DICACIM programme A.Y. 2024–25
Numerical Methods for Partial Differential Equations

Numerical solution of 2D elliptic problems with Finite Elements and MATLAB PDEtool

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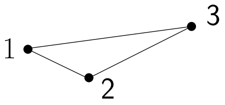
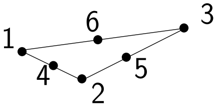
The PDETool toolbox of MATLAB implements meshes of:

- triangles in 2D and tetrahedra in 3D,
- FEM-$\mathbb{P}_1$ (linear) and FEM-$\mathbb{P}_2$ (quadratic).

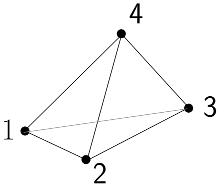
Degrees of Freedom ordering



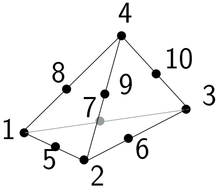
2D $\mathbb{P}_1$ (linear)



2D $\mathbb{P}_2$ (quadratic)



3D $\mathbb{P}_1$ (linear)



3D $\mathbb{P}_2$ (quadratic)

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Workflow

1. define the **geometry** (Ω),
2. select the **model** (specific PDE, coefficients of the problems, source term),
3. define **boundary conditions**,
4. build the **mesh** ($\mathcal{T}_h$),
5. **solve** the linear system (compute u_h),
6. **plot** the solution,
7. analyse the **errors** (if you have a **test solution**).

Two different paths:

1. use **pdeModeler** (graphic interface): only $\mathbb{P}_1$ in 2D ,
2. create a **PDE model object** and write script/function: $\mathbb{P}_1$ and $\mathbb{P}_2$, both 2D and 3D).

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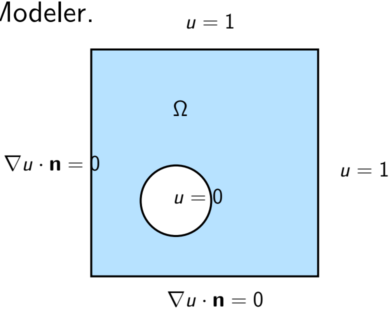
Problem 1

Let $P = (0.4, 0.3)$ a point and C the circle with center P and radius $r = 0.2$. Approximate the solution u of the Poisson problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega = (0,1)^2 \setminus C \\ u = 0 & \text{on } \partial C \\ u = 1 & \text{if } x = 1, \ o \ y = 1 \\ \nabla u \cdot \mathbf{n} = 0 & \text{if } x = 0, \ o \ y = 0 \end{cases}$$

with FEM-$\mathbb{P}_1$ using pdeModeler.



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PDEModeler (graphic interface) – only 2D

Write the command

```
>> pdeModeler
```

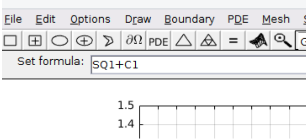
Initialization of the workflow

From the horizontal menu (on top):

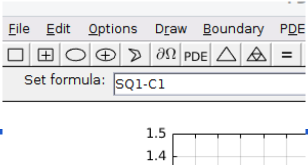
- Options > Axes Limits, to define the 2D area in which to draw the computational domain,
- Options > Grid Spacing, to define the how much fine the grid should be,
- Options > Snap, to facilitate the drawing (the pixels are attracte towards the points of the grid)
- Options > Grid, to draw the grid inside the window

PDE Modeler – define the geometry

- The geometry of the computational domain can be defined as the union/intersection/difference of rectangles, ellipses, polygons, circles,
- click on the icon to draw the chosen form with the mouse,
- by default, Ω is the union of all the elementary shapes you have drawn (the union must be a connected region of the plane). n the "Set formula" box we can read "+"

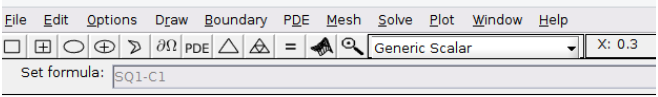


- if you want that Ω is the difference between the two shapes, then replace "+" with "-"



PDE Modeler – Choose the model problem

Open the menu on the right of the magnifying glass to select the problem:



By choosing Generic Scalar, you can solve the elliptic 2nd–order problem

$$-\nabla \cdot (c\nabla u) + au = f$$

where c, a, and f can be either constant or functions of (x, y).

From the top horizontal menu, select PDE > PDE Specification, to define c, a, and f.
Check that "Type of PDE" is **elliptic**.

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PDE Modeler – boundary conditions

From the top horizontal menu, select
Boundary > Boundary Mode, to define boundary conditions.
The edges of the domain are highlighted (in red) with their own direction of motion.
Each arrow is an edge.
Circles (and ellipses) are split in four arcs corresponding to 4 clock faces.

1. **Shift+click** on one edge to selct it,
2. Boundary > Specify Boundary Conditions..
3. select either Neumann (actually it is a Robine b.c.) or Dirichlet
4. write the coefficients
   - Dirichlet: $u = g_D$, set $h = 1$ and $r = g_D$
   - Neumann: $c\frac{\partial u}{\partial \mathbf{n}} = c\nabla u \cdot \mathbf{n} = g_N$, set $q = 0$ and $g = g_N$.
5. if you do not make any choice on one edge, then homogeneous Dirichlet condition is imposed on it.
6. Neumann edges are colored in blue, Dirichlet edges in red.

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PDE Modeler – Mesh generation

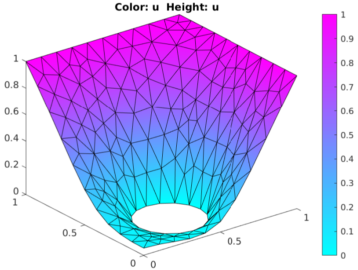
From the top horizontal menu, select

1. Mesh > Mesh Mode, to build the mesh,
2. Mesh > Parameter...
3. to define the "Maximum edge size" (h)
4. Mesh > Initialize Mesh
5. Mesh > Refine Mesh refines the mesh by splitting each triangle in 4 triangles.

PDE Modeler – solution and plot

From the top horizontal menu, select

1. Solve > Solve PDE to solve the problem (matrix and r.h.s are assembled and the linear systems is solved)
2. Plot > Plot Solution to draw the 2D solution
3. Plot > Parameters... if you want a 3D plot, if you want to draw the mesh, ∇u, the conoutlines.



Solution computed with $h = 0.1$.