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Numerical Methods for Partial Differential Equations

Some elements of functional analysis

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The Poisson equation

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega = (0,1)^2 \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (1)$$

At the vertices: $\Delta u = 0$,

$\forall x \in \Omega$: $\Delta u = -1$

$$\Downarrow$$

$$\Delta u \notin C^0(\overline{\Omega}) \quad \Rightarrow \quad u \notin C^2(\overline{\Omega})$$

We cannot look for the solution of (1) in $C^2(\overline{\Omega})$.

We weaken the formulation of our PDE

1d case

Strong form of the Poisson equation:

$$\begin{cases} -u'' = 1 & \text{in } \Omega = (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{2}$$

Test space. $C^1_0([0,1]) = \{v \in V^1([0,1]) : \ v(0) = v(1) = 0\}$.

Weak form of the Poisson equation:

$$\int_0^1 u'v' = \int_0^1 fv \qquad \forall v \in C^1_0([0,1]) \tag{3}$$

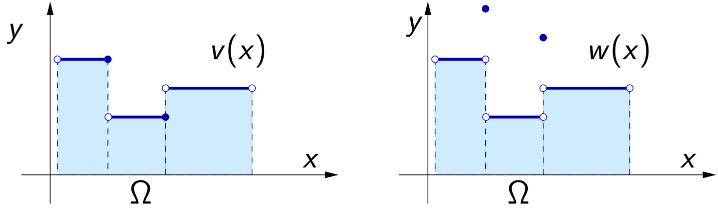
Remark: Even if f is regular, we cannot guarantee that $u \in C^1_0([0,1])$. We need to consider larger spaces: the **Sobolev spaces**.

The $L^2(\Omega)$ space

Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $d = 1, 2, 3$.

$$L^2(\Omega) = \{v : \Omega \to \mathbb{R} : \int_\Omega v^2 d\Omega < +\infty\}.$$

The elements of $L^2(\Omega)$ are classes of functions.
We identify in a unique function all those functions that differ on a set of null measure.



v and w only differ at two points, then they represent the same element of $L^2(\Omega)$.

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Examples of functions of $L^2(\Omega)$

- If Ω is bounded, any bounded functions (also discontinuous) are in $L^2(\Omega)$,
- some unbounded functions belong to $L^2(\Omega)$, e.g.:

$$\Omega = (0,1), \qquad u(x) = \frac{1}{x^\alpha}, \quad \forall \alpha < \frac{1}{2}$$

$u(x) = x^{-1/3} \in L^2(0,1)$,

$u(x) = x^{-2} \notin L^2(0,1)$

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The space $L^2(\Omega)$

- **Inner product:** $(u,v)_{L^2(\Omega)} = \int_\Omega uv \; d\Omega.$
- **Norm:** $\|u\|_{L^2(\Omega)} = \sqrt{(u,u)_{L^2(\Omega)}} = \left(\int_\Omega u^2 \; d\Omega\right)^{1/2}.$
- $(L^2(\Omega), \; \|\cdot\|_{L^2(\Omega)})$ is a Hilbert space (the norm is induced by an inner product and it is complete).
- A normed space $(V, \|\cdot\|_V)$ is complete if every Cauchy sequence in V converges to an element of the space V.
- In $L^2(\Omega)$ it holds the **Cauchy-Schwarz inequality**:

$$\left|\int_\Omega uv \; d\Omega\right| \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \quad \forall u,v \in L^2(\Omega)$$

Other L^p– spaces

- The space of **summable functions** in Ω:

$$L^1(\Omega) = \{v : \Omega \to \mathbb{R} : \int_\Omega |v| d\Omega < +\infty\}.$$

- The space of **locally summable functions** in Ω:

$$L^1_{loc}(\Omega) = \{v : \Omega \to \mathbb{R} : \int_E |v| d\Omega < +\infty, \ \forall E \subset \Omega, \text{ with } E \text{ measurable and bounded}\}.$$

- The space of **essentially bounded functions**:

$$L^\infty(\Omega) = \{v : \Omega \to \mathbb{R} : \operatorname*{esssup}_{x\in\Omega} |v(x)| < +\infty\}$$

where $\operatorname*{esssup}_{x\in\Omega} |v(x)| = \inf\{c \geq 0 : \ |v(x)| \leq c \ a.e. \ in \ \Omega\}$

Define the norm

$$\|v\|_\infty = \operatorname*{esssup}_{x\in\Omega} |v(x)|$$

Weak derivatives

Goal: extend the concept of derivative to functions non-derivable in the classical meaning.

Let

$$C_0^\infty(\Omega) = \{v \in C^\infty(\Omega) : \ \exists K \subset \Omega \ compact : \ supp(v) \subset K\}$$

be the space of **infinite derivable functions with compact support in Ω**.

If $u \in C^1(\Omega)$, $\forall \varphi \in C_0^\infty(\Omega)$ it holds:

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} = - \int_\Omega \frac{\partial u}{\partial x_i} \varphi + \underbrace{[u\varphi]_{\partial \Omega}}_{0}$$

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Weak derivatives

We say that $w \in L^1_{loc}(\Omega)$ is a **weak partial derivative** of u along the $i-$th direction if

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega w\varphi \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Remark. For sake of simplicity, we keep using the same notation to indicate weak deirivatives.

Theorem. If w exists, then it is unique.

Examples

1. $\Omega = (0,2)$,

$$u(x) = \begin{cases} 1-x & x \in (0,1] \\ x-1 & x \in (1,2) \end{cases}$$

u admits the 1st order weak derivative and it is

$$w(x) = \begin{cases} -1 & x \in (0,1] \\ 1 & x \in (1,2) \end{cases}$$

2. $\Omega = (0,2)$,

$$u(x) = \begin{cases} 1 & x \in (0,1] \\ 2 & x \in (1,2) \end{cases}$$

u does not admits the 1st order weak derivative.
(Proof by contradiction, by using the Lebesgue dominated convergence theorem)

Weak derivatives (cont'd)

Define the **multi–index** $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ and the $\alpha-$th derivative

$$\mathcal{D}^{\boldsymbol{\alpha}} u = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_d} u}{\partial x^{\alpha_d}}.$$

Set $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_d$.

$\alpha-$th weak derivative of u:
If $\exists w \in L^1_{loc}(\Omega)$ s.t.

$$\int_\Omega (\mathcal{D}^{\boldsymbol{\alpha}} \varphi)\, u = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega w\, \varphi \qquad \forall \varphi \in C_0^\infty(\Omega),$$

we say that u admits the weak derivative $\mathcal{D}^{\boldsymbol{\alpha}} u$ and $\mathcal{D}^{\boldsymbol{\alpha}} u = w$.

Sobolev Spaces

Sobolev spaces are the right framework in which to solve PDEs.

Remark. From now on, all derivatives are meant in the weak sense.

We define the **Sobolev space of order 1**

$$H^1(\Omega) = \{v \in L^2(\Omega) : \ \nabla u \in [L^2(\Omega)]^d\}$$

- **inner product:** $(u, v)_{H^1(\Omega)} = \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v$
- **norm:**

$$\begin{aligned} \|u\|_{H^1(\Omega)} &= \left(\int_\Omega u^2 + \int_\Omega |\nabla u|^2\right)^{1/2} \\ &= (\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)})^{1/2} \end{aligned}$$

- $(H^1(\Omega), \|\cdot\|_{H^1(\Omega)})$ is a Hilbert space.

Examples of functions of $H^1(\Omega)$

- $\Omega = (0,2)$,
$$u(x) = \begin{cases} 1-x & x\in(0,1] \\ x-1 & x\in(1,2) \end{cases}$$
belongs to $H^1(0,2)$.

- $\Omega = B_1(0) \in \mathbb{R}^d$. $\quad u(x) = \frac{1}{|x|^\alpha}$ belongs to $H^1(B_1(0))$ iff $\alpha < \frac{d}{2} - 1$

- When $d = 1$, $\nexists$ unbounded functions in $H^1(\Omega)$.
$H^1(\Omega) \subset C^0(\overline{\Omega})$.

- When $d = 2, 3$, $\exists$ unbounded functions in $H^1(\Omega)$.
If $\Omega = B_1(0) \subset \mathbb{R}^2$: $u(x) = \left|\log\left(\frac{1}{|x|}\right)\right|^\beta$ with $0 < \beta < \frac{1}{2}$ is in $H^1(\Omega)$ (it is unbounded).
$H^1(\Omega) \not\subset C^0(\overline{\Omega})$

Weak form

Set

$$H_0^1(0,1) = \{v \in H^1(0,1) : \ v(0) = v(1) = 0\}.$$

The correct **weak form** of the Poisson equation with homogeneous Dirichlet conditions reads:

$$?u \in H_0^1(0,1) : \qquad \int_0^1 u'v' = \int_0^1 fv \qquad \forall v \in H_0^1(0,1) \qquad (4)$$

We need to understand under which assumptions on the data this problem admits a unique solution.

The trace operator and $H^1_0(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ an open and bounded set with Lipschitz boundary.
$\exists!$ linear and continuous map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ s.t.

$$\gamma_0 v = v|_{\partial\Omega} \qquad \forall v \in H^1(\Omega) \cap C^0(\overline{\Omega})$$

γ_0 is called the **trace operator**.

Definition. $H^{1/2}(\partial\Omega) = \{v \in L^2(\partial\Omega) : \; \exists u \in H^1(\Omega) : \gamma_0 u = v\}$ is the **space of traces** of functions of $H^1(\Omega)$.

Definition. $H^1_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $H^1(\Omega)$.

Theorem. $H^1_0(\Omega) = \{v \in H^1(\Omega) : \; \gamma_0 v = 0 \; on \; \partial\Omega\}$.

Remark. $H^1_0(\Omega)$ is the right candidate to replace $C^1_0(\overline{\Omega})$.
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Other Sobolev spaces

- $H^0(\Omega) = L^2(\Omega)$;
- Let $k \in \mathbb{N}$.
  $H^k(\Omega) = \{v \in L^2(\Omega) : \ \mathcal{D}^{\boldsymbol{\alpha}} u \in [L^2(\Omega)]^d, \ \textit{forall} \ |\boldsymbol{\alpha}| \leq k\}$ is the **Sobolev space of order** k.
- **norm** in $H^k(\Omega)$: $\|u\|_{H^k(\Omega)} = \left(\sum_{|\boldsymbol{\alpha}|\leq s} \|\mathcal{D}^{\boldsymbol{\alpha}} u\|^2_{L^2(\Omega)}\right)^{1/2}$
- $(H^k(\Omega), \|\cdot\|_{H^k(\Omega)})$ is a Hilbert space.

Poincarè inequality.

$$\exists C_\Omega > 0 : \ \|u\|_{L^2(\Omega)} \leq C_\Omega \|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in H^1_0(\Omega).$$

Abstract setting

- Let $(V, \|\cdot\|_V)$ a Hilbert space;
- $a : V \times V \to \mathbb{R}$ is a **bilinear form** if it is linear w.r.t. each argument, i.e.,

$$a(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1 a(u_1, v) + \alpha_2 a(u_2, v)$$
$$a(u, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 a(u, v_1) + \alpha_2 a(u, v_2);$$

- $a : V \times V \to \mathbb{R}$ is a **continuous form** if $\exists M > 0$ s.t.

$$|a(u, v)| \leq M \|u\|_V \|v\|_V \qquad \forall u, v \in V;$$

- $a : V \times V \to \mathbb{R}$ is a **coercive form** if $\exists \alpha > 0$ s.t.

$$a(v, v) \geq \alpha \|v\|_V^2 \qquad \forall v \in V;$$

Example: $a(u, v) = \int_\Omega \nabla u \cdot \nabla v + \alpha uv$ is a bilinear, continuous and coercive form on $V = H^1(\Omega)$.

Abstract setting (2)

- Let $(V, \|\cdot\|_V)$ a Hilbert space;
- $F : V \to \mathbb{R}$ is a **linear functional** if it is linear, i.e.,

$$F(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 F(v_1) + \alpha_2 F(v_2);$$

- $F : V \to \mathbb{R}$ is a **continuous functional** if $\exists C > 0$ s.t.

$$|F(v)| \leq C\|v\|_V \qquad \forall v \in V;$$

Example: Given $f \in L^2(\Omega)$, $F(v) = \int_\Omega fv$ is a linear, continuous functional on $V = L^2(\Omega)$.

Lax Milgram Lemma

Lemma. Let $(V, \|\cdot\|_V)$ a Hilbert space, $a : V \times V \to \mathbb{R}$ a bilinear, continuous and coercive form, $F : V \to \mathbb{R}$ a linear and continuous functional, then the weak problem

$$\textit{find } u \in V : \qquad a(u,v) = F(v) \qquad \forall v \in V$$

admits a unique solution.
Morever, if α is the coercivity constant of a, then

$$\|u\|_V \leq \frac{1}{\alpha} \sup_{v \in V} \frac{|F(v)|}{\|v\|_V}.$$