Analisi Matematica 1 – Esercitazione del 24 ottobre 2024

1. Calcolare i limiti
$$\lim_{x\to 0}\frac{\sin(\cos x)}{\cos x}, \qquad \lim_{x\to 0}\frac{\log(\cos x)}{x^2}, \qquad \lim_{x\to 0}\frac{\arctan x}{x}, \qquad \lim_{x\to 0}\frac{\arcsin x}{x}.$$

2. (Prova in itinere 13 novembre 2009) Calcolare il limite
$$\lim_{n\to+\infty}\sqrt[n]{\frac{n^3+1}{n^2-1}}\left(1+\frac{1}{2n}\right)^{3n}.$$

3. Calcolare il limite $\lim\limits_{x\to-\infty}\left(\sqrt{x^2-5x+6}+x\right)$.

4. Calcolare il limite $\lim\limits_{x\to+\infty}(\sqrt[3]{2x-3}-\sqrt[3]{2x})$.

5. Fissati $\alpha>0$ e $\beta>0$, calcolare il limite $\lim\limits_{x\to+\infty}\dfrac{e^{\alpha x}}{x^\beta}$.

6. (Prova d'esame 30 agosto 2019) Siano $\alpha>0$ e $f\colon\mathbb{R}\to\mathbb{R}$ la funzione definita da
$$f(x)=\begin{cases}\arctan\left(x^2\left(\dfrac{\alpha}{3}\right)^{1/x}\right) & \text{se } x\neq 0\\ 0 & \text{se } x=0.\end{cases}$$
Discutere, al variare di $\alpha>0$, la continuità di f in $x=0$, classificando l'eventuale discontinuità.

7. (Prova d'esame 16 gennaio 2014) Calcolare il limite
$$\lim_{n\to+\infty}\frac{\log((n+2)!)-\log(n!)}{n\left(\sqrt[n]{n^3}-1\right)}.$$

8. (Prova d'esame 6 febbraio 2014) Siano $\alpha\in\mathbb{R}$ e $f\colon\mathbb{R}\to\mathbb{R}$ la funzione definita da
$$f(x)=\begin{cases}(2+x^2)^\alpha & \text{se } x\leqslant 0\\ \dfrac{\log(1+\sin^2(2x))}{\arctan x^\alpha} & \text{se } x>0.\end{cases}$$
Discutere la continuità di f in $x=0$ al variare di α e classificare le eventuali discontinuità.

9. (Prova d'esame 7 settembre 2016) Calcolare il limite
$$\lim_{n\to+\infty}\left(\frac{1}{2}n^{1/n}+\frac{\sin(n!)}{n}\right)\frac{\sqrt{1+8n^2}-n}{\log(1+e^{n+2})-\frac{n}{2}}.$$

10. (Prova d'esame 29 giugno 2009) Calcolare al variare di $\alpha\in\mathbb{R}$ il limite
$$\lim_{n\to+\infty}\frac{[1-\cos\frac{1}{n}]^2\log\left[\left(e^3+\frac{1}{n}\right)^n\right]}{\left(\sqrt{n+4}\right)^\alpha}.$$

Calcolare i limiti

$$\lim_{x\to 0}\frac{\sin(\cos x)}{\cos x},\qquad \lim_{x\to 0}\frac{\log(\cos x)}{x^2},\qquad \lim_{x\to 0}\frac{\arctan x}{x},\qquad \lim_{x\to 0}\frac{\arcsin x}{x}.$$

continuità delle funzioni seno e coseno

$$\lim_{x\to 0}\frac{\sin(\cos x)}{\cos x} = \frac{\sin 1}{1} = \sin 1$$

$$\lim_{x\to 0}\cos x = \cos 0 = 1$$

continuità della funzione coseno

$$\lim_{x\to 0}\frac{\log(\cos x)}{x^2}$$

$\log(\cos 0) = \log 1 = 0$

$\to 0$

$$\lim_{t\to 0}\frac{\log(1+t)}{t} = 1,$$ equivalentemente $\log(1+t)\sim t$ per $\to 0$

$$\log(\cos x) = \log(1 + \underbrace{(\cos x - 1)}) \sim (\cos x - 1)$$

$\downarrow$ per $x\to 0$

0

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2},$$ cioè $(1-\cos x)\sim +\frac{1}{2}x^2$ per $x\to 0$

$$\log(\cos x)\sim(\cos x - 1)\sim -\frac{1}{2}x^2 \text{ per } x\to 0$$

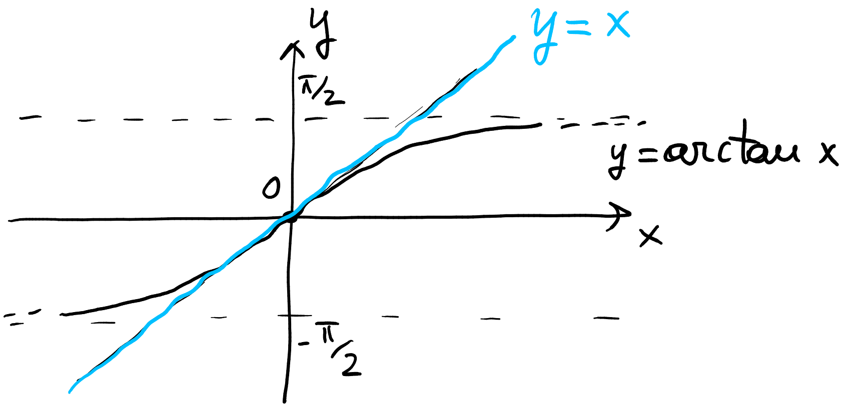
$$\lim_{x\to 0} \frac{\log(\cos x)}{x^2} = \lim_{x\to 0} \frac{-\frac{1}{2}\cancel{x^2}}{\cancel{x^2}} = -\frac{1}{2}$$

$$\lim_{x\to 0} \frac{\arctan x}{x} \underset{\substack{\uparrow \\ x=\tan y \\ y=\arctan x}}{=}$$

$$\boxed{\lim_{y\to 0} \frac{\tan y}{y} = 1}$$

$$= \lim_{y\to 0} \frac{y}{\tan y} = \lim_{y\to 0} \left(\frac{\tan y}{y}\right)^{-1} = 1$$

Abbiamo ricavato che $\arctan x \sim x$ per $x \to 0$

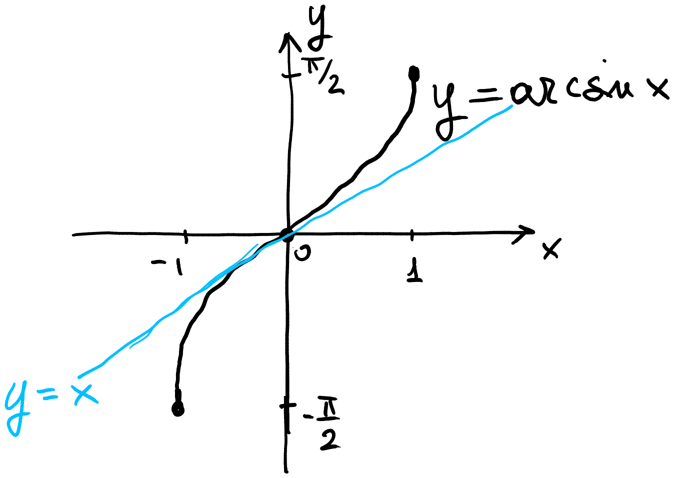


$$\lim_{x\to 0} \frac{\arcsin x}{x} \underset{\substack{\uparrow \\ x=\sin y \\ y=\arcsin x}}{=}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$= \lim_{y\to 0} \frac{y}{\sin y} = \lim_{y\to 0} \left(\frac{\sin y}{y}\right)^{-1} = 1$$

Quindi $\arcsin x \sim x$ per $x \to 0$



Calcolare il limite

$$\lim_{n\to+\infty} \sqrt[n]{\frac{n^3+1}{n^2-1}} \cdot \left(1+\frac{1}{2n}\right)^{3n}.$$

$$\sqrt[n]{\frac{n^3+1}{n^2-1}} \qquad n^3+1 \to +\infty, \quad n^2-1 \to +\infty$$

$(n^3+1) \sim n^3$ per $n\to+\infty$

$(n^2-1) \sim n^2$ per $n\to+\infty$

$$\frac{n^3+1}{n^2-1} \sim \frac{n^3}{n^2} = n \quad \text{per } n\to+\infty$$

$$\sqrt[n]{\frac{n^3+1}{n^2-1}} = \left(\frac{n^3+1}{n^2-1}\right)^{1/n} = e^{\log\left(\frac{n^3+1}{n^2-1}\right)^{1/n}} = e^{\frac{1}{n}\log\left(\frac{n^3+1}{n^2-1}\right)} \sim$$

$$\underbrace{\left(\frac{n^3+1}{n^2-1}\right)}_{\sim n}$$

$$e^{\underbrace{\frac{1}{n}\log n}_{\to 0}} \longrightarrow e^0 = 1$$

per continuità di $y=e^x$

$$\boxed{\lim_{n\to+\infty} \frac{\log n}{n} = 0}$$

$$\left(1+\frac{1}{2n}\right)^{3n} = \Bigg[\underbrace{\left(1+\frac{1}{2n}\right)^{2n}}_{\to e}\Bigg]^{3/2} \longrightarrow e^{3/2}$$

$$\boxed{\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e}$$

$$\lim_{n\to+\infty} \sqrt[n]{\frac{n^3+1}{n^2-1}} \cdot \left(1+\frac{1}{2n}\right)^{3n} = 1 \cdot e^{3/2} = e\sqrt{e}$$

Calcolare $\lim_{x\to-\infty} \left(\underbrace{\sqrt{x^2-5x+6}}_{\downarrow +\infty} + \underbrace{x}_{\downarrow -\infty}\right)$

$$\lim_{x\to-\infty} \left(\sqrt{x^2-5x+6}+x\right) = \lim_{x\to-\infty} \left(\sqrt{x^2-5x+6}+x\right) \cdot \frac{\sqrt{x^2-5x+6}-x}{\sqrt{x^2-5x+6}-x} =$$

$$= \lim_{x\to-\infty} \frac{\cancel{x^2}-5x+6-\cancel{x^2}}{\sqrt{x^2-5x+6}-x} = \lim_{x\to-\infty} \frac{\overbrace{-5x+6}^{+\infty}}{\underbrace{\sqrt{x^2-5x+6}-x}_{+\infty}} =$$

$$= \lim_{x\to-\infty} \frac{-5x+6}{\sqrt{x^2\left(1-\frac{5}{x}+\frac{6}{x^2}\right)}-x} = \lim_{x\to-\infty} \frac{\cancel{x}\left(-5+\frac{6}{x}\right)}{-\cancel{x}\left(\sqrt{1-\frac{5}{x}+\frac{6}{x^2}}+1\right)} =$$

$$\boxed{\sqrt{x^2} = |x| \ \forall x \in \mathbb{R}}$$

$$= \lim_{x\to-\infty} \frac{-5+\overbrace{\frac{6}{x}}^{\to 0}}{-\left(\sqrt{1\underbrace{-\frac{5}{x}}_{\downarrow 0}+\underbrace{\frac{6}{x^2}}_{\downarrow 0}}+1\right)} \overset{A.L.}{=} \frac{-5}{-2} = \frac{5}{2}$$

$$\boxed{\begin{array}{l} \text{Se } x \geq 0, \text{ allora } \sqrt{x^2} = x \\ \text{Se } x < 0, \text{ allora } \sqrt{x^2} = -x \end{array}}$$

Calcolare $\lim_{x\to-\infty} \left(\underbrace{\sqrt[3]{2x-3}}_{\downarrow\ -\infty} - \underbrace{\sqrt[3]{2x}}_{\downarrow\ -\infty}\right)$

$$\boxed{\begin{array}{l} A^3 - B^3 = (A-B)(A^2+AB+B^2) \quad \forall A,B \in \mathbb{R} \\ A^3 + B^3 = (A+B)(A^2-AB+B^2) \quad \forall A,B \in \mathbb{R} \end{array}}$$

$$\lim_{x\to-\infty} \left(\sqrt[3]{2x-3} - \sqrt[3]{2x}\right) = \lim_{x\to-\infty} \left(\overset{A}{\overset{\parallel}{\sqrt[3]{2x-3}}} - \overset{B}{\overset{\parallel}{\sqrt[3]{2x}}}\right) \cdot \frac{\sqrt[3]{(2x-3)^2} + \sqrt[3]{2x(2x-3)} + \sqrt[3]{(2x)^2}}{\sqrt[3]{(2x-3)^2} + \sqrt[3]{2x(2x-3)} + \sqrt[3]{(2x)^2}}$$

$$= \lim_{x\to-\infty} \frac{\cancel{2x} - 3 - \cancel{2x}}{\underbrace{\sqrt[3]{(2x-3)^2}}_{\downarrow\ +\infty} + \underbrace{\sqrt[3]{2x(2x-3)}}_{\downarrow\ +\infty} + \underbrace{\sqrt[3]{(2x)^2}}_{\downarrow\ +\infty}} = 0$$

$$\boxed{\lim_{x\to-\infty} \sqrt[3]{(2x-3)^2} = \lim_{x\to-\infty} \sqrt[3]{4x^2} = +\infty \qquad (2x-3)^2 \sim (2x)^2 = 4x^2 \text{ per } x\to-\infty}$$

$$\boxed{\lim_{x\to+\infty} \frac{(\log x)^{\alpha}}{x^{\beta}} = 0 \quad \text{per ogni } \alpha>0 \text{ e } \beta>0}$$

Calcolare $\lim_{x\to+\infty} \frac{e^{\alpha x}}{x^{\beta}}$, con $\alpha>0$ e $\beta>0$ fissati

$$\lim_{x\to+\infty} \frac{e^{\alpha x}}{x^{\beta}} \underset{e^x = y}{\overset{\uparrow}{=}} \lim_{y\to+\infty} \frac{y^{\alpha}}{(\log y)^{\beta}} = +\infty$$

se $x\to+\infty$, allore $y\to+\infty$

$x = \log y$

(30/08/2019)

Sia $\alpha > 0$ e sia $f : \mathbb{R} \to \mathbb{R}$ funzione def. da

$$f(x) = \begin{cases} 0 & \text{se } x = 0 \\ \arctan\left(x^2 \left(\frac{\alpha}{3}\right)^{1/x}\right) & \text{se } x \neq 0 \end{cases}$$

Studiare la continuità di f in $x = 0$.

$$\left(\frac{\alpha}{3}\right)^{1/x} = \begin{cases} 1 & \text{se } \alpha = 3 \\ \left(\frac{\alpha}{3}\right)^{1/x} & \text{se } \alpha > 3 \quad \text{(funzione esponenziale con base > 1)} \\ \left(\frac{\alpha}{3}\right)^{1/x} & \text{se } 0 < \alpha < 3 \quad \text{(funzione esponenziale con base in } (0,1)\text{)} \end{cases}$$

- Caso $\alpha = 3$

$$f(x) = \begin{cases} 0 & \text{se } x = 0 \\ \arctan(x^2) & \text{se } x \neq 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \arctan(x^2) = \arctan 0 = 0$$

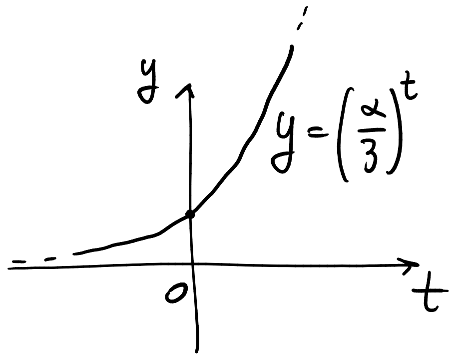
$$f(0) = 0$$

Quindi f è continua in $x = 0$ se $\alpha = 3$

- Caso $\alpha > 3$

$$\lim_{x\to 0^+} \left(\frac{\alpha}{3}\right)^{\overset{\to +\infty}{1/x}} \underset{t=\frac{1}{x}}{=} \lim_{t\to+\infty} \left(\frac{\alpha}{3}\right)^{t} = +\infty$$

se $x\to 0^+$, allora $t\to+\infty$



$$\lim_{x\to 0^-} \left(\frac{\alpha}{3}\right)^{\overset{\to -\infty}{1/x}} \underset{t=\frac{1}{x}}{=} \lim_{t\to-\infty} \left(\frac{\alpha}{3}\right)^{t} = 0$$

se $x\to 0^-$, allora $t\to-\infty$

per continuità di $y=\arctan x$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \arctan\left(\overset{\to 0}{x^2}\underbrace{\left(\frac{\alpha}{3}\right)^{1/x}}_{\to 0}\right) = \arctan 0 = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \arctan\left(\overset{\to 0}{x^2}\underbrace{\left(\frac{\alpha}{3}\right)^{1/x}}_{\to +\infty}\right)$$

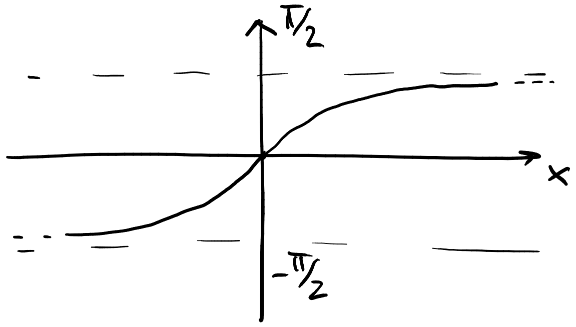
$$\lim_{x\to 0^+} x^2\left(\frac{\alpha}{3}\right)^{1/x} = \lim_{x\to 0^+} x^2 e^{\frac{1}{x}\log\left(\frac{\alpha}{3}\right)} \underset{y=\frac{1}{x}}{=}$$

se $x\to 0^+$, allora $y\to+\infty$

$x = \frac{1}{y}$

$$= \lim_{y\to+\infty} \left(\frac{1}{y}\right)^2 e^{y\log\left(\frac{\alpha}{3}\right)} = \lim_{y\to+\infty} \frac{e^{y\overset{>0}{\log\left(\frac{\alpha}{3}\right)}}}{y^2} = +\infty$$

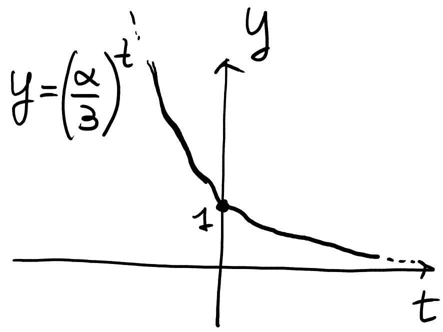
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \arctan\Big(\underbrace{x^2\left(\frac{\alpha}{3}\right)^{1/x}}_{+\infty}\Big) = \frac{\pi}{2}$$



$f(0)=0$, $\lim_{x\to 0^-} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \frac{\pi}{2}$

Se $\alpha > 3$, allora f è continua a sinistra in $x=0$ e $x=0$ è punto di salto.

- Caso $0<\alpha<3$



$$\lim_{x\to 0^+} \left(\frac{\alpha}{3}\right)^{\overset{\to +\infty}{1/x}} \underset{t=\frac{1}{x}}{=} \lim_{t\to+\infty} \left(\frac{\alpha}{3}\right)^t = 0$$

$$\lim_{x\to 0^-} \left(\frac{\alpha}{3}\right)^{\overset{\to -\infty}{1/x}} \underset{t=\frac{1}{x}}{=} \lim_{t\to-\infty} \left(\frac{\alpha}{3}\right)^t = +\infty$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \arctan\Big(\underbrace{x^2\left(\frac{\alpha}{3}\right)^{1/x}}_{0}\Big) = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \arctan\left(x^2\left(\frac{\alpha}{3}\right)^{1/x}\right)$$

$$\lim_{x\to 0^-} x^2\left(\frac{\alpha}{3}\right)^{1/x} = \lim_{x\to 0^-} x^2 e^{\frac{1}{x}\log\left(\frac{\alpha}{3}\right)} =$$

$$\boxed{y = \frac{1}{x} \quad \text{se } x\to 0^- \text{ allora } y\to -\infty}$$

$$= \lim_{y\to -\infty} \frac{e^{\overbrace{\underbrace{y}_{-\infty}\,\underbrace{\log\left(\frac{\alpha}{3}\right)}_{<0}}^{\to +\infty}}}{y^2} = +\infty$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \arctan\Big(\underbrace{x^2\left(\frac{\alpha}{3}\right)^{1/x}}_{\to +\infty}\Big) = \frac{\pi}{2}$$

Se $0<\alpha<3$, allora la funzione f è continua a destra in $x=0$ e $x=0$ è punto di salto.

(16/01/2014) Calcolare

$$\lim_{n\to+\infty} \frac{\log((n+2)!) - \log(n!)}{n\left(\sqrt[n]{n^3}-1\right)} = a_n$$

$$\log((n+2)!) - \log(n!) = \log\left(\frac{(n+2)!}{n!}\right) =$$

$$= \log\left(\frac{(n+2)(n+1)\cancel{n!}}{\cancel{n!}}\right) = \log\left((n+2)(n+1)\right) \sim \log(n^2) =$$
$$= 2\log(n)$$
per $n\to+\infty$

$$\boxed{(n+2)(n+1) \sim n^2 \text{ per } n\to+\infty}$$

$$\sqrt[n]{n^3} - 1 = n^{3/n} - 1 = e^{\frac{3}{n}\log n} - 1 \sim 3\frac{\log n}{n} \quad \text{per } n\to+\infty$$

$$(e^t - 1) \sim t \text{ per } t\to 0 \qquad \lim_{n\to+\infty} \frac{\log n}{n} = 0$$

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{2\cancel{\log(n)}}{\cancel{n}\cdot 3\frac{\cancel{\log(n)}}{\cancel{n}}} = \frac{2}{3}$$

(06/02/2014)

Siano $\alpha \in \mathbb{R}$ e $f : \mathbb{R} \to \mathbb{R}$ definita da

$$f(x) = \begin{cases} (2+x^2)^\alpha & \text{se } x \leq 0 \\ \dfrac{\log(1+\sin^2(2x))}{\arctan x^\alpha} & \text{se } x > 0 \end{cases}$$

Studiare la continuità di f in $x=0$

- f è continua in $\mathbb{R} \setminus \{0\}$
- Studiamo la continuità di f in $x=0$
- $f(0) = (2+0^2)^\alpha = 2^\alpha$
- $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2+x^2)^\alpha = 2^\alpha$

per continuità delle funzioni coinvolte

- Calcoliamo $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \dfrac{\log(1+\sin^2(2x))}{\arctan x^\alpha}$ $\to 0$, $\to 0$

$\arctan t \sim t$ per $t \to 0$

Se $x \to 0^+$, allora $x^\alpha \longrightarrow \begin{cases} 0 & \text{se } \alpha > 0 \\ 1 & \text{se } \alpha = 0 \\ +\infty & \text{se } \alpha < 0 \end{cases}$

• <u>Caso $\alpha > 0$</u>

$\arctan x^\alpha \sim x^\alpha$ per $x \to 0^+$

$$\lim_{x\to 0^+} \frac{\log(1+\sin^2(2x))}{\arctan x^\alpha} = \lim_{x\to 0^+} \frac{4x^2}{x^\alpha}$$

$\sin t \sim t$ per $t \to 0$, $\log(1+y) \sim y$ per $y \to 0$

$\log(1+\sin^2(2x)) \sim \sin^2(2x) \sim (2x)^2 = 4x^2$ per $x \to 0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 4x^{2-\alpha} = \begin{cases} +\infty & \text{se } \alpha > 2 \\ 4 & \text{se } \alpha = 2 \\ 0 & \text{se } 0 < \alpha < 2 \end{cases}$$

Se $\alpha = 2$, allora $f(0) = \lim_{x\to 0^-} f(x) = 4$ e

$$\lim_{x\to 0^+} f(x) = 4,$$

quindi f è continua in $x = 0$

Se $0 < \alpha < 2$, allora $f(0) = \lim_{x\to 0^-} f(x) = 2^\alpha$ e

$\lim_{x\to 0^+} f(x) = 0$, quindi f ha un punto di salto in $x = 0$

Se $\alpha > 2$, allora $f(0) = \lim_{x\to 0^-} f(x) = 2^\alpha$ e

$\lim_{x\to 0^+} f(x) = +\infty$, quindi f ha un punto di infinito in $x = 0$.

Caso $\alpha = 0$

$$f(x) = \begin{cases} (2+x^2)^0 = 1 & \text{se } x \leq 0 \\ \dfrac{\log(1+\sin^2(2x))}{\underbrace{\arctan x^0}_{\arctan 1 = \frac{\pi}{4}}} = \dfrac{4}{\pi}\log(1+\sin^2(2x)) & \text{se } x > 0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = 1, \quad f(0) = 1$$

$$\lim_{x\to 0^+} f(x) = 0$$

Se $\alpha = 0$, allora f ha un punto di salto in $x=0$

Caso $\alpha < 0$

$$\lim_{x\to 0^-} f(x) = 2^\alpha \qquad f(0) = 2^\alpha$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\log(1+\sin^2(2x))}{\underbrace{\arctan \underbrace{x^\alpha}_{\to +\infty}}_{\to \frac{\pi}{2}}} \overset{AL}{=} \frac{0}{\frac{\pi}{2}} = 0$$

Se $\alpha < 0$, f ha un punto di salto in $x=0$

In conclusione

$\alpha < 2 \Rightarrow x=0$ punto di salto

$\alpha = 2 \Rightarrow x=0$ punto di continuità

$\alpha > 2 \Rightarrow x=0$ punto di infinito

(07/09/2016) Calcolare

$$\lim_{n\to+\infty} \underbrace{\left(\frac{1}{2}n^{1/n}+\frac{\sin(n!)}{n}\right)\frac{\sqrt{1+8n^2}-n}{\log\left(1+e^{n+2}\right)-\frac{n}{2}}}_{=a_n}$$

succ. limitata

$$\frac{1}{2}n^{1/n}+\underbrace{\frac{\sin(n!)}{n}}_{\to 0} = \underbrace{\frac{1}{2}e^{\frac{1}{n}\log n}}_{\to \frac{1}{2}}+\underbrace{\frac{\sin(n!)}{n}}_{\to 0} \to \frac{1}{2}$$

$\frac{1}{n}\log n \to 0$

per $n\to+\infty$

$$\sqrt{1+8n^2}-n = \left(\sqrt{1+8n^2}-n\right)\cdot\frac{\sqrt{1+8n^2}+n}{\sqrt{1+8n^2}+n} =$$

$$=\frac{1+8n^2-n^2}{\sqrt{1+8n^2}+n} = \frac{7n^2+1}{\sqrt{1+8n^2}+n} \sim \frac{7n^2}{n\sqrt{8}+n} =$$

$$\boxed{\sqrt{1+8n^2}\sim\sqrt{8n^2}=n\sqrt{8} \quad \text{per } n\to+\infty}$$

$$=\frac{7n^2}{n(1+\sqrt{8})} = \frac{7n}{\sqrt{8}+1}\cdot\frac{\sqrt{8}-1}{\sqrt{8}-1} = \frac{\cancel{7}n}{\cancel{7}}\left(2\sqrt{2}-1\right)$$

per $n\to+\infty$

$$\left(\sqrt{1+8n^2}-n\right)\sim n\left(2\sqrt{2}-1\right) \quad \text{per } n\to+\infty$$

$$\log\left(1+e^{n+2}\right)-\frac{n}{2}=$$

$$=\log\left(e^{n+2}\left(e^{-n-2}+1\right)\right)-\frac{n}{2}=$$

$$=\log\left(e^{n+2}\right)+\log\left(1+e^{-n-2}\right)-\frac{n}{2}=$$

$$=n+2+\log\left(1+e^{-n-2}\right)-\frac{n}{2}=$$

$$=\frac{n}{2}+2+\underbrace{\log\Big(1+\underbrace{e^{\overset{\to -\infty}{-n-2}}}_{\to 0}\Big)}_{\to 0}\sim\frac{n}{2}\quad \text{per } n\to+\infty$$

$$\lim_{n\to+\infty}a_n=\lim_{n\to+\infty}\underbrace{\left(\frac{1}{2}n^{1/n}+\frac{\sin(n!)}{n}\right)}_{\to\frac{1}{2}}\cdot\frac{\cancel{n}\left(2\sqrt{2}-1\right)}{\frac{\cancel{n}}{2}}=$$

$$=\frac{1}{\cancel{2}}\cdot\cancel{2}\left(2\sqrt{2}-1\right)=2\sqrt{2}-1$$

(29/06/2009) Calcolare, al variare di $\alpha \in \mathbb{R}$,

$$\lim_{n\to+\infty} \frac{\left[1-\cos\frac{1}{n}\right]^2 \log\left[\left(e^3+\frac{1}{n}\right)^n\right]}{\left(\sqrt{n+4}\right)^\alpha} \qquad = a_n$$

$$\left(\sqrt{n+4}\right)^\alpha = (n+4)^{\alpha/2} \sim n^{\alpha/2} \quad \text{per } n\to+\infty$$

$$\left[1-\cos\frac{1}{n}\right]^2 \sim \left[\frac{1}{2}\cdot\frac{1}{n^2}\right]^2 = \frac{1}{4n^4} \quad \text{per } n\to+\infty$$

$$1-\cos t \sim \frac{1}{2}t^2 \quad \text{per } t\to 0$$

$$\log\left[\left(e^3+\frac{1}{n}\right)^n\right] = \log\left[\left(e^3\left(1+\frac{1}{n\cdot e^3}\right)\right)^n\right] =$$

$$= n\cdot\log\left[e^3\left(1+\frac{1}{n\cdot e^3}\right)\right] =$$

$$= n\cdot\left[\log(e^3)+\log\left(1+\frac{1}{ne^3}\right)\right] =$$

$$= n\left[3+\underbrace{\log\left(1+\overset{\to 0}{\frac{1}{n\cdot e^3}}\right)}_{\to 0}\right] \sim 3n \quad \text{per } n\to+\infty$$

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty}\frac{\frac{1}{4n^4}\cdot 3n}{n^{\alpha/2}} = \lim_{n\to+\infty}\frac{1}{4\cancel{n^4}n^3}\cdot 3\cancel{n}\cdot\frac{1}{n^{\alpha/2}} =$$

$$= \lim_{n \to +\infty} \frac{3}{4} \cdot n^{-3-\frac{\alpha}{2}} = \begin{cases} +\infty & \text{se } \alpha < -6 \text{ (esponente di } n \text{ positivo)} \\ \frac{3}{4} & \text{se } \alpha = -6 \text{ (esponente di } n \text{ uguale a } 0) \\ 0 & \text{se } \alpha > -6 \text{ (esponente di } n \text{ negativo)} \end{cases}$$

$$\boxed{-3 - \frac{\alpha}{2} > 0 \iff \frac{\alpha}{2} < -3 \iff \alpha < -6}$$