# DICACIM programme A.Y. 2023–24 Numerical Methods for Partial Differential Equations

## 1dimensional fem: interpolation and error estimates

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# Lagrange composite interpolation

Assumptions/notations:

- Given $\Omega = (a, b)$ and $u \in C^0(\overline{\Omega})$,
- let $a = x_1 < x_2 < \ldots < x_{N_h} = b$ be $N_h$ distinct and ordered points in $\overline{\Omega}$,
- set $T_k = [x_k, x_{k+1}]$, and $h = \max_k (x_{k+1} - x_k)$.


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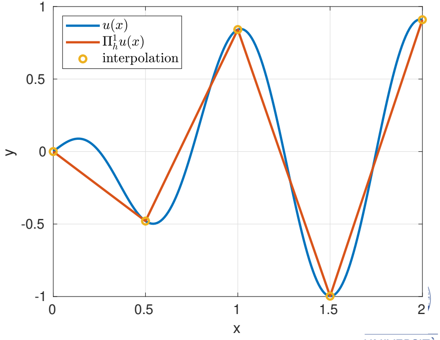
The **linear piece-wise Lagrange interpolation of $u$ at the nodes $x_k$** is the (unique) function $\Pi_h^1 u(x)$ s.t.:

1. $\Pi_h^1 u(x) \in C^0(\overline{\Omega})$
   (global continuity)
2. $\Pi_h^1 u|_{T_k} \in \mathbb{P}_1, \ \forall T_k$
   (local polynomial of degree 1)
3. $\Pi_h^1 u(x_k) = u(x_k)$ for $k = 1, \ldots, N_h$
   (interpolation conditions)



# Interpolation estimates

**Theorem.** Let $\Omega = (a, b)$ and $s > \frac{1}{2}$. If $u \in H^{s+1}(\Omega)$, then

$$\|u - \Pi_h^1 u\|_{H^1(\Omega)} \leq C h^{\min(s,1)} \|u\|_{H^{s+1}(\Omega)}$$

$$\|u - \Pi_h^1 u\|_{L^2(\Omega)} \leq C h^{\min(s,1)+1} \|u\|_{H^{s+1}(\Omega)}$$

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**Example.** If $u \in H^2(\Omega)$, then $s = 1$ and

$$\|u - \Pi_h^1 u\|_{H^1(\Omega)} \leq C h \|u\|_{H^2(\Omega)}$$ **(linear convergence)**

$$\|u - \Pi_h^1 u\|_{L^2(\Omega)} \leq C h^2 \|u\|_{H^2(\Omega)}$$ **(quadratic convergence)**

If $h$ reduces of one order, then $\|u - \Pi_h^1 u\|_{H^1(\Omega)}$ reduces of one order too, while $\|u - \Pi_h^1 u\|_{L^2(\Omega)}$ reduces of two orders.

Remark. The error in the $L^2-$ norm is lower than the error in $H^1-$norm, it doesn't measure the discrepancy on derivatives.

# Approximation error of $\mathbb{P}_1$−fem

**1d, 2nd-order elliptic PDE**

If $u$ is the solution of the **continuous weak problem**

$$?u \in V : \qquad a(u,v) = F(v) \qquad \forall v \in V$$

and $u_h$ is the **Galerkin solution**, i.e. the solution of

$$?u_h \in V_h : \qquad a(u_h,v_h) = F(v_h) \qquad \forall v_h \in V_h$$

the **Céa Lemma** states that:

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V .$$

# Approximation error of $\mathbb{P}_1-$fem

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Recalling that $V = H^1(\Omega)$ and because

$$\inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} \leq \|u - \Pi_h^1 u\|_{H^1(\Omega)},$$

if $s > 1/2$ and $u \in H^{s+1}(\Omega)$, it holds

$$\|u - u_h\|_{H^1(\Omega)} \leq C \frac{M}{\alpha} h^{\min(s,1)} \|u\|_{H^{s+1}(\Omega)}.$$


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# Approximation error of $\mathbb{P}_1-$fem (cont'd)

It can also be measured the error in $L^2-$norm:

$$\|u-u_h\|_{L^2(\Omega)} \leq C\frac{M}{\alpha}h^{\min(s,1)+1}\|u\|_{H^{s+1}(\Omega)}.$$

**Example.** If $u \in H^2(\Omega)$, then $s = 1$ and

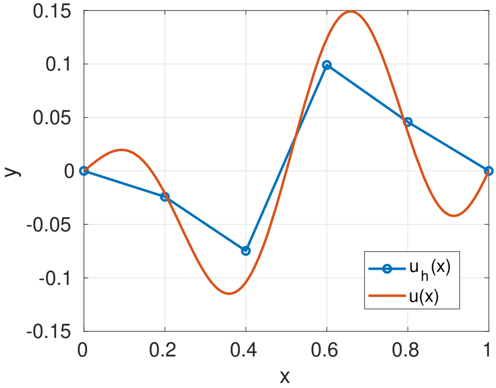
$$\|u-u_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^2(\Omega)} \qquad \text{(linear convergence)}$$

$$\|u-u_h\|_{L^2(\Omega)} \leq Ch^2\|u\|_{H^2(\Omega)} \qquad \text{(quadratic convergence)}$$

If $h$ reduces of one order, then $\|u-u_h\|_{H^1(\Omega)}$ reduces of one order too, while $\|u-u_h\|_{L^2(\Omega)}$ reduces of two orders.

# Warning

Notice that $u_h$ does not necesseraly interpolate $u$ at the mesh nodes.



# 1-dimensional $\mathbb{P}_2$ fem

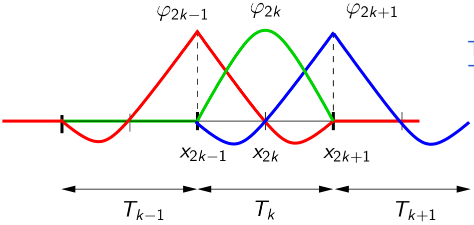
**Mesh.** Let $\mathcal{T}_h$ be a partition of $\Omega = (x_a, x_b)$ in $N_e$ intervals:

$$\mathcal{T}_h = \{\, T_k, k = 1, \ldots, N_e \, : \, T_k \text{ are dsjoint and } \overline{\cup_k T_k} = \overline{\Omega}\}$$

**Finite elements space.** Set

$$X_h^2 = \{v \in C^0(\overline{\Omega}) : \; v|_{T_k} \in \mathbb{P}_2, \quad \forall T_k \in \mathcal{T}_h\} \qquad \text{and} \qquad V_h = X_h^2 \cap V.$$

**Basis.** Lagrangian quadratic piece-wise functions:




Total number of points: $N_h = 2N_e + 1$.
Total number of basis functions: $N_h$.

In each element $T_k$, consider the midpoint and define 3 Lagrangian basi functions of degree 2.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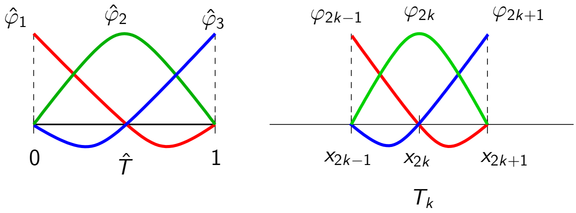
# $\mathbb{P}_2$ fem

The basis functions on the reference element are:

$$\hat{\varphi}_1(\hat{x}) = 2(\hat{x}-1)(\hat{x}-0.5) \qquad \hat{\varphi}_2(\hat{x}) = 4\hat{x}(1-\hat{x}) \qquad \hat{\varphi}_3(\hat{x}) = 2\hat{x}(\hat{x}-0.5)$$

$\mathbb{P}_2 \ni \hat{\varphi}_j(\hat{x}_i) = \delta_{ij}$ (Lagrangian basis functions)




**Local Mass matrix**:

$$M^{(k)} = \frac{h_k}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

$$M_{ij}^{(k)} = \int_{T_k} \varphi_j \varphi_i$$

**Local Stiffness matrix**:

$$K^{(k)} = \frac{1}{3h_k}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$$K_{ij}^{(k)} = \int_{T_k} \varphi_j' \varphi_i'$$

# Lagrange composite interpolation of degree $r$

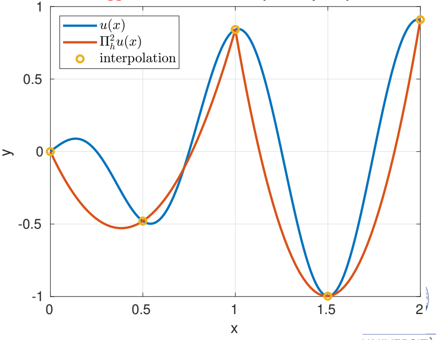
Let $\mathcal{T}_h$ be a partition of $\Omega = (x_a, x_b)$ in $N_e$ intervals:

$$\mathcal{T}_h = \{T_k, k = 1, \ldots, N_e : T_k \text{ are dsjoint and } \overline{\cup_k T_k} = \overline{\Omega}\}$$

In each element add $(r-1)$ internal (and equispaced) points.
$N_h$ = total number of points in $\overline{\Omega}$.

The **piece-wise Lagrange interpolation of $u$ of degree $r$** is the (unique) function $\Pi_h^r u(x)$ s.t.:

1. $\Pi_h^r u(x) \in C^0(\overline{\Omega})$
   (global continuity)
2. $\Pi_h^r u|_{T_k} \in \mathbb{P}_r, \ \forall T_k$
   (local polynomial of degree $r$)
3. $\Pi_h^r u(x_k) = u(x_k)$ for $k = 1, \ldots, N_h$
   (interpolation conditions)



# Interpolation estimates for local degree $r \geq 1$

**Theorem.** Let $\Omega = (a, b)$ and $s > \frac{1}{2}$. If $u \in H^{s+1}(\Omega)$, then

$$\|u - \Pi_h^r u\|_{H^1(\Omega)} \leq C h^{\min(s,r)} \|u\|_{H^{s+1}(\Omega)}$$

$$\|u - \Pi_h^r u\|_{L^2(\Omega)} \leq C h^{\min(s,r)+1} \|u\|_{H^{s+1}(\Omega)}$$

# Error estimates for $\mathbb{P}_r$ fem (with $r \geq 1$)

**1d, 2nd-order elliptic PDE**

**Theorem**. If $u$ is the exact solution and $u_h$ is the $\mathbb{P}_r$ fem solution

$$\|u - u_h\|_{H^1(\Omega)} \leq C \frac{M}{\alpha} h^{\min(s,r)} \|u\|_{H^{s+1}(\Omega)}$$

$$\|u - u_h\|_{L^2(\Omega)} \leq C \frac{M}{\alpha} h^{\min(s,r)+1} \|u\|_{H^{s+1}(\Omega)}$$

| $r$ | $u \in H^1(\Omega)$ | $u \in H^2(\Omega)$ | $u \in H^3(\Omega)$ | $u \in H^4(\Omega)$ | $u \in H^5(\Omega)$ |
|---|---|---|---|---|---|
| 1 | converges | $h^1$ | $h^1$ | $h^1$ | $h^1$ |
| 2 | converges | $h^1$ | $h^2$ | $h^2$ | $h^2$ |
| 3 | converges | $h^1$ | $h^2$ | $h^3$ | $h^3$ |
| 4 | converges | $h^1$ | $h^2$ | $h^3$ | $h^4$ |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