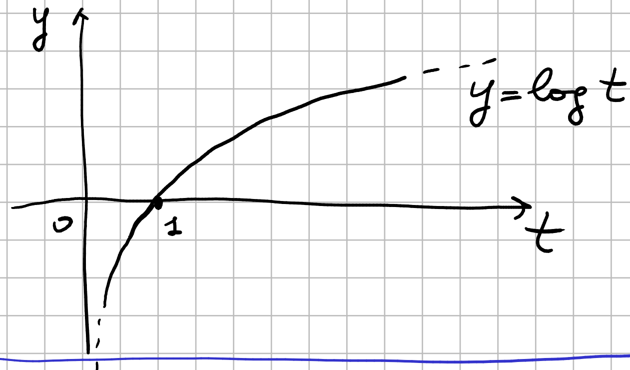
Analisi Matematica 1 – Esercitazione del 15 ottobre 2025

1. Calcolare il limite $\lim_{x\to 0^+} \frac{\log 3x}{\log 7x}$.
2. Calcolare il limite $\lim_{x\to +\infty} \frac{2x^{2/3} - 5x^{4/3} + \sqrt{x}}{3x + 6x^{2/3} - 2x\sqrt[3]{x}}$.
3. Calcolare i limiti $\lim_{x\to +\infty} \frac{\arctan x}{x}$ e $\lim_{x\to -\infty} \frac{\arctan x}{x}$.
4. Calcolare i limiti $\lim_{x\to 0^+} \frac{\cos x}{x}$, $\lim_{x\to 0^-} \frac{\cos x}{x}$ e $\lim_{x\to 0} \frac{\cos x}{x^2}$.
5. Calcolare i limiti $\lim_{x\to 0^+} \frac{\arccos x}{x}$ e $\lim_{x\to 0^-} \frac{\arccos x}{x}$.
6. Calcolare il limite $\lim_{x\to +\infty} \frac{\log x - 2\arctan x}{3\sqrt[4]{x} + 6x^{1/\sqrt{2}} - \log x}$.
7. Calcolare il limite $\lim_{x\to +\infty} \frac{e}{\sqrt{2x} - \sqrt{2x+3}}$.
8. Calcolare il limite $\lim_{x\to 0} \frac{\sqrt{1-3x} - \sqrt{1-x}}{8x}$.
9. Calcolare il limite $\lim_{x\to +\infty} \frac{\left(3 + \frac{3}{x}\right)^x}{e^x - \left(\frac{3}{2}\right)^x}$.
10. Calcolare il limite $\lim_{x\to +\infty} \frac{x^x}{(x+1)^{x+3}}$.
11. Calcolare il limite $\lim_{x\to 0} \frac{\tan x}{x}$.
12. Calcolare il limite $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.
13. Calcolare il limite $\lim_{x\to 0} \frac{x\tan x}{1-\cos x}$.
14. Calcolare il limite $\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$.
15. Calcolare il limite $\lim_{x\to 0} x \sin \frac{2}{x}$.
16. Calcolare il limite $\lim_{x\to -\infty} \frac{4x - 9x^{3/4}}{5x + 6\sin x}$.

$$\lim_{x\to 0^+} \frac{\log 3x}{\log 7x} =$$

$\log 3x \to -\infty$, $\log 7x \to -\infty$



$$= \lim_{x\to 0^+} \frac{\log 3 + \log x}{\log 7 + \log x} =$$

$$\boxed{\begin{array}{c}\log A + \log B = \log(AB)\\ \text{per ogni } A, B > 0\end{array}}$$

$$= \lim_{x\to 0^+} \frac{\cancel{\log x}\left(1 + \frac{\log 3}{\log x}\right)}{\cancel{\log x}\left(1 + \frac{\log 7}{\log x}\right)} =$$

$$= \lim_{x\to 0^+} \frac{1 + \frac{\log 3}{\log x}}{1 + \frac{\log 7}{\log x}} \overset{A.L.}{=} \frac{1+0}{1+0} = 1$$

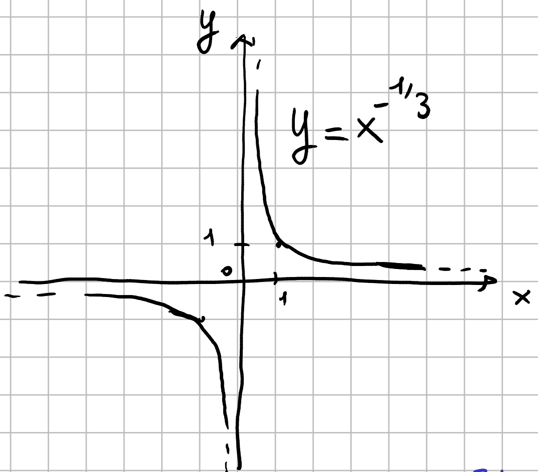
$\frac{\log 3}{\log x} \to 0$, $\frac{\log 7}{\log x} \to 0$

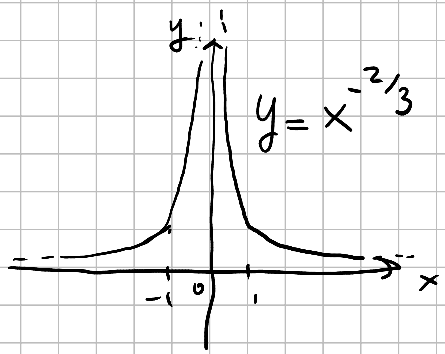
<u>Osservazione</u>. Abbiamo mostrato che $\log(k\cdot x) \sim \log x$ per $x \to 0^+$ se k è un numero positivo

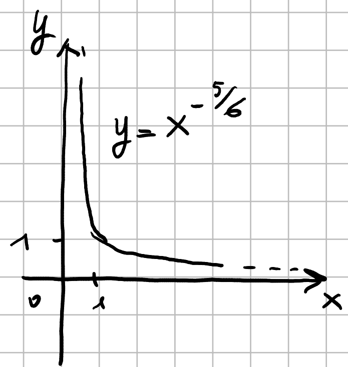
$$\lim_{x \to +\infty} \frac{2x^{2/3} - 5x^{4/3} + \sqrt{x}}{3x + 6x^{2/3} - 2x\sqrt[3]{x}} =$$

$$= \lim_{x \to +\infty} \frac{2x^{2/3} - 5x^{4/3} + x^{1/2}}{3x + 6x^{2/3} - 2x^{4/3}} =$$

$$= \lim_{x \to +\infty} \frac{\cancel{x^{4/3}}\left(2\overset{\to 0}{x^{-2/3}} - 5 + \overset{\to 0}{x^{-\frac{5}{6}}}\right)}{\cancel{x^{4/3}}\left(3\underset{\to 0}{x^{-1/3}} + 6\underset{\to 0}{x^{-2/3}} - 2\right)} \overset{AL}{=} \frac{-5}{-2} = \frac{5}{2}$$



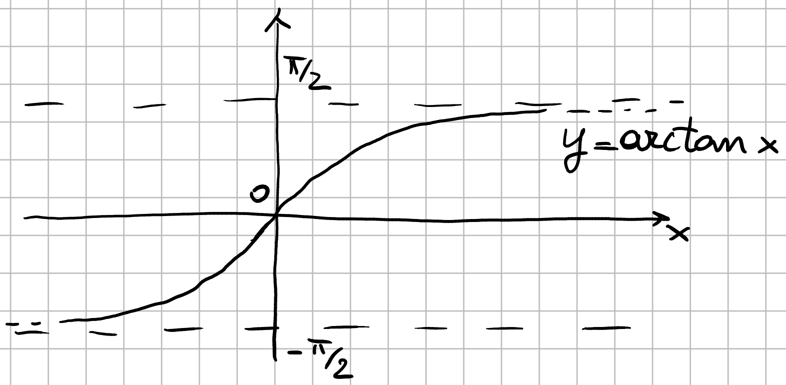




<u>Osservazione</u>. $2x^{2/3} - 5x^{4/3} + \sqrt{x} \sim -5x^{4/3}$ per $x \to +\infty$

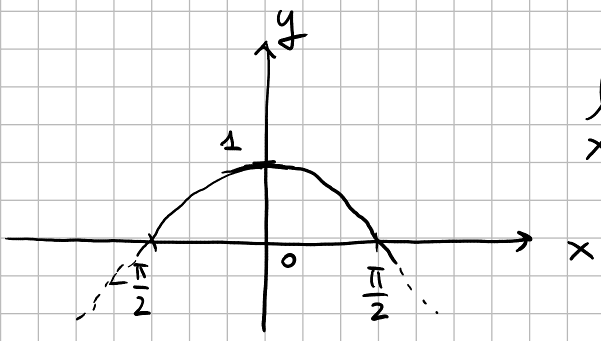
$3x + 6x^{2/3} - 2x\sqrt[3]{x} \sim -2x^{4/3}$ per $x \to +\infty$

$$\lim_{x\to+\infty} \frac{\overset{\to \frac{\pi}{2}}{\arctan x}}{\underset{\to +\infty}{x}} \overset{AL}{=} 0 \qquad \lim_{x\to-\infty} \frac{\overset{\to -\frac{\pi}{2}}{\arctan x}}{\underset{\to -\infty}{x}} \overset{AL}{=} 0$$



<u>Osservazione</u> $f(x) = \dfrac{\arctan x}{x}$ è una funzione pari quindi è sufficiente uno dei due limiti sopra per conoscere anche l'altro.

Calcolare $\displaystyle\lim_{x\to0^+} \frac{\cos x}{x}$, $\displaystyle\lim_{x\to0^-} \frac{\cos x}{x}$, $\displaystyle\lim_{x\to0} \frac{\cos x}{x^2}$



$$\lim_{x\to0} \cos x = 1$$

$$\lim_{x\to0^+} \frac{\overset{\to 1}{\cos x}}{\underset{\to 0^+}{x}} \overset{AL}{=} +\infty$$

$$\lim_{x\to0^-} \frac{\cos x}{x} = -\infty$$

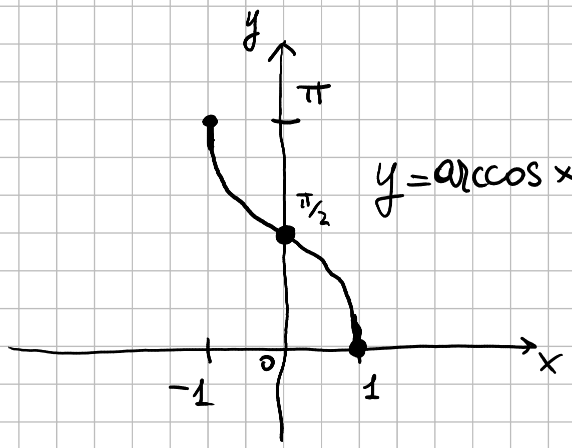
perché $y = \dfrac{\cos x}{x}$ è una funzione dispari

<u>Non</u> esiste $\displaystyle\lim_{x\to0} \frac{\cos x}{x}$

$$\lim_{x\to 0} \frac{\cos x}{x^2} = +\infty$$

Algebra dei limiti ($\cos x \to 1$, $x^2 \to 0^+$)

Calcolare $\lim_{x\to 0^+} \frac{\arccos x}{x}$ e $\lim_{x\to 0^-} \frac{\arccos x}{x}$



$$\lim_{x\to 0} \arccos x = \frac{\pi}{2}$$

$$\lim_{x\to 0^+} \frac{\arccos x}{x} = +\infty \quad \text{e} \quad \lim_{x\to 0^-} \frac{\arccos x}{x} = -\infty$$

($\arccos x \to \frac{\pi}{2}$, $x \to 0^+$; $\arccos x \to \frac{\pi}{2}$, $x \to 0^-$)

Calcolare $\lim_{x\to +\infty} \frac{\log x - 2\arctan x}{3\sqrt[4]{x} + 6x^{1/\sqrt{2}} - \log x} = \circledast$

• COMPORTAMENTO DEL NUMERATORE

$$\log x - 2\arctan x = \log x \left(1 - \underbrace{\frac{2\arctan x}{\log x}}_{\to 0}\right) \sim \log x$$

($\arctan x \to \frac{\pi}{2}$, $\log x \to +\infty$)

per $x \to +\infty$

• COMPORTAMENTO DEL DENOMINATORE

$$3x^{1/4} + 6x^{1/\sqrt{2}} - \log x = x^{1/\sqrt{2}}\left(\underbrace{3x^{\frac{1}{4}-\frac{1}{\sqrt{2}}}}_{\to 0} + 6 - \underbrace{\frac{\log x}{x^{1/\sqrt{2}}}}_{\to 0}\right) \sim 6x^{1/\sqrt{2}}$$

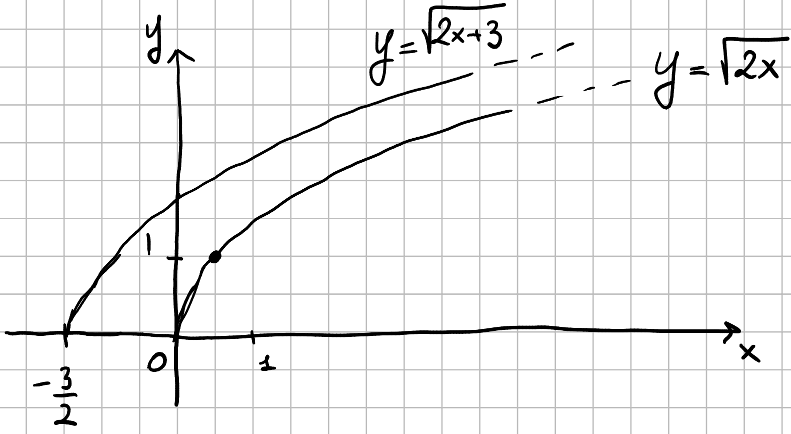
per $x \to +\infty$

$$\boxed{4 > \sqrt{2} \qquad \frac{1}{4} < \frac{1}{\sqrt{2}}, \text{ quindi } \frac{1}{4} - \frac{1}{\sqrt{2}} < 0}$$

$$\boxed{\lim_{x\to +\infty} \frac{\log x}{x^\alpha} = 0 \quad \forall \alpha > 0}$$

$$\circledast = \lim_{x\to+\infty} \frac{\log x}{6x^{1/\sqrt{2}}} = 0$$

Calcolare $\displaystyle\lim_{x\to+\infty} \frac{e}{\underbrace{\sqrt{2x}}_{\downarrow +\infty} - \underbrace{\sqrt{2x+3}}_{\downarrow +\infty}}$



$y=\sqrt{2x}$

$y=\sqrt{2\left(x+\frac{3}{2}\right)}$

Traslazione di vettore $\left(0,-\frac{3}{2}\right)$

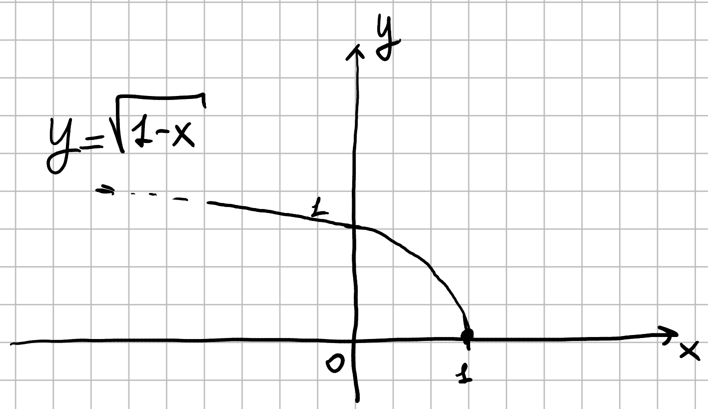
$$(A-B)(A+B) = A^2-B^2 \quad \forall A,B\in\mathbb{R}$$

$$\lim_{x\to+\infty}\frac{e}{\sqrt{2x}-\sqrt{2x+3}} = \lim_{x\to+\infty}\frac{e}{\underbrace{\sqrt{2x}}_{A}-\underbrace{\sqrt{2x+3}}_{B}}\cdot\frac{\sqrt{2x}+\sqrt{2x+3}}{\underbrace{\sqrt{2x}}_{A}+\underbrace{\sqrt{2x+3}}_{B}} =$$

$$= \lim_{x\to+\infty}\frac{e\left(\overset{\nearrow +\infty}{\sqrt{2x}}+\overset{\nearrow +\infty}{\sqrt{2x+3}}\right)}{\cancel{2x}-\cancel{2x}-3} \overset{AL}{=} \frac{e\cdot(+\infty)}{-3} = -\infty$$

Calcolare $\lim_{x\to 0} \frac{\sqrt{1-3x}-\sqrt{1-x}}{8x}$

$\sqrt{1-3x} \to 1$, $\sqrt{1-x} \to 1$



$$\lim_{x\to 0} \frac{\sqrt{1-3x}-\sqrt{1-x}}{8x} \cdot \frac{\sqrt{1-3x}+\sqrt{1-x}}{\sqrt{1-3x}+\sqrt{1-x}} =$$

$$= \lim_{x\to 0} \frac{\cancel{1}-3x-\cancel{1}+x}{8x(\sqrt{1-3x}+\sqrt{1-x})} = \lim_{x\to 0} \frac{-2\cancel{x}}{8\cancel{x}(\underbrace{\sqrt{1-3x}}_{\downarrow\, 1}+\underbrace{\sqrt{1-x}}_{\downarrow\, 1})} =$$

$$\overset{AL}{=} \frac{-2}{8\cdot(1+1)} = -\frac{1}{8}$$

$$\boxed{\lim_{x\to +\infty} \left(1+\frac{1}{x}\right)^x = e}$$ LIMITE NOTEVOLE (VEDI LEZIONE SCORSA)

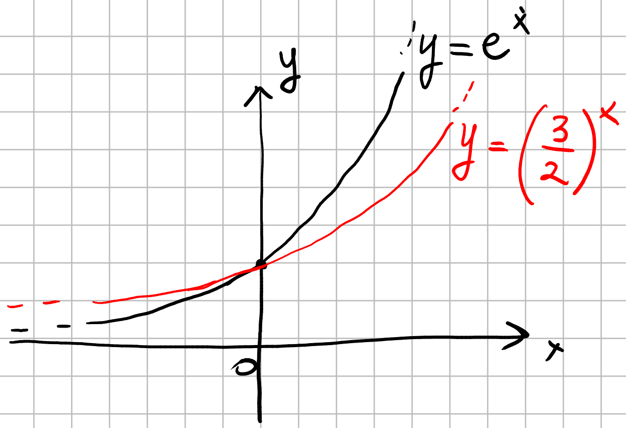
Calcolare $\lim_{x\to +\infty} \frac{\left(3+\frac{3}{x}\right)^x}{e^x-\left(\frac{3}{2}\right)^x}$

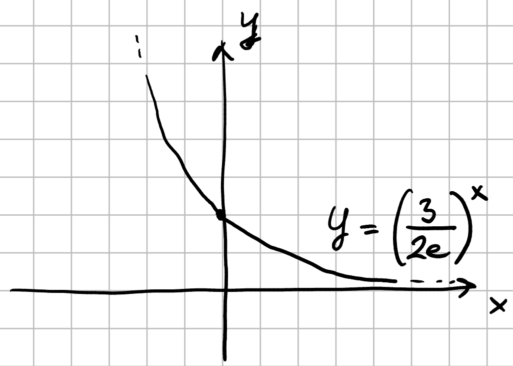
- COMPORTAMENTO DEL NUMERATORE

$$\left(3+\frac{3}{x}\right)^x = 3^x\left(1+\frac{1}{x}\right)^x \sim e\cdot 3^x \quad \text{per } x\to +\infty$$

• COMPORTAMENTO DEL DENOMINATORE

$$e^x - \left(\frac{3}{2}\right)^x = e^x\left(1-\underbrace{\left(\frac{3}{2e}\right)^x}_{\downarrow\, 0}\right) \sim e^x \quad \text{per } x \to +\infty$$





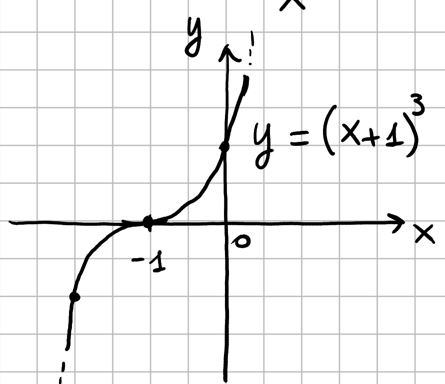
$$\lim_{x\to+\infty} \frac{\left(3+\frac{3}{x}\right)^x}{e^x - \left(\frac{3}{2}\right)^x} = \lim_{x\to+\infty} \frac{e\cdot 3^x}{e^x} = \lim_{x\to+\infty} e\cdot\underbrace{\left(\frac{3}{e}\right)^x}_{\downarrow\, +\infty} \overset{AL}{=} +\infty$$

Calcolare $\displaystyle\lim_{x\to+\infty} \frac{x^x}{(x+1)^{x+3}}$

$$\boxed{\begin{array}{l} \displaystyle\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e \\ \text{osservazione: } \\ \displaystyle 1+\frac{1}{x} = \frac{x+1}{x} \quad \forall x\neq 0 \end{array}}$$

$$\lim_{x\to+\infty} \frac{x^x}{(x+1)^{x+3}} = \lim_{x\to+\infty} \frac{x^x}{(x+1)^x\cdot(x+1)^3} =$$

$$= \lim_{x\to+\infty} \frac{1}{\frac{(x+1)^x}{x^x}} \cdot \frac{1}{(x+1)^3} = \lim_{x\to+\infty} \frac{1}{\underbrace{\left(1+\frac{1}{x}\right)^x}_{\searrow\, e}} \cdot \frac{1}{\underbrace{(x+1)^3}_{\downarrow\, +\infty}} \overset{AL}{=} 0$$



$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

LIMITE NOTEVOLE

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{1} \cdot \underbrace{\frac{1}{\cos x}}_{1} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-\cos x}{x^2} \cdot \frac{1+\cos x}{1+\cos x} =$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2} \cdot \frac{1}{1+\cos x} =$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1+\cos x} =$$

$$= \lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{1} \cdot \underbrace{\frac{\sin x}{x}}_{1} \cdot \frac{1}{1+\underbrace{\cos x}_{1}} \overset{AL}{=} \frac{1}{2}$$

Definizione (INFINITESIMI DELLO STESSO ORDINE)

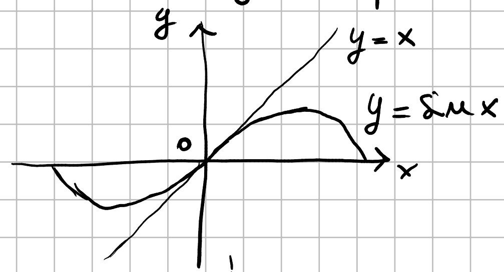
Siano $x_0 \in \overline{\mathbb{R}}$ e f e g funzione definite in un intorno I di x_0, escluso eventualmente x_0.

Se f e g sono funzioni infinitesime per $x \to x_0$ (cioè $\lim_{x\to x_0} f(x) = 0$ e $\lim_{x \to x_0} g(x) = 0$) e $g(x) \neq 0 \ \forall x \in I \setminus \{x_0\}$, allora si dice che f e g SONO INFINITESIME CON LO STESSO ORDINE per $x \to x_0$ se e solo se

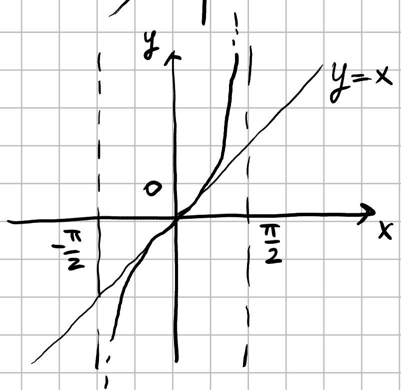
$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \ell \in \mathbb{R} \setminus \{0\}.$$

In questo caso, si scrive $f(x) \sim \ell \cdot g(x)$ per $x \to x_0$

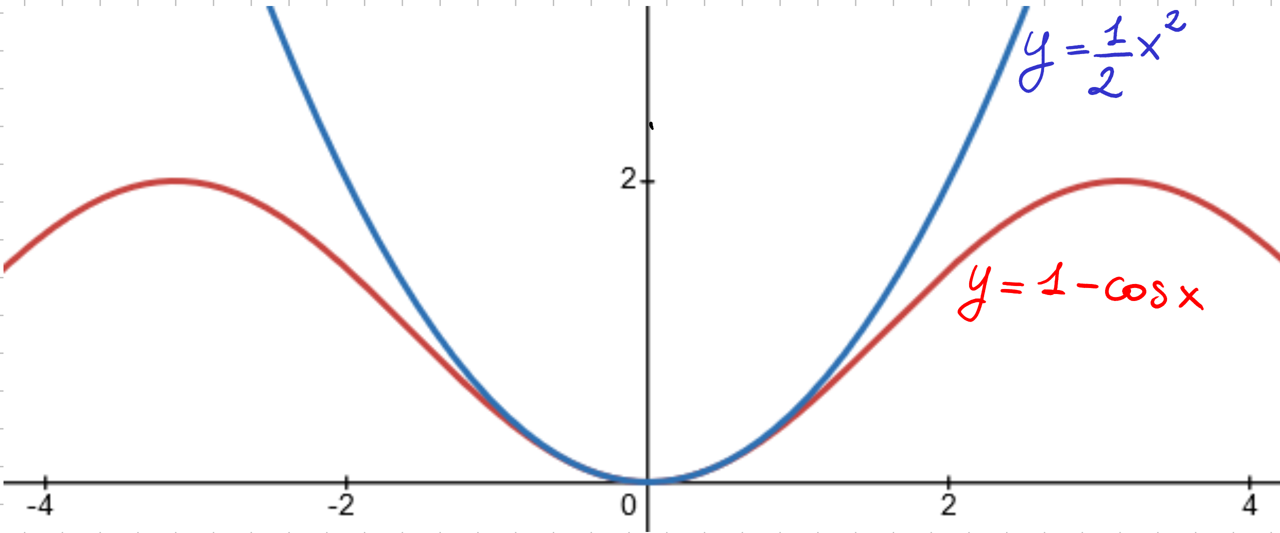
$\sin x \sim x$ per $x \to 0$



$\tan x \sim x$ per $x \to 0$



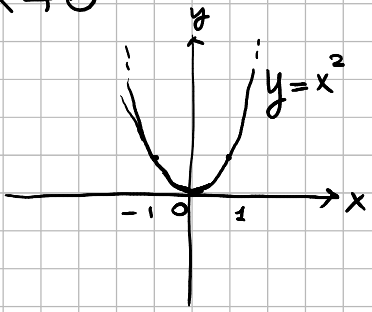
$$1-\cos x \sim \frac{1}{2}x^2 \quad \text{per } x \to 0$$

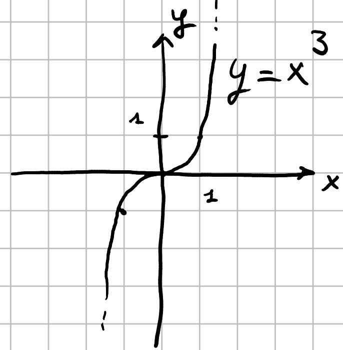


Esempi

- $f(x) = x^2 + x^3$ e $g(x) = 5x^3$, $\quad x_0 = 0$

$$\lim_{x\to 0} f(x) = 0 \quad \text{e} \quad \lim_{x\to 0} g(x) = 0$$





$$\lim_{x\to 0} \frac{x^2+x^3}{5x^3} = \lim_{x\to 0} \frac{x^2(1+x)}{5x^3} = \lim_{x\to 0} \frac{\overset{\to 1}{1+x}}{\underset{\to 0}{5x}}$$

Il limite non esiste

perché

$$\lim_{x\to 0^+} \frac{1+x}{5x} = +\infty \qquad \lim_{x\to 0^-} \frac{1+x}{5x} = -\infty$$

$f(x)$ e $g(x)$ NON hanno lo stesso ordine di infinitesimo per $x \to 0$

- $f(x) = x^4 + x^3$ e $g(x) = 5x^3$, $x_0 = 0$

$\lim_{x\to 0} f(x) = 0$ e $\lim_{x\to 0} g(x) = 0$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^3(x+1)}{5x^3} = \lim_{x\to 0} \frac{x+1}{5} = \frac{1}{5}$$

f e g sono infinitesimi con lo stesso ordine per $x\to 0$

$f(x) \sim \frac{1}{5} g(x)$ per $x \to 0$

Proposizione. Siano $x_0 \in \overline{\mathbb{R}}$, f_1, f_2, g_1, g_2 funzioni definite in un intorno I di x_0, escluso eventualmente x_0, con $g_1(x) \neq 0$ e $g_2(x) \neq 0$ per ogni $x \in I \setminus \{x_0\}$.
Se f_1, f_2, g_1, g_2 sono funzioni infinitesime per $x \to x_0$ e $f_1 \sim f_2$ per $x \to x_0$, $g_1 \sim g_2$ per $x \to x_0$, allora

$$f_1 g_1 \sim f_2 g_2 \text{ per } x \to x_0 \quad \text{e} \quad \frac{f_1}{g_1} \sim \frac{f_2}{g_2} \text{ per } x \to x_0.$$

Calcolare $\lim_{x\to 0} \frac{x \tan x}{1-\cos x}$

$\tan x \sim x$ per $x \to 0$ $\qquad$ $1-\cos x \sim \frac{1}{2}x^2$ per $x \to 0$

$$\lim_{x\to 0} \frac{x \tan x}{1-\cos x} = \lim_{x\to 0} \frac{\cancel{x}\cdot\cancel{x}}{\frac{1}{2}\cancel{x^2}} = 2$$

Calcolare $\lim\limits_{x \to 0} \dfrac{\sin x - \tan x}{x^3}$

$$\lim_{x \to 0} \frac{\sin x - \tan x}{x^3} = \lim_{x \to 0} \frac{\sin x - \dfrac{\sin x}{\cos x}}{x^3} =$$

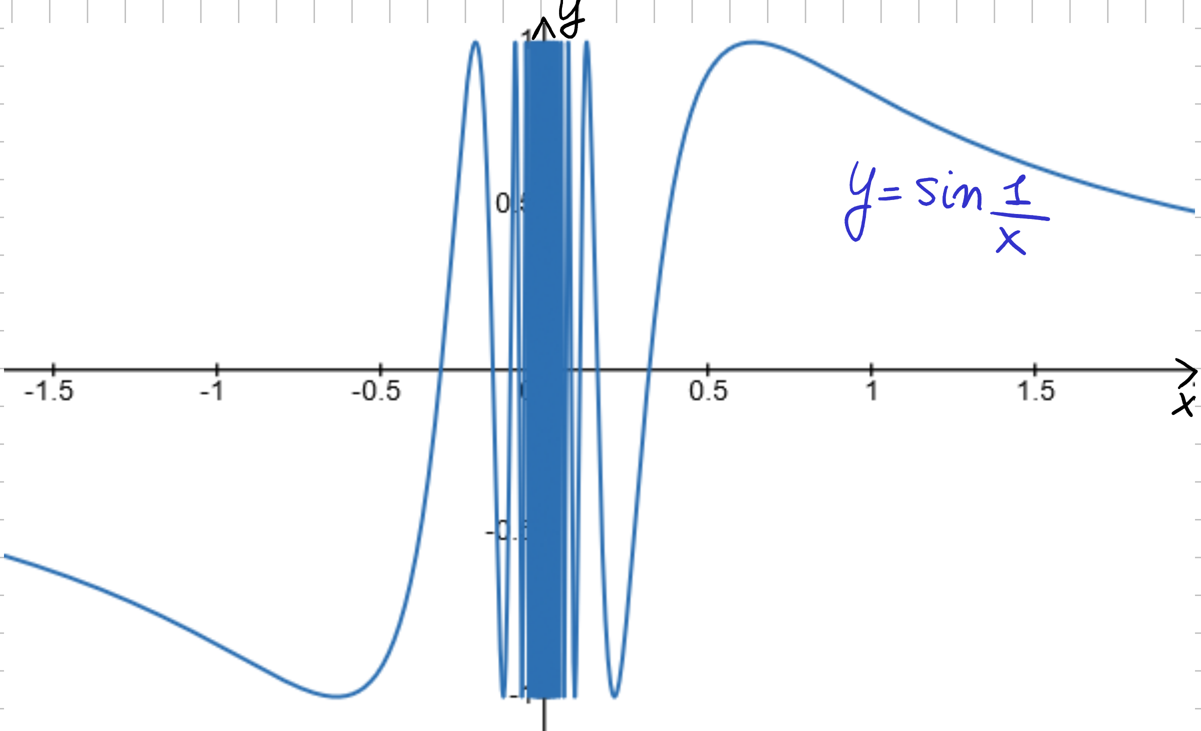
$$= \lim_{x \to 0} \frac{\sin x}{x^3}\left(1 - \frac{1}{\cos x}\right) = \lim_{x \to 0} \frac{\sin x}{x^3} \cdot \frac{\cos x - 1}{\underbrace{\cos x}_{\to 1}} = \circledast$$

$\sin x \sim x$ per $x \to 0$

$(\cos x - 1) \sim -\dfrac{1}{2}x^2$ per $x \to 0$

$$\circledast = \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x^3}} \cdot \left(-\frac{1}{2}\cancel{x^2}\right) \cdot \frac{1}{\underbrace{\cos x}_{\to 1}} = -\frac{1}{2}$$

Calcolare $\lim\limits_{x \to 0} x \cdot \sin \dfrac{2}{x}$



$$\lim_{x\to 0} \sin\frac{2}{x} \quad \text{NON esiste}$$

- Osserviamo però che $y=\sin\frac{2}{x}$ è una funzione limitata

$$-1 \leq \sin\frac{2}{x} \leq 1 \quad \forall x \in \mathbb{R}\setminus\{0\}$$

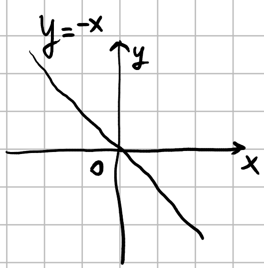
- Osserviamo anche che $f(x) = x\cdot\sin\frac{2}{x}$ è una funzione pari, quindi è sufficiente calcolare

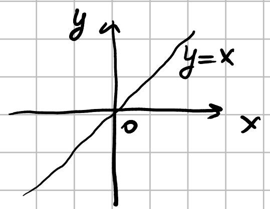
$$\lim_{x\to 0^+} x\cdot\sin\frac{2}{x}$$

- Applichiamo il II teorema del confronto

$$-1 \leq \sin\frac{2}{x} \leq 1 \quad \forall x \in (0,+\infty)$$

$$\underset{\downarrow\, 0}{-x} \leq x\cdot\sin\frac{2}{x} \leq \underset{\downarrow\, 0}{x} \quad \forall x \in (0,+\infty)$$

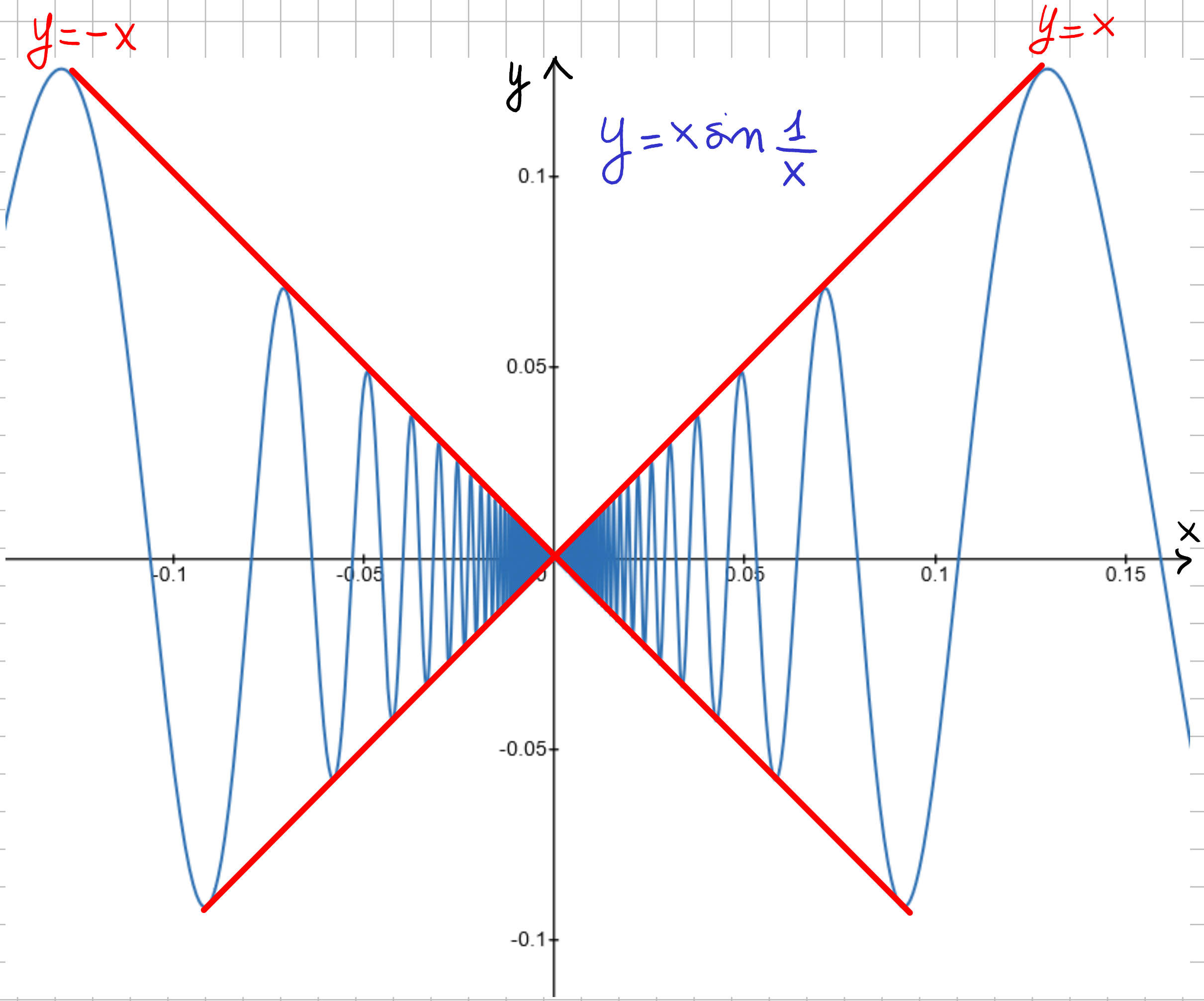




Dato che $\lim_{x\to 0^+}(-x) = 0$ e $\lim_{x\to 0^+} x = 0$, per il II teorema del confronto, si ha $\lim_{x\to 0^+} x\cdot\sin\frac{2}{x} = 0$

Inoltre, $\lim_{x\to 0^-} x\sin\frac{2}{x} = 0$ perché la funzione è pari.

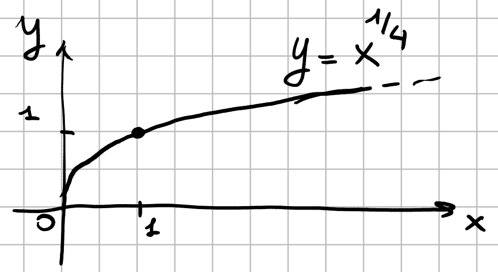
Quindi $\lim_{x\to 0} x\sin\frac{2}{x} = 0$



$y=-x$
$y=x$
y
x
$y = x\sin\frac{1}{x}$
0.1
0.05
-0.05
-0.1
-0.1
-0.05
0
0.05
0.1
0.15

Calcolare $\lim\limits_{x\to+\infty} \dfrac{4x-9x^{3/4}}{5x+6\sin x}$

- $4x-9x^{3/4} = x\left(4-\dfrac{9}{\underbrace{x^{1/4}}_{\to+\infty}}\right) \sim 4x$ per $x\to+\infty$



$5x+6\sin x = x\left(5+6\dfrac{\sin x}{x}\right)$

$\lim\limits_{x\to+\infty} \sin x$ NON ESISTE

Calcoliamo $\lim\limits_{x\to+\infty} \dfrac{\sin x}{x}$

$-1 \leq \sin x \leq 1 \quad \forall x \in (0,+\infty)$

$\underbrace{-\dfrac{1}{x}}_{\downarrow\, 0} \leq \dfrac{\sin x}{x} \leq \underbrace{\dfrac{1}{x}}_{\downarrow\, 0} \quad \forall x \in (0,+\infty)$

per $x\to+\infty$

Per il II teorema del confronto, si ha $\lim\limits_{x\to+\infty} \dfrac{\sin x}{x} = 0$.

$5x+6\sin x = x\left(5+6\underbrace{\dfrac{\sin x}{x}}_{\downarrow\, 0}\right) \sim 5x$ per $x\to+\infty$

$$\lim_{x \to +\infty} \frac{4x - 9x^{3/4}}{5x + 6\sin x} = \lim_{x \to +\infty} \frac{4x}{5x} = \frac{4}{5}$$