DICACIM programme A.Y. 2023–24
Numerical Methods for Partial Differential Equations

Numerical solution of 1D elliptic problems with Finite Elements and MATLAB

Paola Gervasio

DICATAM, Università degli Studi di Brescia (Italy)

uniBS, May, 2024

UNIVERSITÀ
DEGLI STUDI
DI BRESCIA

Download the file

https://paola-gervasio.unibs.it/NMPDE/FEM1d.zip

It contains:

- `fem_1d_solver.m`: it solves 1d elliptic problems by linear or quadratic FEM;
- `fem_1d_errors.m`: it computes the errors between the exact and the numerical solutions in the H^1 and L^2 norms;
- `fem_1d_setting.m`: it defines quadrature nodes and weights, evaluates the basis functions at the quadrature nodes, computes the jacobian of the maps transforming the reference interval to the generic element. It is referenced inside the two previous functions;
- `xwlg.m`: it computes Legendre-Gauss quadrature nodes and weights; it is referenced inside `fem_1d_setting.m`;
- `xwlgl.m`: it computes Legendre-Gauss-Lobatto quadrature nodes and weights; it is referenced inside `fem_1d_setting.m`;

UNIVERSITÀ DEGLI STUDI DI BRESCIA

fem_1d_solver.m

```
>> help fem_1d_solver

fem_1d_solver: solve -mu*u''+sigma u=f in Omega
with Dirichlet and/or Neumann boundary conditions
by either P1-fem or P2-fem on a uniform grid.

[nodes,uh]=fem_1d_solver(geom,problem_data,p,Ne)

Input: geom: struct with fields:
           geom.xa, geom.xb = end-points of Omega
       problem_data: struct with coefficients and ...
           ....
       p = local polynomial degree (1 or 2)
       Ne =  number of elements of the partition
Output: nodes = column array with nodes of the mesh
        uh = column array of the numerical solution
```

fem_1d_errors.m

```
>> help fem_1d_errors

fem_1d_errors Computes errors for 1d b.v.p.
[errors]=fem_1d_errors(uh,geom,p,Ne,uex,u1ex,errtype)
Input:
 uh= numerical solution
 geom = struct with fields:
   geom.xa, geom.xb = end-points of Omega
 p = local polynomial degree (1 or 2)
 Ne =  number of elements of the partition
 uex =  function handle @(x) with the exact solution
 u1ex =  function handle @(x) with the derivative
             of the exact solution
optional input: errtype = 0: absolute errors (default)
                errtype = 1: relative errors
Output:
  errors.h1 = ||u-uh||_H1
  errors.l2 = ||u-uh||_L2
```

Problem 1

Let us consider the **Poisson problem with homogeneous Dirichlet conditions**:

$$\begin{cases} -u''(x) = f(x) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

with $\Omega = (0,1)$ and $f(x) = 9\pi^2 \sin(3\pi x) + 2$.

1. **Compute the numerical solution** $u_h(x)$ by $\mathbb{P}_1$ fem on a uniform mesh with `Ne=5, 10, 20, 40`. Then plot the numerical solution.

2. **Compute the errors** between the numerical solution u_h and the exact solution $u(x) = \sin(3\pi x) - x^2 + x$ in $H^1(\Omega)$ norm and in $L^2(\Omega)$ norm, then verify that, when $h \to 0$, it holds

$$\|u - u_h\|_{H^1(\Omega)} \simeq h \qquad \|u - u_h\|_{L^2(\Omega)} \simeq h^2$$

3. **Repeat** the work with $\mathbb{P}_2$ fem and verify that the H^1 and L^2 errors vanish like h^2 and h^3 (respectively) when $h \to 0$.

Problem 2

Let us consider the **reaction diffusione equation with non–homogeneous Dirichlet conditions**:

$$\begin{cases} -u''(x) + 10u(x) = 9e^x & \text{in } (-1,1) \\ u(-1) = 1/e, u(1) = e \end{cases}$$

Repeat the work done to solve the Problem 1. Now the exact solution is $u(x) = e^x$.

Problem 3. Heat transfer in a thin rod

Let us consider a thin rod of length L and circular section of radius r. At the end point $x = 0$ the temperature is fixed and equal to T_0, while at the right end point $x = L$ the rod is insulated. La temperature T of the rod at a point $x \in (0, L)$ is the solution of the following elliptic boundary value problem:

$$\begin{cases} -kAT'' + \tilde{\sigma} pT = 0, & x \in (0, L), \\ T(0) = T_0, & T'(L) = 0, \end{cases} \tag{1}$$

where k is the thermal conductivity, $\tilde{\sigma}$ is the heat transfer coefficient, A and p are the area and perimenter (respectively) of the circular section of the rod.

1. **Approximate the solution** of (1) by using $\mathbb{P}_2$ FEM on a uniform grid with `Ne=10, 20, 40, 80` elements and by taking:
$L = 1m$, $r = 10^{-2}m$, $k = 200\frac{W}{mK}$, $\tilde{\sigma} = 2\frac{W}{m^2K}$, $T_0 = 10K$.

2. **Verify** that the exact solution of (1) is

$$T(x) = T_0 \frac{\cosh(\alpha(L - x))}{\cosh(\alpha L)},$$

with $\alpha = \sqrt{\frac{\tilde{\sigma} p}{kA}}$, and verify that the approximation errors behave like the theoretical estimates.

UNIVERSITÀ DEGLI STUDI DI BRESCIA
Paola Gervasio - DICATAM - unibs.it

Problem 4. Tension of a string

Consider a string with tension T and length L, blocked at its end points. The function $u(x)$, which measures the vertical displacement of the string when it is subject to a transversal load of intensity f, is the solution of the boundary value problem:

$$\begin{cases} -Tu'' + ku = f & \text{in } (0, L), \\ u(0) = 0, & u(L) = 0, \end{cases}$$

where k is the elastic coefficient of the string.

Approximate u by $\mathbb{P}_1$ fem, using a uniform grid with the following data:

1. $L = 1,\ T = 1,\ k = 1,\ f(x) \equiv 1$, `Ne=5, 10, 20, 40;`
2. $L = 1,\ T = 10^{-3},\ k = 1,\ f(x) \equiv 1$, `Ne=5, 10, 20, 40,`
3. $L = 1,\ T = 10^{-4},\ k = 1,\ f(x) \equiv 1$, `Ne=10, 20, 40, 80,`

Problem 4: drawback

When $T << k$ and h is not small enough, the numerical solution features **spurious oscillations** that are not physical. They are due to the coarse discrertization.

The exact solution exhibits two **boundary layers** of size $\sim \sqrt{T/k}$. **To capture the layers, h must be small enough** with respect to the size of the region where the layer occurs.

More precisely, it is required that

$$h < \sqrt{6\frac{T}{k}}. \tag{2}$$

Verify that if h satisfies (2), then the numerical solution is oscillations–free.

Problema 4: cure

To overcome the latter problem, the mass matrix M ($M_{ij} = \int_\Omega \varphi_j \varphi_i$) can be replaced with the **lumped mass matrix**

$$\hat{M} : \quad \hat{M}_{ij} = \delta_{ij} \sum_j M_{ij}.$$

$\hat{M}$ is diagonal, the diagonal element on the row i is the sum of all the element of the row i of M.

Test:
open the file `fem_1d_solver.m` and uncomment line 97.

Then recompute the numerical solution.