BOUNDARY VALUE PROBLEMS

30/03/12 the 20'

1D: Given $[a,b] \subset \mathbb{R}$, $f:[a,b) \to \mathbb{R}$

? $u(x)$: $\begin{cases} -u''(x) = f(x) & a<x<b \\ u(a) = u_a \\ u(b) = u_b \end{cases}$ $\Big\}$ boundary conditions. of Dirichlet type

Poisson equation

x denotes the space-variable.
~~and we can compute u~~
Boundary conditions guarantee the uniqueness of the solution.

2D: the 2D Poisson eq reads:
given $\Omega \subset \mathbb{R}^2$ open and bounded (regular enough)

$$\begin{cases} -\Delta u(\underline{x}) = f(\underline{x}) & \underline{x} \in \Omega \\ u(\underline{x}) = g(\underline{x}) & \underline{x} \in \partial\Omega \end{cases}$$

given $f: \Omega \to \mathbb{R}$ (regular enough)
given $g: \partial\Omega \to \mathbb{R}$ (reg enough)

$$\Delta u(\underline{x}) = \frac{\partial^2 u}{\partial x_1^2}(\underline{x}) + \frac{\partial^2 u}{\partial x_2^2}(\underline{x})$$

Advection-diffusion-reaction eq:

$$\begin{cases} -\nu \Delta u(\underline{x}) + \nabla \cdot (\underline{b} u(\underline{x})) + \gamma u(\underline{x}) = f(\underline{x}) & \underline{x} \in \Omega \\ u(\underline{x}) = g(\underline{x}) & \text{on } \partial\Omega \end{cases}$$

$\nu > 0$ const. diffusivity

$\underline{b} = [b_1(\underline{x}), b_2(\underline{x})]$ advective field

$\gamma \in \mathbb{R}$ absorbtion coeff

this eq models, e.g. the motion of a concentration u inside a region Ω of the space, subject to an external vector field $\underline{b}$, a diffusion ν and a reaction γ.

The Dirichlet condition $u(x) = g(x)$ can be replaced by a Neumann condition: $\underline{n}(\underline{x}) \cdot \nabla u(\underline{x}) = h(\underline{x})$ where h must satisfy the compatibility condition

$$\int_{\partial\Omega} h = -\int_{\Omega} f$$

where $\underline{n} = \underline{n}(\underline{x})$ is the outward unit normal vector to $\partial\Omega$

$\underline{n} \cdot \nabla u$ expresses the flux across $\partial\Omega$.

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time dependent problems

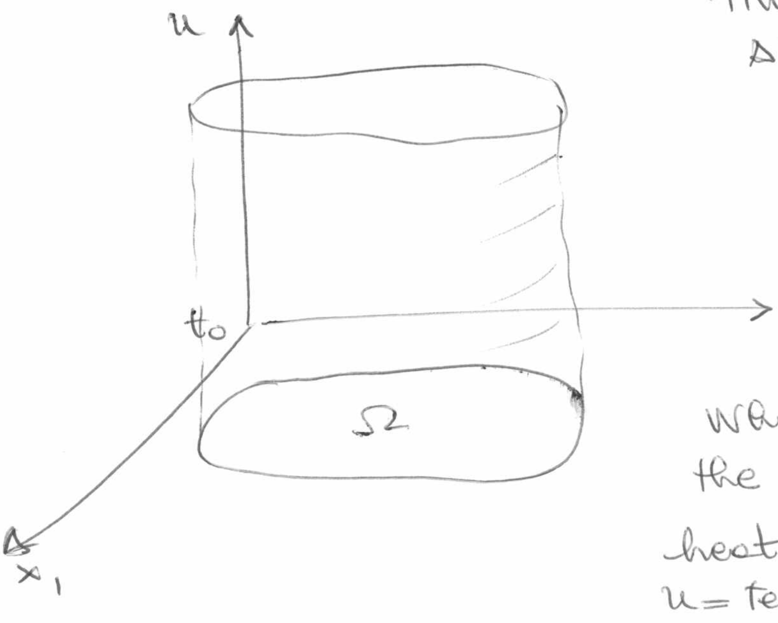
$\Omega \subset \mathbb{R}^2$ space

$[t_0, T] \subset \mathbb{R}$ time

? $u = u(\underline{x}, t)$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + \operatorname{div}(\underline{b}\, u) + \gamma u = f & \text{in } \Omega \times [t_0, T] \\ u = g & \text{on } \partial\Omega \times [t_0, T] \\ u = u_0 & \text{in } \Omega \times \{t_0\} \end{cases}$$

time-dependent ADR eqs.

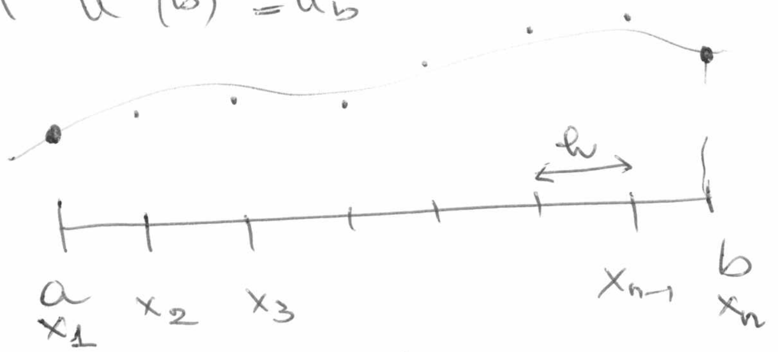


When $\underline{b} = \underline{0}$, $\gamma = 0$ the eq is named heat equation
u = temperature

We discretize the time derivative by BE, FE, CN, and so on, while the space derivative can be appx by finite differences, finite elements, spectral elements, finite volumes, ...

Finite differences for 1D-Poisson problem

$$\begin{cases} -u''(x) = f(x) & a < x < b \\ u(a) = u_a \\ u(b) = u_b \end{cases}$$



We fix $n \in \mathbb{N}$ $\quad h = \dfrac{b-a}{n-1}$

? $u_h(x_i)$ for $i = 2, \ldots, n-1$ $\quad \boxed{u_i = u_h(x_i)}$ notation

such that $\underset{\text{appx}}{u_h(x_i)} \sim \underset{\text{exact}}{u(x_i)}$

We appx u'' by a finite difference formula

$$u''(x_i) \simeq \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$ 2nd order centered Finite difference.

$$\left| u''(x_i) - \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \right| \leq C h^2 \quad \text{if } u \in C^4(I)$$

i.e. it is a 2-order appx of the second derivative.
the formula is symmetric w.r.t x_i and it is dubbed "centered".

$$\begin{cases} -u''(x_i) = f(x_i) & i=2,\dots,n-1 \\ u(x_1) = u_a \\ u(x_n) = u_b \end{cases}$$

$$\begin{cases} -\dfrac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f(x_i) & i=2,\dots,n-1 \\ u(x_1) = u_a \\ u(x_n) = u_b \end{cases}$$

$$-u_{i-1} + 2u_i - u_{i+1} = h^2 f(x_i)$$

$$\underbrace{\begin{bmatrix} 1 & 0 & & & & & 0 \\ -1 & 2 & -1 & 0 & \dots & & 0 \\ 0 & -1 & 2 & -1 & \dots & & 0 \\ & & \ddots & \ddots & \ddots & & \\ 0 & \dots & & 0 & -1 & 2 & -1 \\ 0 & & & & & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}}_{\underline{u}} = \underbrace{\begin{bmatrix} u_a \\ h^2 f(x_2) \\ \vdots \\ h^2 f(x_{n-1}) \\ u_b \end{bmatrix}}_{\underline{b}}$$

To look for $u_h(x_i)$ means to solve $A\underline{u} = \underline{b}$ - (a linear system)

If $f \in C^2([a,b]) \Rightarrow$

$$\boxed{\max_{0 \le i \le n} |u(x_i) - u_i| \le \frac{h^2}{96} \max_{a \le x \le b} |f''(x)|}$$

i.e. when $h \to 0$ $u_i \to u(x_i)$ and the error decays as h^2. (2° order convergence) -

When Neumann b.c. are considered, we can use: BDF formulas

$$-u'(x_1) \approx \frac{3u_1 - 4u_2 + u_3}{2h}$$

$$u'(x_N) \approx \frac{3u_N - 4u_{N-1} + u_{N-2}}{2h}$$

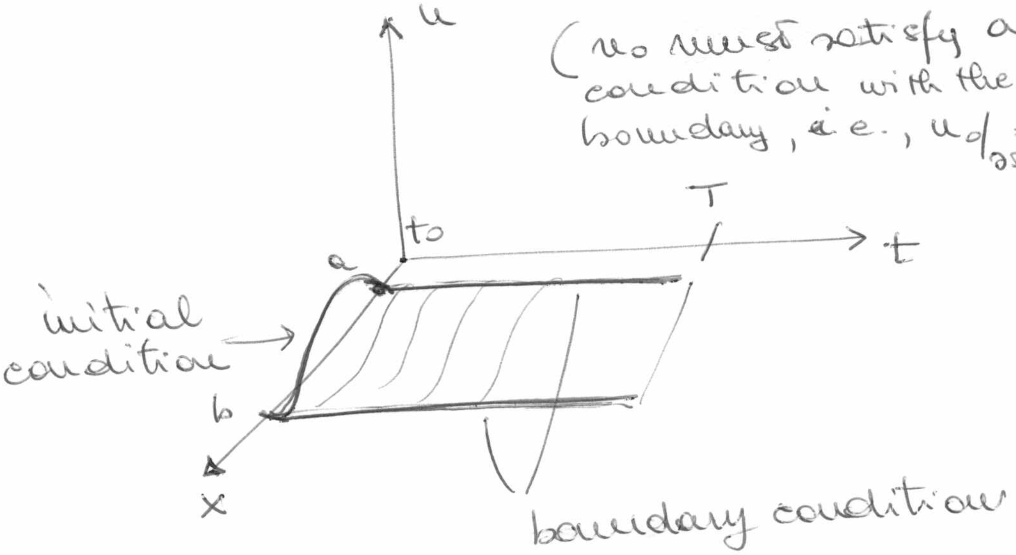
1D Heat equation.

Given $\Omega = [a,b] \subset \mathbb{R}$

$[t_0, T] \subset \mathbb{R}$

? $u(x,t)$ sol of
$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u = f & \text{in } \Omega \times (t_0, T) \\ u = 0 & \text{on } \partial\Omega \times (t_0, T) \\ u = u_0 & \text{in } \Omega \times \{t_0\} \end{cases}$$

(u_0 must satisfy a compatibility condition with the data on the boundary, i.e., $u_0|_{\partial\Omega} = 0$)



we discretize $[a,b]$ as before.

$$\underline{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} u(x_1,t) \\ u(x_2,t) \\ \vdots \\ u(x_n,t) \end{bmatrix} \qquad u_j(t) = u(x_j, t)$$

each $u_j(t)$ depends only on t.

$$\frac{\partial u_j}{\partial t}(t) = \frac{d u_j}{dt}(t)$$

$$-\Delta u_j(t) = -\frac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{h^2}$$

$$\rightarrow \quad \frac{du_j}{dt}(t) - \nu \frac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{h^2} = f_j(t) \qquad j = 2, \dots, n-1$$

$$\begin{cases} u_1(t) = 0 \\ \dfrac{du_j(t)}{dt} - \nu \dfrac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{h^2} = f_j(t) \qquad j = 2, \dots, n-1 \\ u_n(t) = 0 \end{cases}$$

A is the matrix obtained by discretizing $-u''$ by F.D.

$$\frac{d\underline{u}(t)}{dt} + \frac{\nu}{h^2} A \underline{u}(t) = \underline{f}(t)$$

$$\begin{cases} \dfrac{d\underline{u}(t)}{dt} = \underbrace{\underline{f}(t) - \dfrac{\nu}{h^2} A \underline{u}(t)}_{\underline{F}(t, \underline{u}(t))} \\ \underline{u}(t_0) = \underline{u}_0 \end{cases}$$

BE $\begin{cases} \underline{u}_{n+1} = \underline{u}_n + \Delta t \left[\underline{f}(t_{n+1}) - \dfrac{\nu}{h^2} A \cdot \underline{u}_{n+1} \right] \\ \underline{u}_0 \quad \text{given} \end{cases}$

$\forall n$:

$$\underbrace{\left(I + \frac{\nu}{h^2} \Delta t A \right)}_{\tilde{A}} \underline{u}_{n+1} = \underbrace{\underline{u}_n + \Delta t \, \underline{f}(t_{n+1})}_{\tilde{\underline{b}}}$$

$\tilde{A}$ is computed once (it is indep of n)

$\forall n$: $\underline{u}_n$ is known $\rightarrow$ compute $\tilde{\underline{b}}$

solve $\tilde{A} \underline{u}_{n+1} = \tilde{\underline{b}}$

$\tilde{A}$ can be factorized once, before the loop on n

Stability is guaranteed $\forall \Delta t > 0$.

(we are using Backward Euler in time)