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Numerical Methods for Partial Differential Equations

Legendre–Gauss Quadrature formulas

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Legendre–Gauss (LG) quadrature formulas

Quadrature formulas approximate integrals by using some values of the function at a suitable points in the interval of integration:

$$\int_{-1}^{1} \hat{g}(\hat{x}) d\hat{x} \simeq \sum_{j=0}^{n} \hat{g}(\hat{x}_j)\hat{w}_j$$

where:

- $\hat{x}_j$ are $(n+1)$ distinct points in $(-1,1)$ named **quadrature nodes**,
- $\hat{w}_j$ are $(n+1)$ positive real values named **quadrature weights**.

In the LG quadrature formulas:

nodes: the roots of the Legendre polynomial $L_{n+1}(\hat{x})$,

weights: $\hat{w}_j = \frac{2}{(1-\hat{x}_j^2)[L'_{n+1}(\hat{x}_j)]^2}$.

n	$\hat{x}_j$	$\hat{w}_j$
1	$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$	1, 1
2	$-\frac{\sqrt{15}}{5}, 0, \frac{\sqrt{15}}{5}$	$\frac{5}{9}, \frac{8}{9}, \frac{5}{9}$
3	$\pm\frac{\sqrt{525-70\sqrt{30}}}{35}, \pm\frac{\sqrt{525+70\sqrt{30}}}{35},$	$\frac{(18+\sqrt{30})}{36}, \frac{(18-\sqrt{30})}{36}$

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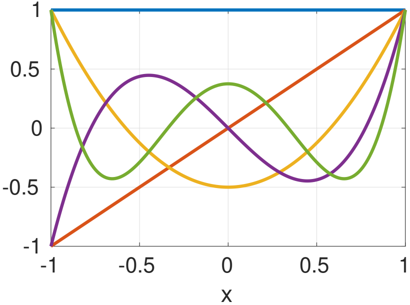
Legendre polynomials

The Legendre polynoamials $\{L_n(x) \in \mathbb{P}_n,\ n = 0, 1, \dots,\}$ satify the recursive relation:

$$L_0(x) = 1, \quad L_1(x) = x, \text{ and for } n \geq 1$$
$$L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x).$$

They are orthogonal w.r.t. the L^2-product: $(L_n, L_m)_{L^2(-1,1)} = 0$ if $n \neq m$.



LG quadrature formulas (cont'd...)

The LG quadrature formulas with $n+1$ nodes have **degree of exactness** equal to $2n+1$:
if $\hat{g}$ is a polynomial of degree $\leq 2n+1$, then it holds:

$$\int_{-1}^{1} \hat{g}(\hat{x})d\hat{x} = \sum_{j=0}^{n} \hat{g}(\hat{x}_j)\hat{w}_j.$$

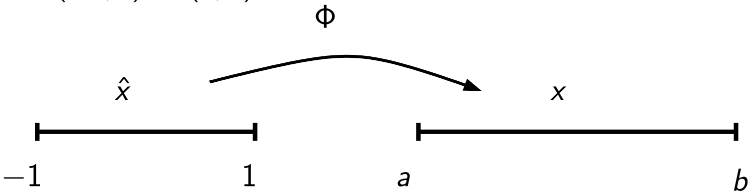
Approximation error: if $\hat{g} \in H^{s+1}(-1,1)$,[1] then

$$\left| \int_{-1}^{1} \hat{g}(\hat{x})d\hat{x} - \sum_{j=0}^{n} \hat{g}(\hat{x}_j)\hat{w}_j \right| \leq \left(\frac{1}{n}\right)^{s+1} \|\hat{g}\|_{H^{s+1}(-1,1)}$$

[1] $H^{s+1}(-1,1)$ is the Sobolev space of order $(s+1)$, i.e., it is the space of functions defined in $(-1,1)$ whose derivative (in weak sense) up to order $(s+1)$ belong to $L^2(-1,1)$.

Nodes and weights on generic intervals

Let $\hat{x}_j,\ \hat{w}_j$ (with $j = 0, \ldots, n$) be the nodes and weights on $(-1, 1)$.
Let $\Phi : (-1, 1) \to (a, b)$:



Then we have:

$$x_j = \Phi(\hat{x}_j) = \frac{b-a}{2}\hat{x}_j + \frac{a+b}{2}, \quad w_j = \frac{b-a}{2}\hat{w}_j$$

and

$$\int_a^b g(x)dx \simeq \sum_{j=0}^{n} g(x_j)w_j.$$

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