 2)$ schemes; in particular for the results of Figure 4.2 we have used BDF4, so that, for $p = 0, 1, 2$, splitting errors prevail over the error of BDF. For the results of Figure 4.3 we have coupled *Yosida*-2 with BDF2, *Yosida*-3 with BDF3, and *Yosida*-4 with BDF4 and have obtained a global approximation error of order $p + 2 = q$ for the velocity and of order $p + 3/2$ for the pressure. We refer to [8, 7, 15] for numerical results about the approximation of Navier–Stokes equations.

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