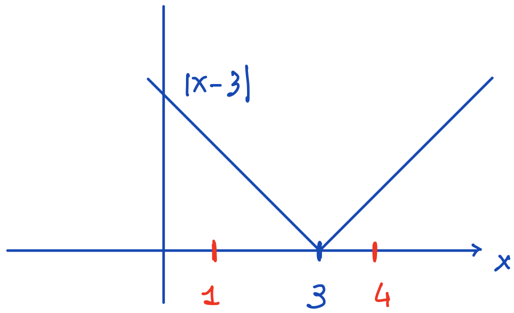
• $\int_1^4 |x-3|\,dx =$

$$|x-3| = \begin{cases} x-3 & \text{se} \quad x-3 \geq 0 \\ -(x-3) & \text{se} \quad x-3 < 0 \end{cases}$$




$$\left( \quad |f(x)| = \begin{cases} +f(x) & \text{se} \quad f(x) \geq 0 \\ -f(x) & \text{se} \quad f(x) < 0 \end{cases} \right.$$

$$= \int_1^3 -(x-3)\,dx + \int_3^4 (x-3)\,dx =$$

$$= \int_1^3 -x+3\,dx + \int_3^4 x-3\,dx$$

$$= \underbrace{\left[-\frac{1}{2}x^2+3x\right]_1^3} + \underbrace{\left[\frac{1}{2}x^2-3x\right]_3^4} = \underbrace{-\frac{1}{2}9+9-\left(-\frac{1}{2}+3\right)} + \underbrace{\frac{1}{2}\cdot 16-12-\left(\frac{1}{2}9-9\right)}$$

$$= -\cancel{\frac{9}{2}} + \cancel{9} + \frac{1}{2} - 3 + 8 - 12 - \cancel{\frac{9}{2}} + 9 = 2 + \frac{1}{2} = \frac{5}{2}$$

• $\int \frac{1}{2x^2+3}\,dx \rightsquigarrow \int \frac{1}{t^2+1}\,dt = \text{arctg}(t) + c$

$a^2-b^2=(a+b)(a-b)$

$a^2+b^2$ non è scomponibile

$$2x^2+3 = 3\left(\frac{2}{3}x^2+1\right) =$$

$$= 3\left(\left(\sqrt{\frac{2}{3}}x\right)^2+1\right)$$

$$\int \frac{1}{2x^2+3}\,dx = \sqrt{\frac{3}{2}} \int \frac{1}{3\left(\left(\sqrt{\frac{2}{3}}\,x\right)^2+1\right)} \sqrt{\frac{2}{3}}\,dx = \frac{1}{3}\sqrt{\frac{3}{2}} \int \frac{1}{t^2+1}\,dt =$$

$dt$ ; $\sqrt{\frac{2}{3}}\,x = t$

$$t = \sqrt{\frac{2}{3}}\,x$$

$$dt = \sqrt{\frac{2}{3}}\cdot dx$$

$$= \frac{1}{3}\sqrt{\frac{3}{2}}\,\operatorname{arctg}(t) = \underbrace{\frac{1}{3}\sqrt{\frac{3}{2}}}\,\operatorname{arctg}\left(\sqrt{\frac{2}{3}}\cdot x\right) + C$$

$$\frac{1}{\sqrt{3}\cdot\sqrt{2}} = \frac{1}{\sqrt{6}} = \frac{6}{\sqrt{6}}$$

- Calcolare la media integrale di $f(x) = \sqrt{x}\cdot \operatorname{atan}\left(\sqrt{x^3}\right)$ su $[1,3]$

$$⨍_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx$$

media integrale $⨍_1^3 \sqrt{x}\cdot\operatorname{atan}\left(\sqrt{x^3}\right)dx = \frac{1}{3-1}\underbrace{\int_1^3 \underbrace{\sqrt{x}}_{\frac{2}{3}\varphi'(x)}\cdot\operatorname{atan}\big(\underbrace{\sqrt{x^3}}_{\varphi(x)}\big)dx}_{I_1}$

$$\int f'g = fg - \int fg'$$

$$\int \underbrace{f(\varphi(x))}\cdot\underbrace{\varphi'(x)}\,dx$$

$$\varphi(x) = \sqrt{x^3} = x^{3/2} \qquad \varphi'(x) = \frac{3}{2}x^{3/2-1} = \frac{3}{2}\sqrt{x}$$

faccio la sostituzione $y = \sqrt{x^3} = \varphi(x)$

$$dy = \varphi'(x)\,dx = \frac{3}{2}\sqrt{x}\,dx$$

$$I_1 = \frac{2}{3}\int_1^3 \underbrace{\frac{3}{2}\sqrt{x}\,\operatorname{atan}(\underbrace{\sqrt{x^3}}_{y})\,dx}_{dy} = \frac{2}{3}\int_1^{\sqrt{3^3}} 1\cdot\operatorname{atan}(y)\,dy$$

se $x = 1 \Rightarrow y = \sqrt{x^3} = 1$

se $x = 3 \Rightarrow y = \sqrt{x^3} = \sqrt{3^3}$

$$\int \underset{f'}{1}\cdot \underbrace{\operatorname{atan}(y)}_{g}\,dy = y\cdot\operatorname{atan}(y) - \underbrace{\int \frac{y}{y^2+1}\,dy}_{I_2}$$

$f(y) = y \qquad g'(y) = \frac{1}{y^2+1}$

$$\boxed{\int f'g = fg - \int fg'}$$

$$I_2 = \frac{1}{2}\int \frac{2y}{y^2+1}\,dy \underset{\substack{t = y^2+1\\ dt = 2y\,dy}}{=} \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| \overset{(*)}{=}$$

$$\boxed{\int \frac{1}{y^2+1}\,dy = \operatorname{atan}(y) + c}$$

$$\boxed{\int \frac{\overbrace{\varphi'(y)\,dy}^{dt}}{\underbrace{\varphi(y)}_{t}} = \int \frac{1}{t}\,dt = \log|t| = \log|\varphi(y)| + c \qquad \begin{aligned} t &= \varphi(y)\\ dt &= \varphi'(y)\,dy\end{aligned}}$$

$$\overset{(*)}{=} \frac{1}{2}\log|y^2+1| = I_2$$

$$\frac{1}{2}\log(y^2+1)$$

$$\Rightarrow \int \operatorname{atan}(y)\,dy = y\cdot\operatorname{atan}(y) - I_2 = \underbrace{y\cdot\operatorname{atan}(y) - \frac{1}{2}\log(y^2+1)}_{G(y)}$$

$$I_1 = \frac{2}{3}\int_1^3 \underbrace{\frac{3}{2}\sqrt{x}\,\operatorname{atan}(\underbrace{\sqrt{x^3}}_{y})\,dx}_{dy} = \frac{2}{3}\int_1^{3\sqrt{3}} 1\cdot\operatorname{atan}(y)\,dy =$$

2° thm fond del calcolo

$$= \frac{2}{3}\left[ y \cdot \operatorname{atan}(y) - \frac{1}{2}\log\left(y^2+1\right)\right]_1^{3\sqrt{3}} =$$

$$= \frac{2}{3}\left(3\sqrt{3}\cdot \operatorname{atan}\left(3\sqrt{3}\right) - \frac{1}{2}\log\left(\overset{3^3+1}{\cancel{4}}\right) - \left(1\cdot \operatorname{atan}(1) - \frac{1}{2}\log(2)\right)\right)$$

$$\underbrace{= \frac{2}{3}\left(3\sqrt{3}\cdot \operatorname{atan}\left(3\sqrt{3}\right) - \frac{1}{2}\log 28 - \frac{\pi}{4} + \frac{1}{2}\log 2\right)}_{\text{media} = \frac{1}{2}} = I_2$$

• Sia $f(x) = 2x\sqrt{x^2+1}$

1) $\int_0^1 f(x)\,dx$ , 2) calcolare area sottesa a $f(x)$ in $[-1,1]$

1) $$\int_0^1 \underbrace{2x\sqrt{\underbrace{x^2+1}_{y}}\,dx}_{dy} =$$

sost $y = x^2+1 = \varphi(x)$ se $x=0 \Rightarrow y = x^2+1 = 1$

$dy = \varphi'(x)\,dx = 2x\,dx$ se $x=1 \Rightarrow y=2$

$$= \int_1^2 \sqrt{y}\,dy =$$

$\sqrt{y} = y^{1/2}$

$$\int y^\alpha\,dy = \frac{1}{\alpha+1}y^{\alpha+1} + c \qquad \alpha \neq -1$$

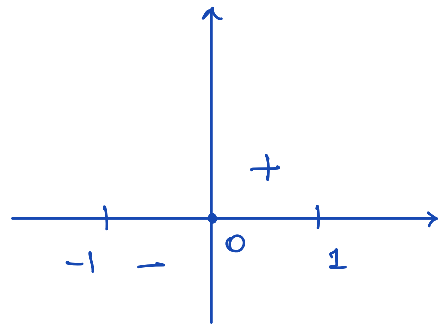
primitiva di $y^{1/2}$ è $\frac{1}{\frac{1}{2}+1}\cdot y^{\frac{1}{2}+1} = \frac{2}{3}y^{3/2}$

$$= \left[\frac{2}{3} y^{3/2}\right]_1^2 = \frac{2}{3}\sqrt{2^3} - \frac{2}{3} = \frac{2}{3}\left(2\sqrt{2}-1\right)$$

2) Area sottesa ad $f$ in $[-1,1]$

$$\text{Area} = \int_{-1}^{1} |f(x)|\,dx \overset{(*)}{=}$$

$$f(x) = \underbrace{2x}_{\substack{\text{positivo se } x>0 \\ \text{negativo se } x<0}} \cdot \underbrace{\sqrt{x^2+1}}_{>0}$$




$$|f(x)| = \begin{cases} f(x) & \text{se } f(x) \geq 0 \\ -f(x) & \text{se } f(x) < 0 \end{cases} = \begin{cases} 2x\sqrt{x^2+1} & \text{se } x \geq 0 \\ -2x\sqrt{x^2+1} & \text{se } x < 0 \end{cases}$$

$$\overset{(*)}{=} \underbrace{\int_{-1}^{0} -2x\sqrt{x^2+1}\,dx}_{-\int_{-1}^{0} 2x\sqrt{x^2+1}\,dx} + \underbrace{\int_{0}^{1} 2x\sqrt{x^2+1}\,dx}_{\frac{2}{3}\left(2\sqrt{2}-1\right)}$$

Se $f$ è disp $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

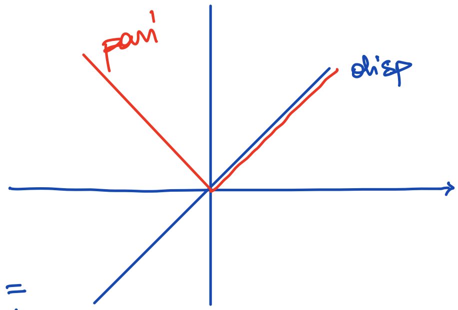
Se $f$ è pari $\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

per calcolare $-\int_{-1}^{0} 2x\sqrt{x^2+1}\,dx$ posso sfruttare $\int_{0}^{1} 2x\sqrt{x^2+1}\,dx$

vedo se $f$ è disp o pari

$$f(-x) = 2(-x)\sqrt{(-x)^2+1} = -2x\sqrt{x^2+1} = -f(x)$$

$f$ è dispari $\Rightarrow$ $|f(x)|$




$$\text{Area} = \int_{-1}^{1} \underbrace{|f(x)|}_{\text{è pari}}\,dx = 2\int_0^1 |f(x)|\,dx$$

$$= 2\int_0^1 f(x)\,dx = 2\cdot\frac{2}{3}\left(2\sqrt{2}-1\right) = \frac{4}{3}\left(2\sqrt{2}-1\right)$$

se $x>0$

$\Rightarrow |f(x)| = f(x)$

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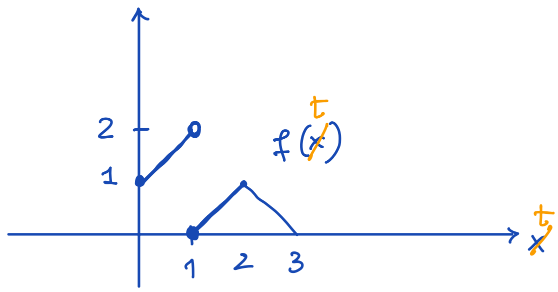
Disegnare

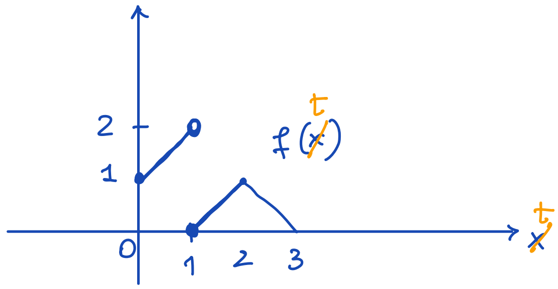
~~Costruire~~ la funzione integrale di $f(x)$ disegnata di seguito con $x_0 = 0$, nell'intervallo $[0,3]$:



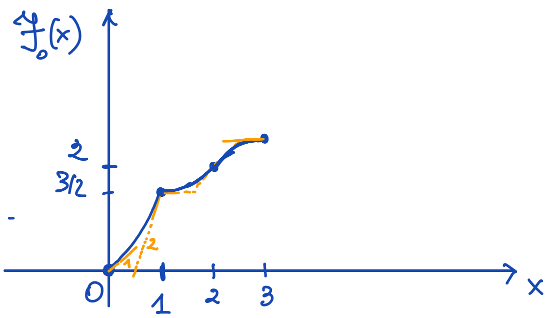

$$F_{x_0}(x) = \int_{x_0}^{x} f(t)\,dt \qquad x_0 = 0$$

$$F_0(x) = \int_0^x f(t)\,dt$$

Il 1° teorema del calcolo dice che se $f$ è cont su $[a,b]$ allora $F_{x_0}(x)$ è derivabile e $F'_{x_0}(x) = f(x)$




$$f(t) = \begin{cases} t+1 & \text{in } (0,1) \\ t-1 & \text{in } (1,2) \end{cases}$$




$$F_0(0) = \int_0^0 f(t)\,dt = 0$$

? in $(0,1)$ $\quad f$ è cont

$\Rightarrow \quad F_0' = f > 0 \Rightarrow F_0$ è cresc

$F_0'' = f' > 0 \Rightarrow F_0$ è conv.

$$F_0(1) = \int_0^1 f(t)\,dt \qquad f(t) = t+1$$

$$= \int_0^1 t+1\,dt = \left[\frac{1}{2}t^2 + t\right]_0^1 = \frac{1}{2} + 1 = \frac{3}{2}$$

Su $(1,2)$ $\quad f \geq 0 \Rightarrow F_0' \geq 0 \Rightarrow F_0$ cresc

$\underbrace{f' > 0 \Rightarrow F_0'' > 0}_{=} \Rightarrow F_0$ è convessa

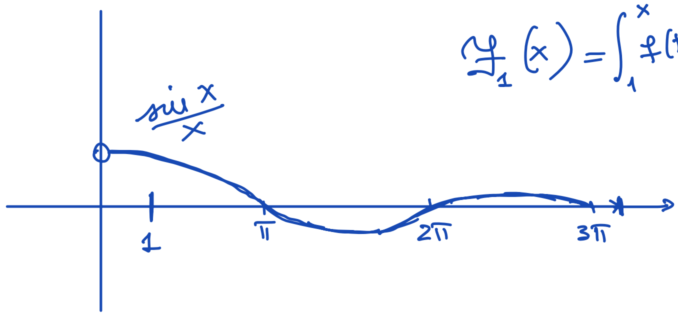
$$F_0(2) = \int_0^2 f(t)\,dt \underset{\text{add}}{=} \underbrace{\int_0^1 f(t)\,dt}_{\frac{3}{2}} + \underbrace{\int_1^2 f(t)\,dt}_{\int_1^2 t-1\,dt} = \frac{3}{2} + \frac{1}{2} = 2$$

$$\int_1^2 t-1\,dt = \left[\frac{1}{2}t^2 - t\right]_1^2 =$$

$$= \cancel{\frac{1}{2}\cdot 4} - \cancel{2} - \left(\frac{1}{2} - 1\right) =$$

$$= \frac{1}{2}$$

Disegnare la funzione integrale di $f(x) = \frac{\sin x}{x}$ su $[1, 10]$




$$y_1(x) = \int_1^x f(t)\,dt$$

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• $$\int_0^4 \frac{\underset{\| \atop y}{\sqrt{x}}\overset{y^2 \atop /\!/}{x} - 1}{\sqrt{x}+2}\,\overset{2y\,dy}{dx} =$$

$y = \sqrt{x} = \varphi(x)$

$dy = \frac{1}{2\sqrt{x}}\,dx \iff 2\overset{y}{\sqrt{x}}\,dy = dx$

$x = y^2$ se $x = 0 \Rightarrow y = \sqrt{x} = 0$

se $x = 4 \Rightarrow y = \sqrt{x} = 2$

$$= \int_0^2 \frac{y^2-1}{y+2}\,2y\,dy =$$

$$= \int_0^2 \frac{2y^3-2y}{y+2}\,dy = 2\int_0^2 \frac{y^3-y}{y+2}\,dy$$

se grado num $\geq$ grado denominatore $\Rightarrow$ fare divisione di polinomi

numeratore | denom.

$$\begin{array}{rrrr|l}
y^3 & +\,0\cdot y^2 & -\,y & +\,0 & y+2 \\ \hline
-y^3 & -\,2y^2 & +\,0y & +\,0 & y^2-2y+3 \\ \hline
/\!/ & -2y^2 & -\,y & +\,0 & \\
 & +\,2y^2 & +\,4y & & \\ \hline
 & /\!/ & 3y & +\,0 & \\
 & & -3y & -\,6 & \\ \hline
 & & /\!/ & -6 &
\end{array}$$

quoziente: $y^2-2y+3$

resto: $-6$

$$\frac{\text{numeratore}}{\text{denominatore}} = \text{quoziente} + \frac{\text{resto}}{\text{denominatore}}$$

$$\frac{y^3-y}{y+2} = y^2-2y+3 - \frac{6}{y+2}$$

$$\int \frac{y^3-y}{y+2}\,dy = \int y^2-2y+3\,dy - \int \frac{6\cdot 1}{y+2}\,dy$$

$$= \frac{1}{3}y^3 - \frac{2}{2}y^2 + 3y - 6\underbrace{\int \frac{1}{y+2}\,dy}_{\log|y+2|} =$$

$$\Rightarrow$$

$$\int_0^2 \frac{y^3-y}{y+2}\,dy = \left[\frac{1}{3}y^3 - y^2 + 3y - 6\log|y+2|\right]_0^2$$

$$= \frac{1}{3}\cdot 8 - 4 + 6 - 6\log 4 + 6\log 2 =$$

$$= \frac{8}{3} + 2 - 12\log 2 + 6\log 2 = \frac{14}{3} - 6\log 2$$

$$\Rightarrow \quad I = 2\left(\frac{14}{3} - 6\log 2\right)$$

Area sottesa a $f(x) = \dfrac{1}{x\left(\log^2 x + 4\log x + 5\right)}$ su $[1, e]$