DICACIM programme A.Y. 2023–24
Numerical Methods for Partial Differential Equations

A summary of Direct Methods for Linear Systems

Paola Gervasio

DICATAM, Università degli Studi di Brescia (Italy)

UNIVERSITÀ
DEGLI STUDI
DI BRESCIA

Introduction

Given $A \in \mathbb{R}^{n\times n}$ not singular and $\mathbf{b} \in \mathbb{R}^n$, find $\mathbf{x} \in \mathbb{R}^n$ solution of $A\mathbf{x} = \mathbf{b}$

Direct methods	**Iterative methods**
A and $\mathbf{b}$ are modified to achieve a system $\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}$ equivalent to $A\mathbf{x} = \mathbf{b}$, but easier to solve.	A and $\mathbf{b}$ are not modified, A is used to compute matrix–vector products.
find the exact solution $\mathbf{x}$ up to rounding errors in a finite number of operations.	look for $\mathbf{x} = \lim_{k\to\infty} \mathbf{x}^{(k)}$, so that the exact solution could only be found after infinite iterations and operations. Actually, a stopping test is used.
$\dfrac{\|\mathbf{x}-\hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq C\epsilon_M$	$\dfrac{\|\mathbf{x}-\hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq C\epsilon_M + \varepsilon$

where:
$\mathbf{x}$ is the exact solution (on paper)
$\hat{\mathbf{x}}$ is the numerical solution
ϵ_M is the machine precision (typically $\epsilon_M \sim 10^{-16}$)
ε is the tolerance for the stopping test of the iterative method.

Direct methods for square linear systems

- **Gauss Elimination Method**: trasforms the linear system $A\mathbf{x} = \mathbf{b}$ in an equivalent one $\tilde{A}\mathbf{x} = \tilde{\mathbf{b}}$ with $\tilde{A}$ upper triangular. The latter is solved by backward subsitutions.
  Total cost: $\sim \frac{2}{3}n^3$ floating point operations

- **LU factorization**: computes L lower triangular and U upper triangular s.t. $A = L \cdot U$, then solves $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$.
  Total cost: $\sim \frac{2}{3}n^3$ floating point operations.
  More efficient than GEM if you need to solve more systems with the same matrix but different right hand sides.

- **Cholesky factorization:** (only if A is symmetric positive definite) computes an upper triangular matrix R s.t. $A = R^T R$, then solves $R^T\mathbf{y} = \mathbf{b}$ and $R\mathbf{x} = \mathbf{y}$.
  Total cost: $\sim \frac{1}{3}n^3$ floating point operations.

MATLAB instructions

- **Gauss Elimination Method**:

```
A=[1,-2, 3; 2, 1, -1; -2,4,2];
b=[1;3;-1];
x=A\b
```

- **LU factorization**:

```
A=[1,-2, 3; 2, 1, -1; -2,4,2];
b=[1;3;-1];
[L,U]=lu(A);
y=L\b;
x=U\y
```

- **Cholesky factorization:**

```
A=[4,-2, 3; -2, 4, -1; 3,-1,5];
b=[1;3;-1];
R=chol(A);
y=R'\b;
x=R\y
```