DICACIM programme A.Y. 2024–25 Numerical Methods for Partial Differential Equations

A short review of Spectral Element Methods (SEM)

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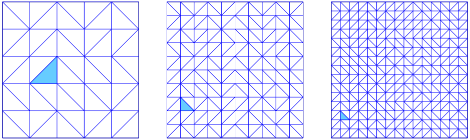
A step back: FEM

h-FEM: fixed low degree
refinement in h
(simplices and quads)
One parameter:
h =mesh size
(the same on quads)



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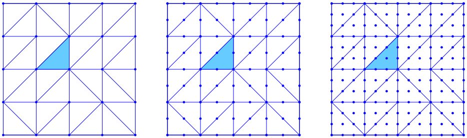
p-FEM: fixed h
refinement in p
(simplices and quads)
One parameter:
p =pol. degree
(the same on quads)



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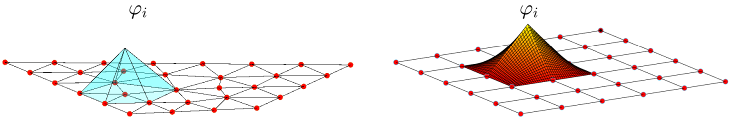
hp-FEM: refinement in both h and p (simplices and quads)

Two parameters: p = pol. degree ($\nearrow$)
h = mesh size (=elements diameter) ($\searrow 0$)

FEM: features

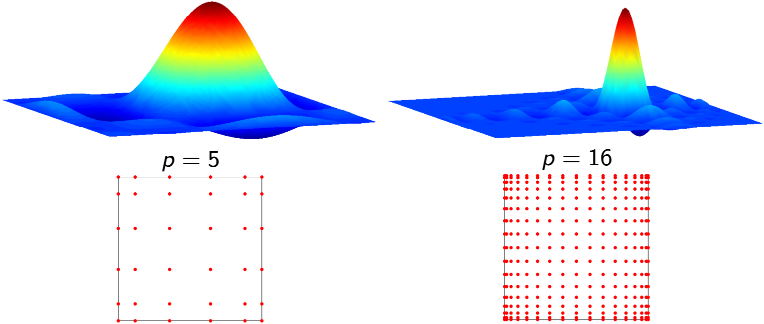
1. Low polynomial degree p (tipically $p = 1$ or $p = 2$),
2. Lagrangian basis functions $\varphi_i(x)$ related to the nodes of the mesh,
3. local support basis functions,
4. integrals are computed "exactly" or almost-exactly (with high-order degree quadrature formulas)



Spectral Methods

Gottlieb & Orszag (1977),
Canuto, Hussaini, Quarteroni, & Zang (1988),
Bernardi & Maday (1992)

Spectral Methods: one quad element Ω and global support of the polynomial basis functions on Ω.
One parameter: $p =$ polynomial degree ($\nearrow$)

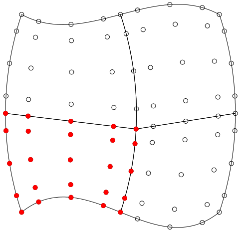


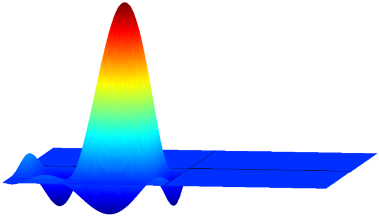
Spectral Element Methods

Patera (1984)

Spectral Element Methods: conformal partition of quads in Ω, global C^0 basis functions (local polynomials) with local support.

Two parameters: $p =$ pol. degree ($\nearrow$)
$h =$ mesh size (=elements diameter) ($\searrow 0$)





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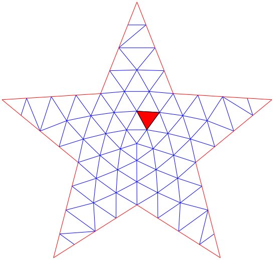
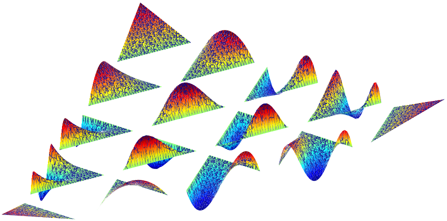
Spectral/hp

Patera (1984) for SEM on quads,
Dubiner (1991), Sherwin & Karniadakis (1995) for SEM on simplices

spectral/*hp* conformal partition of quads/simplices in Ω,
global C^0 basis functions (local polynomials) with local support.

Two parameters: p = pol. degree ($\nearrow$)
h = mesh size (=elements diameter) ($\searrow 0$)



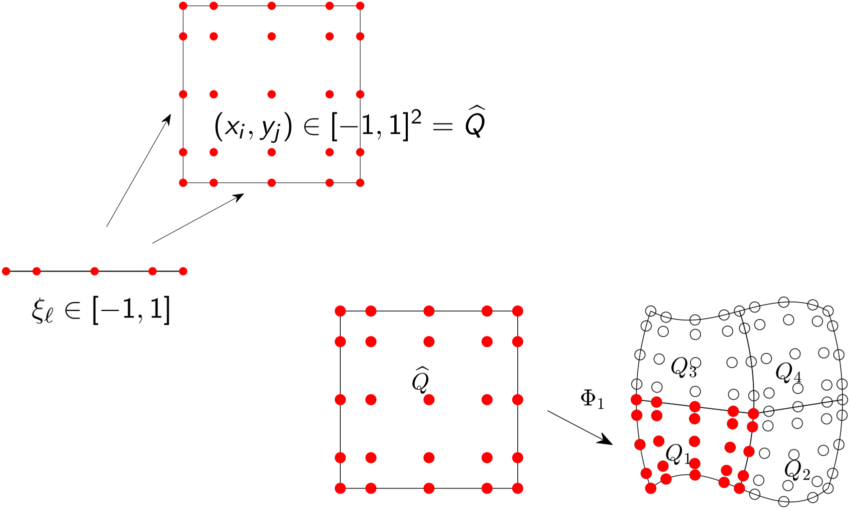


Spectral Element Methods (SEM) on quads

Strong points (of the most used form nowadays: SEM-GNI)

1. Moderate/high polynomial degree;
2. local support basis functions, but typically wider elements than in FEM;
3. Lagrangian basis functions related to the Legendre Gauss Lobatto (LGL) nodes in each element;
4. exact and almost-exact integration are too expensive, then LGL **quadrature formulas** are used. The quadrature nodes coincide with the nodes associated with the dof's. (Very cheap approach).
   (SEM-GNI = Galerkin with Numerical Integration);
5. the mass matrix is diagonal (consequence of the previous choice);
6. tensorial structure of the basis functions in $\mathbb{R}^d$ (with $d \geq 2$) $\Longrightarrow$ high computational efficiency

Tensorization and remapping of LGL nodes



Finite dimensional spaces

Set $\delta = (h, p)$,

$$X_\delta = \{v \in C^0(\overline{\Omega}) : \; v|_{T_k} = \hat{v} \circ \Phi_k^{-1}, \; with \; \hat{v} \in \mathbb{Q}_p, \; \forall T_k \in \mathcal{T}_h\}$$

$$V_\delta = X_\delta \cap V$$

If $a(u, v) = \int_\Omega \mu \nabla u \cdot \nabla v + \int_\Omega \sigma uv$, then the discrete bilinear form reads:

$$a_\delta(u, v) = \sum_q \mu(x_q) \nabla u(x_q) \nabla v(x_q) w_q + \sum_q \sigma(x_q) u(x_q) v(x_q) w_q$$

while the discrete functional is

$$F_\delta(v) = \sum_q f(x_q) v(x_q) w_q$$

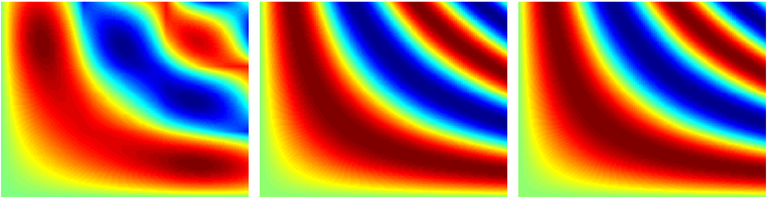
The LGL quadrature nodes x_q coincide with the nodes associated with the dof's.

Convergence analysis for SEM-GNI

$$?u_\delta \in V_\delta : \quad a_\delta(u_\delta, v_\delta) = (f, v_\delta)_{\delta,\Omega} \qquad \forall v_\delta \in V_\delta$$

u_δ converges with spectral accuracy (with respect to p) to the exact solution when the latter and f are smooth:

$$\|u - u_\delta\|_{H^1(\Omega)} \leq C(s)\Big(h^{\min(p,s)} \left(\frac{1}{p}\right)^s \|u\|_{H^{s+1}(\Omega)} + h^{\min(p,r)} \left(\frac{1}{p}\right)^r \|f\|_{H^r(\Omega)}\Big)$$



$h = 2/3$, $p = 2$	$h = 2/3$, $p = 6$	$h = 2/3$, $p = 16$
$e_{H1} \simeq 3.77e$-01	$e_{H1} \simeq 8.80e$-04	$e_{H1} \simeq 3.64e$-14

$\Omega = (0, 2)^2$

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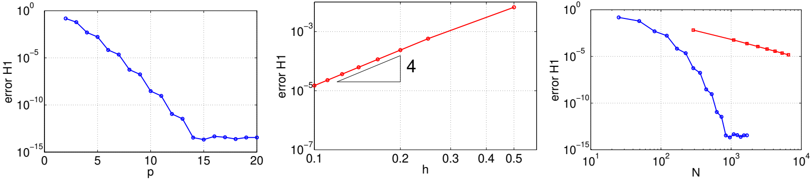
Convergence rate

1. $s,\ r$ large $(s, r > p)$

$$\|u-u_\delta\|_{H^1(\Omega)} \leq C\left(h^p \left(\frac{1}{p}\right)^s \|u\|_{H^{s+1}(\Omega)} + h^p\left(\frac{1}{p}\right)^r \|f\|_{H^r(\Omega)}\right)$$

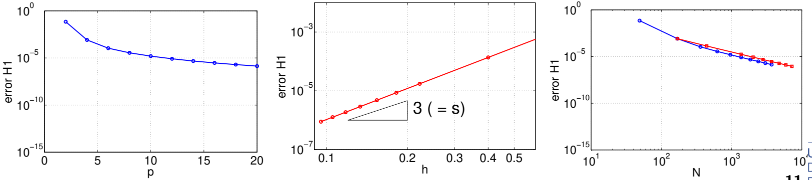
$s, r = \infty$



2. s small $(s \leq p)$

$$\|u-u_\delta\|_{H^1(\Omega)} \leq C\left(\frac{h}{p}\right)^s \|u\|_{H^{s+1}(\Omega)}$$

$s=4,\ r=2,\ f$ composite $\mathbb{Q}_2$, null quadrature error on f, when $p>2$



Essential bibliography (books)

BM C. Bernardi, Y. Maday. *Approximations Spectrales de Problèmes aux Limites Elliptiques.* Springer Verlag (1992)

KS G.E. Karniadakis, S.J. Sherwin. *Spectral/hp Element Methods for Computational Fluid Dynamics*, 2nd ed. Oxford University Press (2005)

CHQZ2 C. Canuto, M.Y. Hussaini, A. Quarteroni, T. Zang. *Spectral Methods. Fundamentals in Single Domains*. Springer (2006)

CHQZ3 C. Canuto, M.Y. Hussaini, A. Quarteroni, T. Zang. *Spectral Methods. Evolution to Complex Geometries and Applications to Fluid Dynamics*. Springer (2007)