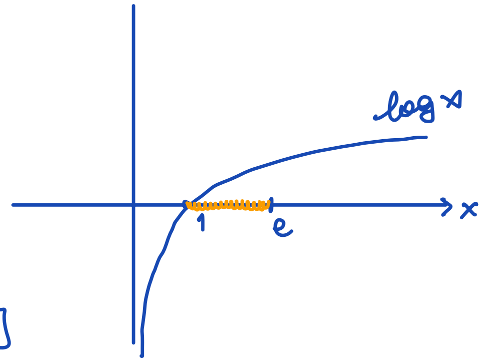
• Calcolare l'area sottesa a $f(x)=\dfrac{1}{x(\log^2 x+4\log x+5)}$ nell'intervallo $[1,e]$

$$A=\int_1^e |f(x)|\,dx = \int_1^e \frac{1}{|x(\log^2 x+4\log x+5)|}\,dx =$$

Su $[1,e]$, $x>0$

$\log x>0$

$\Rightarrow f(x)>0 \quad \forall x\in[1,e]$



$\Rightarrow |f(x)|=f(x)$ su $[1,e]$

$$=\int_1^e \frac{1}{x(\underbrace{\log^2 x}_{y^2}+4\underbrace{\log x}_{y}+5)}\,dx \quad dy =$$

$y=\log x=\varphi(x)$

$dy=\frac{1}{x}\,dx=\varphi'(x)\,dx$

se $x=1 \Rightarrow y=\log x=\log 1=0$

se $x=e \Rightarrow y=\log x=\log e=1$

$$=\int_0^1 \frac{1}{y^2+4y+5}\,dy =$$

? posso scomporre il trinomio nel prodotto di fattori di grado 1

$ax^2+bx+c=0 \qquad \Delta=b^2-4ac \begin{cases} >0 \Rightarrow \exists\, x_1\neq x_2\in\mathbb{R} \\ =0 \quad \exists\, x_1=x_2\in\mathbb{R} \\ <0 \quad \nexists\, x_1,x_2\in\mathbb{R}\end{cases}$

$\Delta=16-20<0$

$\Rightarrow y^2+4y+5 \neq 0$ in $\mathbb{R}$ e non si può scomporre

$$y^2+4y+\underset{\substack{\uparrow\\4+1}}{5} = \underbrace{(y^2+4y+4)}_{(y+2)^2}+1 = (y+2)^2+1$$

$$\frac{1}{(y+2)^2+1} \underset{\substack{\uparrow\\ t=y+2\\ dt=dy}}{=} \frac{1}{t^2+1}$$

se $y=0 \Rightarrow t=2$

se $y=1 \Rightarrow t=3$

$$= \int_0^1 \frac{1}{y^2+4y+5}\,dy = \int_2^3 \frac{1}{t^2+1}\,dt = \Big[\operatorname{arctg}(t)\Big]_2^3 =$$

$$= \operatorname{arctg} 3 - \operatorname{arctg} 2$$

• Sia $f(x) = \dfrac{x^2-3x+3}{x^3-2x^2+x}$? primitiva $F(x)$: $F(2) = \log 8$

$$\int \frac{x^2-3x+3}{x^3-2x^2+x}\,dx$$ grado num < grado den

$$x^3-2x^2+x = x(x^2-2x+1) = x(x-1)^2$$

$$\frac{1\cdot x^2-3x+3}{x^3-2x^2+x} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2} =$$

$$= \frac{Ax^2 - 2Ax + A + Bx^2 - Bx + Cx}{x(x-1)^2} = \frac{(A+B)x^2 + (-2A-B+C)x + A}{x(x-1)^2}$$

$$\begin{cases} A = 3 \\ -2A - B + C = -3 \\ A + B = 1 \end{cases} \qquad \begin{cases} A = 3 \\ A + B = 1 \\ -2A - B + C = -3 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}$$

$$\begin{cases} A = 3 \\ B = -2 \\ C = -3 + 2 \cdot 3 + (-2) = 1 \end{cases}$$

$$\int \frac{x^2 - 3x + 3}{x(x-1)^2} \, dx = \int \left(\frac{3}{x} - \frac{2}{x-1} + \frac{1}{(x-1)^2} \right) dx$$

$$= \int \frac{3}{x} \, dx - 2\int \frac{1}{x-1} \, dx + \int \frac{1}{(x-1)^2} \, dx$$

$$= \underbrace{3 \log|x| - 2 \log|x-1| - \frac{1}{x-1} + C}_{F(x)}$$

$$\boxed{\begin{aligned} \left(\frac{1}{t}\right)' &= -\frac{1}{t^2} \\ \int \frac{1}{t^2} \, dt &= -\frac{1}{t} \end{aligned}}$$

$$? \, c : F(2) = \log 8$$

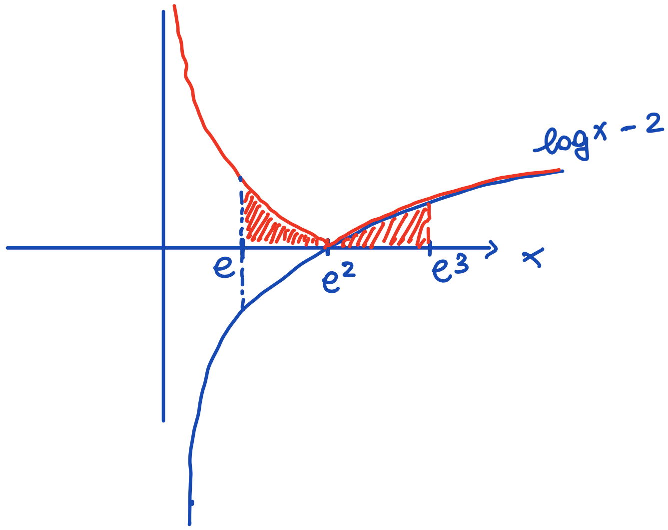
$$F(2) = \cancel{\underbrace{3 \log 2}_{\log 2^3 \;\; \log 8}} - 2 \underbrace{\log 1}_{0} - 1 + c = \cancel{\log 8}$$

$$\Rightarrow c = 1$$

$$\Rightarrow F(x) = 3\log|x| - 2\log|x-1| - \frac{1}{x-1} + 1$$

- Calcolare l'area sottesa a $f(x) = \log(x) - 2$ in $[e, e^3]$

$$A = \int_e^{e^3} \Big|\underbrace{\log x - 2}_{f(x)}\Big|\, dx =$$



$$\log x - 2 = 0$$
$$\log x = 2$$
$$x = e^2$$

$$= \int_e^{e^2} \underbrace{-\log x + 2}_{-f(x)}\, dx + \int_{e^2}^{e^3} \log x - 2\, dx = (*)$$

$$\int \log x\, dx = \int \underbrace{1}_{f'} \cdot \underbrace{\log x}_{g}\, dx = x\log x - \int 1\, dx = x\log x - x$$

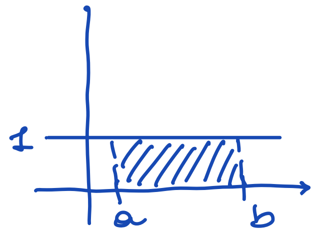
$$f = x \qquad g' = \frac{1}{x}$$

$$\int f'g = fg - \int fg'$$

$$(*) = -\int_e^{e^2} \log x\, dx + \int_e^{e^2} 2\, dx + \int_{e^2}^{e^3} \log x\, dx - 2\int_{e^2}^{e^3} dx$$

$$= -\left[x\log x - x\right]_e^{e^2} + 2\left[x\right]_e^{e^2} + \left[x\log x - x\right]_{e^2}^{e^3} - 2\left(e^3 - e^2\right) =$$

$$\int_a^b dx = \left[x\right]_a^b = b-a$$



$$= -\left(e^2\cdot\overset{2}{\log e^2} - e^2 - e\,\overset{1}{\log e} + e\right) + 2\left(e^2 - e\right) + \left(e^3\cdot\overset{3}{\log e^3} - e^3\right.$$
$$\left. - e^2\underset{2}{\log e^2} + e^2\right) - 2e^3 + 2e^2 =$$

$$= -\cancel{2e^2} + e^2 + \cancel{e} - \cancel{e} + \cancel{2e^2} - 2e + \cancel{3e^3} - \cancel{e^3} - \cancel{2e^2} + e^2 - \cancel{2e^3} + \cancel{2e^2}$$

$$= 2e^2 - 2e = 2e(e-1).$$

- $\displaystyle\int \frac{e^x + 1}{\sqrt{e^x - 1}}\,dx =$

$$y = \sqrt{e^x - 1} \rightarrow y^2 = e^x - 1 \rightarrow e^x = y^2 + 1$$

$$dy = \frac{e^x}{2\sqrt{e^x - 1}}\,dx$$

$$= \underbrace{\int \frac{e^x}{\sqrt{e^x - 1}}\,dx}_{I_1} + \underbrace{\int \frac{1}{\sqrt{e^x - 1}}\,dx}_{I_2}$$

$$I_1 = 2\int \underbrace{\frac{e^x}{2\sqrt{e^x - 1}}\,dx}_{dy} = 2\int dy = 2y = 2\sqrt{e^x - 1} + c$$

$$I_2 = 2\int \frac{1\, e^x}{2\sqrt{e^x-1}\cdot e^x}\,dx = 2\int \frac{1}{y^2+1}\,dy = 2\,\mathrm{arctg}(y) =$$

dy

$e^x = y^2+1$

$$= 2\,\mathrm{arctg}\left(\sqrt{e^x-1}\right) + c$$

$$I = I_1 + I_2 = 2\sqrt{e^x-1} + 2\,\mathrm{arctg}\left(\sqrt{e^x-1}\right) + c$$

- $$\int_{\frac{\pi}{6}}^{\pi/2} 3\sin^2(x)\cdot\underbrace{\cos(x)\cdot\log(7\sin(x))\,dx}_{dy} =$$

$y = \sin x$

$dy = \cos x \cdot dx$

se $x = \frac{\pi}{6} \Rightarrow y = \sin\frac{\pi}{6} = \frac{1}{2}$

se $x = \frac{\pi}{2} \Rightarrow y = \sin\frac{\pi}{2} = 1$

$$= \int_{\frac{1}{2}}^{1} \underbrace{3y^2}_{f'}\cdot\underbrace{\log(7y)}_{g}\,dy \overset{\circledast}{=}$$

$$f = \frac{\cancel{3}}{\cancel{3}}y^3 \qquad g' = \frac{\cancel{7}}{\cancel{7}y} = \frac{1}{y}$$

$$\int 3y^2\cdot\log(7y)\,dy = y^3\cdot\log(7y) - \int \frac{y^{\cancel{3}\,2}}{\cancel{y}}\,dy = y^3\cdot\log(7y) - \frac{1}{3}y^3$$

$$\overset{\circledast}{=} \left[y^3\log(7y) - \frac{1}{3}y^3\right]_{\frac{1}{2}}^{1} = 1\cdot\log 7 - \frac{1}{3} - \left(\frac{1}{8}\log\frac{7}{2} - \frac{1}{3\cdot 8}\right)$$

$$= \log 7 - \frac{1}{3} - \frac{1}{8}\underbrace{\log\frac{7}{2}}_{\log 7 - \log 2} + \frac{1}{24}$$

$$= \log 7 - \frac{1}{8}\log\frac{7}{2} - \frac{7}{24}$$

• $\int_1^4 \frac{1}{x(x^2+1)}\,dx = \frac{1}{4-1}\int_1^4 \frac{1}{x(x^2+1)}\,dx$ ⊛

? $\int \frac{1}{x(x^2+1)}\,dx$

$$0\cdot x^2+0\cdot x+1$$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)} =$$

$$= \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)} = \frac{(A+B)x^2+Cx+A}{x(x^2+1)}$$

$$\begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \qquad \begin{cases} A=1 \\ B=-1 \\ C=0 \end{cases}$$

$$\frac{1}{x(x^2+1)} = \frac{1}{x} + \frac{-x}{x^2+1} = \frac{1}{x} - \frac{x}{x^2+1}$$

$$\int \frac{1}{x(x^2+1)}\,dx = \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$

$$= \log|x| - \frac{1}{2}\log|x^2+1| = \underbrace{\log|x| - \frac{1}{2}\log(x^2+1)}_{G(x)}$$

$$\left(\int \frac{\varphi'(x)}{\varphi(x)}\,dx = \log|\varphi(x)|\right)$$

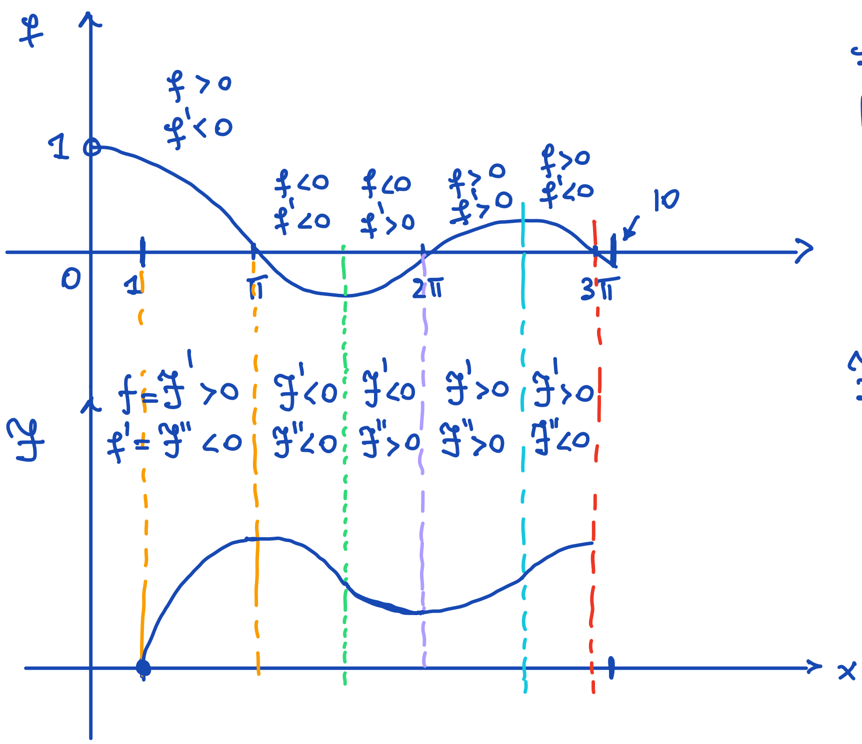
$$\overset{(*)}{=} \frac{1}{3}\left[\log|x| - \frac{1}{2}\log(x^2+1)\right]_1^4 =$$

$$= \frac{1}{3}\left[\underbrace{\log 4}_{2\log 2} - \frac{1}{2}\log(17) - \cancel{\log 1} + \frac{1}{2}\log 2\right]$$

$$= \frac{1}{3}\left(\frac{5}{2}\log 2 - \frac{1}{2}\log 17\right)$$

$$f(x) = \frac{\sin x}{x}$$

? $\mathcal{F}_{x_0}(x) = \int_{x_0}^{x} f(t)\,dt$ disegnarla su $[1, 10]$



f è continua per il 1° th. fond.

$$f = \mathcal{F}_1'$$

$$f' = \mathcal{F}_1''$$

$$\mathcal{F}_1(x) = \int_1^x \frac{\sin t}{t}\,dt$$

Funzioni integrabili non elementarmente

Alcune funzioni sono integrabili in senso indefinito, cioè sono la derivata di una primitiva, ma la loro primitiva non è esprimibile in termini di funzioni elementari. Diciamo allora che queste funzioni sono integrabili ma non elementarmente.

Esempi:

$$\frac{\sin(x)}{x},\ \frac{\cos(x)}{x},\ \frac{e^x}{x},\ e^{x^2},\ e^{-x^2},\ \frac{\log x}{1+x},\ \cos(x^2),\ \sin(x^2),\ \frac{\cos(x)}{x^2},\ \frac{\sin(x)}{x^2}$$

sono tutte funzioni continue sul loro dominio e quindi integrabili (grazie al teorema fondamentale del calcolo integrale), ma non possiamo scrivere le loro primitive in termini di funzioni elementari.

Tuttavia la primitiva è sempre esprimibile mediante una funzione integrale.

Sia $I \subset \mathbb{R}$ un intervallo contenuto nel dominio di f. Scelto $x_0 \in I$, una primitiva di f è

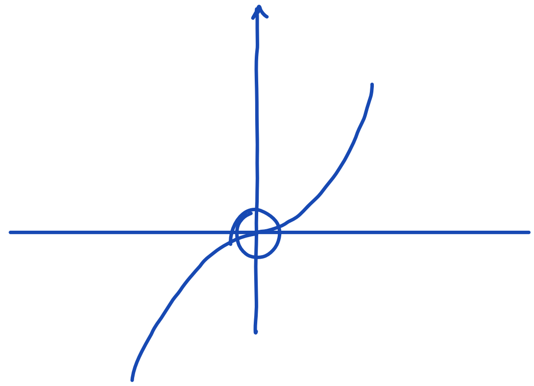
$$F(x) = \mathcal{F}_{x_0}(x) = \int_{x_0}^{x} f(t)dt \qquad \forall x \in I.$$

$f(x) = |x|$? $F(x) = \int |x| \, dx$ su $[-1,1]$

se $x < 0$ $F(x) = \int -x \, dx = -\frac{1}{2} x^2 + c$

se $x > 0$ $F(x) = \int x \, dx = \frac{1}{2} x^2 + c$

$c = 0$



F è primitiva di f se F è derivabile e $F' = f$

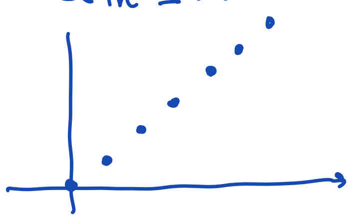
Se a_n è conv $\overset{?}{\Rightarrow}$ $\sum a_n$ è conv

$a_n = \frac{1}{n}$ $\sum \frac{1}{n}$ diverge

$\lim_{n \to \infty} \frac{1}{n} = 0$

Se a_n è div $\underset{\text{sì}}{\Rightarrow}$ $\sum a_n$ div

$a_n = n$



se sommo quantità $\to +\infty$

$\Rightarrow$ anche la loro somma $\to +\infty$

posso applicare il criterio del confronto

$n \geq 1$ $a_n = n \geq b_n = 1$

$$\sum_{n=0}^{\infty} b_n =$$

$$b_n = 1 \qquad s_n = \sum_{k=1}^{n} \underbrace{b_n}_{1} = n \qquad \lim_{n \to \infty} s_n = +\infty$$