DICACIM programme A.Y. 2024–25 Numerical Methods for Partial Differential Equations

1dimensional fem: $\mathbb{P}_2$-fem

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1-dimensional $\mathbb{P}_2$ fem

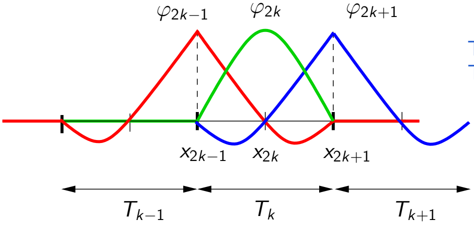
Mesh. Let $\mathcal{T}_h$ be a partition of $\Omega = (x_a, x_b)$ in N_e intervals:

$$\mathcal{T}_h = \{ T_k, k = 1, \ldots, N_e : T_k \text{ are dsjoint and } \overline{\cup_k T_k} = \overline{\Omega} \}$$

Finite elements space. Set

$$X_h^2 = \{ v \in C^0(\overline{\Omega}) : \; v|_{T_k} \in \mathbb{P}_2, \quad \forall T_k \in \mathcal{T}_h \} \qquad \text{and} \qquad V_h = X_h^2 \cap V.$$

Basis. Lagrangian quadratic piece-wise functions:



Total number of points: $N_h = 2N_e + 1$.
Total number of basis functions: N_h.

In each element T_k, consider the midpoint and define 3 Lagrangian basi functions of degree 2.

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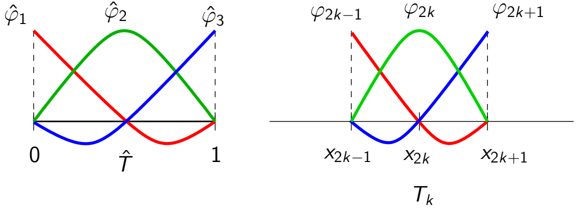
$\mathbb{P}_2$ fem

The basis functions on the reference element are:

$$\hat{\varphi}_1(\hat{x}) = 2(\hat{x}-1)(\hat{x}-0.5) \qquad \hat{\varphi}_2(\hat{x}) = 4\hat{x}(1-\hat{x}) \qquad \hat{\varphi}_3(\hat{x}) = 2\hat{x}(\hat{x}-0.5)$$

$\mathbb{P}_2 \ni \hat{\varphi}_j(\hat{x}_i) = \delta_{ij}$ (Lagrangian basis functions)



Local Mass matrix:

$$M^{(k)} = \frac{h_k}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

$$M_{ij}^{(k)} = \int_{T_k} \varphi_j \varphi_i$$

Local Stiffness matrix:

$$K^{(k)} = \frac{1}{3h_k}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$$K_{ij}^{(k)} = \int_{T_k} \varphi_j' \varphi_i'$$

Lagrange composite interpolation of degree r

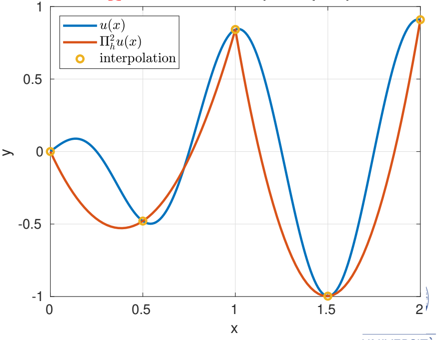
Let $\mathcal{T}_h$ be a partition of $\Omega = (x_a, x_b)$ in N_e intervals:

$$\mathcal{T}_h = \{T_k, k = 1, \dots, N_e : T_k \text{ are dsjoint and } \overline{\cup_k T_k} = \overline{\Omega}\}$$

In each element add $(r-1)$ internal (and equispaced) points.
N_h = total number of points in $\overline{\Omega}$.

The **piece-wise Lagrange interpolation of u of degree r** is the (unique) function $\Pi_h^r u(x)$ s.t.:

1. $\Pi_h^r u(x) \in C^0(\overline{\Omega})$
   (global continuity)
2. $\Pi_h^r u|_{T_k} \in \mathbb{P}_r, \ \forall T_k$
   (local polynomial of degree r)
3. $\Pi_h^r u(x_k) = u(x_k)$ for $k = 1, \dots, N_h$
   (interpolation conditions)



Interpolation estimates for local degree $r \geq 1$

Theorem. Let $\Omega = (a, b)$ and $s > \frac{1}{2}$. If $u \in H^{s+1}(\Omega)$, then

$$\|u - \Pi_h^r u\|_{H^1(\Omega)} \leq Ch^{\min(s,r)} \|u\|_{H^{s+1}(\Omega)}$$

$$\|u - \Pi_h^r u\|_{L^2(\Omega)} \leq Ch^{\min(s,r)+1} \|u\|_{H^{s+1}(\Omega)}$$

Error estimates for $\mathbb{P}_r$ fem (with $r \geq 1$)

1d, 2nd-order elliptic PDE

Theorem. If u is the exact solution and u_h is the $\mathbb{P}_r$ fem solution

$$\|u - u_h\|_{H^1(\Omega)} \leq C\frac{M}{\alpha} h^{\min(s,r)} \|u\|_{H^{s+1}(\Omega)}$$

$$\|u - u_h\|_{L^2(\Omega)} \leq C\frac{M}{\alpha} h^{\min(s,r)+1} \|u\|_{H^{s+1}(\Omega)}$$

r	$u \in H^{s+1}(\Omega)$ $s \in [1/2, 1)$	$u \in H^2(\Omega)$	$u \in H^3(\Omega)$	$u \in H^4(\Omega)$	$u \in H^5(\Omega)$
1	h^s	h^1	h^1	h^1	h^1
2	h^s	h^1	h^2	h^2	h^2
3	h^s	h^1	h^2	h^3	h^3
4	h^s	h^1	h^2	h^3	h^4