2° teor fond del calcolo integrale

Se f è cont su $[a,b]$ e $G(x)$ è una primitiva di f

allora $\int_a^b f(x)\,dx = G(b) - G(a)$

• $\int_0^{\pi/2} \sin(2x)\,dx$

applico sostituzione, 2 approcci

1) cerco $G(x)$ primitiva, risolvo $\int \sin(2x)\,dx$

sostituzione $\quad y = 2x = \varphi(x)$

$\qquad dy = \varphi'(x)\,dx = 2\cdot dx$

$$\boxed{\begin{gathered}\int f(\underbrace{\varphi(x)}_{y})\underbrace{\varphi'(x)\,dx}_{dy} = \int f(y)\,dy \\ y = \varphi(x) \\ dy = \varphi'(x)\,dx\end{gathered}}$$

$$\frac{1}{2}\int 2\sin(\underbrace{2x}_{y})\,dx = \frac{1}{2}\int \sin y\,dy = \frac{1}{2}(-\cos y) + c$$

$$= \underbrace{-\frac{1}{2}\cos(2x)}_{G(x)} + c$$

applico il 2° teor fond del calcolo

$$\int_0^{\pi/2} \sin(2x)\,dx = G\left(\frac{\pi}{2}\right) - G(0) = -\frac{1}{2}\cos(\pi) - \left(-\frac{1}{2}\cos(0)\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

2) applico la sostituzione diretta all' $\int$ definito

$$\frac{1}{2}\int_0^{\pi/2} \underbrace{2\sin(\underbrace{2x}_{y})\,dx}_{dy} = \frac{1}{2}\int_0^{\pi} \sin y \, dy =$$

$y = 2x$ se $x = 0 \Rightarrow y = 2\cdot 0 = 0$

$dy = 2\,dx$ se $x = \frac{\pi}{2} \Rightarrow y = 2\cdot\frac{\pi}{2} = \pi$

$$= \frac{1}{2}\Big[\underbrace{-\cos y}_{G(y)}\Big]_0^{\pi} = \frac{1}{2}\left(-\cos\pi + \cos 0\right) = \frac{1}{2}(1+1) = 1$$

- $$\int_0^{\pi/2} \frac{\sin(2x)}{1+4\cos^2(x)}\,dx =$$

$\sin(2x) \neq 2\sin x$

$$\sin(2x) = 2\cdot\sin x\cdot\cos x$$

$$f(x) = \frac{\sin(2x)}{1+4\cos^2(x)} = \frac{2\sin x\cdot\cos x}{1+4\cdot\cos^2 x}$$

$$y = \cos x = \varphi(x)$$

$$dy = \varphi'(x)\,dx = -\sin x\cdot dx$$

$$=(-1)\int_0^{\pi/2} \frac{\overbrace{-2\sin x\cdot \overbrace{\cos x}^{y}\,dx}^{dy}}{1+4\underbrace{\cos^2 x}_{y^2}} =$$

se $x = 0 \Rightarrow y = \cos(0) = 1$

se $x = \frac{\pi}{2} \Rightarrow y = \cos\left(\frac{\pi}{2}\right) = 0$

$$= -\frac{1}{4}\int_1^0 \frac{\overbrace{4\cdot 2\,y\,dy}^{dt}}{\underbrace{1+4y^2}_{t}} =$$

$t = 1+4y^2$ se $y = 1 \Rightarrow t = 1+4\cdot 1 = 5$

$dt = 8y \cdot dy$ se $y=0 \Rightarrow t = 1+4\cdot 0 = 1$

$$= -\frac{1}{4}\int_5^1 \frac{dt}{t} = -\frac{1}{4}\Big[\log|t|\Big]_5^1 = -\frac{1}{4}\Big[\underbrace{\log|1|}_{0} - \log|5|\Big] = \frac{\log 5}{4}$$

$G(t) = \log|t|$ è una primitiva di $\frac{1}{t}$

$$\int f'g = fg - \int fg'$$
$$\int fg' = fg - \int f'g$$

- $$\int_1^2 \underbrace{(x+3)}_{g}\underbrace{e^{5-x}}_{f'}\,dx =$$ P.P.

$f'(x) = e^{5-x}$ $\qquad f(x) = -e^{5-x}$

$g(x) = x+3$ $\qquad g'(x) = 1$

consiglio cercare $\int (x+3)e^{5-x}dx = G(x)$

poi applicare alla fine il 2° teor fond del calcolo

$$\int (x+3)e^{5-x}dx = \underbrace{-e^{5-x}}_{f}\underbrace{(x+3)}_{g} \overset{+}{-} \int -e^{5-x}dx =$$

$$\int e^{5-x}dx = -e^{5-x}$$

$$= -e^{5-x}(x+3) - e^{5-x} =$$

$$= -e^{5-x}(x+4) = G(x)$$

$$\int_1^2 (x+3)e^{5-x}dx = G(2) - G(1) = \Big[G(x)\Big]_1^2 = \underbrace{-e^{5-2}(2+4)}_{G(2)} - \underbrace{\left(-e^{5-1}\cdot(1+4)\right)}_{G(1)}$$

$$= -6e^3 + 5e^4$$

• ? $G:(-5,+\infty)\to\mathbb{R}$ primitiva di $g(x)=\dfrac{\log(x+5)}{(x+5)^2}$

t.c. $G(-4)=4$

$$\int g(x)\,dx = \int \frac{\log\overbrace{(x+5)}^{y}}{(\underbrace{x+5}_{y})^2}\overbrace{dx}^{dy} = -\int \underbrace{-\frac{1}{y^2}}_{f'}\cdot\underbrace{\log y}_{g}\,dy \overset{(*)}{=}$$

$y = x+5$

$dy = 1\cdot dx$

P.P.

$f'(y) = -\dfrac{1}{y^2}$ $\qquad f(y)=\dfrac{1}{y}$

$-y^{-2}$ $\qquad y^{-1}$

$$\int f'g = fg - \int fg'$$

$g(y)=\log y$ $\qquad g'(y)=\dfrac{1}{y}$

$$\overset{(*)}{=} -\left[\frac{1}{y}\log y - \underbrace{\int \frac{1}{y}\cdot\frac{1}{y}\,dy}_{-\int \frac{1}{y^2}dy}\right] = \overbrace{-\frac{1}{y}\log y - \frac{1}{y}}^{-\frac{1}{y}(\log y+1)} = -\frac{1}{x+5}\left(\log(x+5)+1\right) + C$$

$$-\int \frac{1}{y^2}\,dy = \int -\frac{1}{y^2}\,dy = \frac{1}{y}$$

la generica primitiva è $G(x) = -\dfrac{1}{x+5}\left(\log(x+5)+1\right)+C$

cercare il valore di c t.c. $\boxed{G(-4)=4}$

$$G(-4) = -\frac{1}{(-4+5)}\Big(\underbrace{\log(-4+5)}_{0}+1\Big)+C = 4$$

$-1\cdot 1 + C = 4 \Rightarrow C = 5$ $\Rightarrow$ la primitiva cercata è

$$G(x) = -\frac{1}{x+5}\left(\log(x+5)+1\right)+5$$

$$\bullet \int \frac{\overbrace{e^{2x}}^{y}\,\overbrace{e^{x}\,dx}^{dy}}{4\underbrace{e^{2x}}_{y^2}+4\underbrace{e^{x}}_{y}+1} = \int \frac{y}{4y^2+4y+1}\,dy = \int \frac{y}{(2y+1)^2}\,dy$$

$$y = e^x$$
$$dy = e^x dx$$

voglio riscrivere $f(y) = \dfrac{y}{(2y+1)^2} = \dfrac{A}{(2y+1)} + \dfrac{B}{(2y+1)^2}$? A, B t.c. sia vero $\forall y \in \mathbb{R}$ $y \neq -\frac{1}{2}$

$$\left(\text{se fosse stato } \frac{y}{(2y+1)^4} = \frac{A}{2y+1} + \frac{B}{(2y+1)^2} + \frac{C}{(2y+1)^3} + \frac{D}{(2y+1)^4} \right)$$

$$\frac{A}{(2y+1)} + \frac{B}{(2y+1)^2} = \frac{A(2y+1)+B}{(2y+1)^2} = \frac{2A\cdot y + (A+B)}{(2y+1)^2} = \frac{1\cdot y + 0}{(2y+1)^2}$$

$$\begin{cases} 2A = 1 \\ A+B = 0 \end{cases} \qquad \begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \end{cases}$$

$$\Rightarrow \frac{y}{(2y+1)^2} = \underbrace{\frac{1}{2}}_{A}\cdot\frac{1}{(2y+1)} \underbrace{-\frac{1}{2}}_{B}\cdot\frac{1}{(2y+1)^2}$$

$$\int \frac{y}{(2y+1)^2}\,dy = \int \frac{1}{2}\frac{1}{(2y+1)} - \frac{1}{2}\cdot\frac{1}{(2y+1)^2}\,dy$$

linearità

$$= \frac{1}{2}\underbrace{\int \frac{1}{2y+1}\,dy}_{I_1} - \frac{1}{2}\underbrace{\int \frac{1}{(2y+1)^2}\,dy}_{I_2} \overset{(*)}{=}$$

$$I_1 = \frac{1}{2}\int \frac{\overbrace{2\cdot 1\,dy}^{dt}}{2y+1} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t|$$

$$t = 2y+1$$
$$dt = 2\,dy$$

$$I_2 = \frac{1}{2}\int \underbrace{\frac{2 \cdot 1}{(2y+1)^2}}_{t^2}\, dy = -\frac{1}{2}\int \frac{-dt}{t^2} = -\frac{1}{2}\,\frac{1}{t}$$

$$\overset{(*)}{=} \frac{1}{2}\cdot\frac{1}{2}\log|t| - \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{t} =$$

$$= \frac{1}{4}\log|t| + \frac{1}{4}\,\frac{1}{t} =$$

$t = 2y+1$

$$= \frac{1}{4}\log|2y+1| + \frac{1}{4}\,\frac{1}{2y+1} =$$

$y = e^x$

$$= \frac{1}{4}\log|2e^x+1| + \frac{1}{4}\,\frac{1}{2e^x+1} + C$$

$$= \frac{1}{4}\log(2e^x+1) + \frac{1}{4}\cdot\frac{1}{(2e^x+1)} + C$$

- $$\int_0^1 \sqrt{1-y^2}\, dy = (*)$$ ($y = \sin x$, $dy = \cos x \cdot dx$)

$$\int f(\underbrace{\varphi(x)}_{y})\varphi'(x)\,dx = \int f(y)\,dy \longrightarrow$$

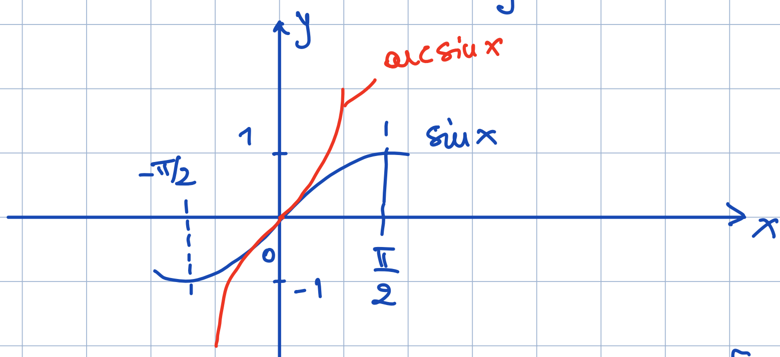
$$\boxed{y = \varphi(x) = \sin x}$$

$$dy = \varphi'(x)\,dx = \cos x \cdot dx$$

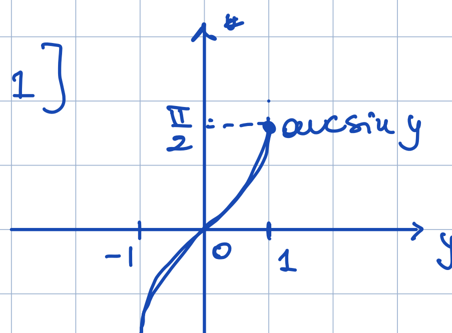
estremi: $y = 0 \quad ?\ x = \arcsin y = 0$

$y = \sin x$

se $y = 1 \Rightarrow x = \arcsin 1 = \frac{\pi}{2}$



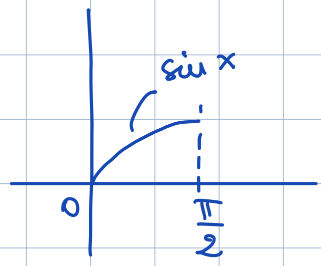
posso costruire $\arcsin y$ con $y \in [0,1]$



$$(*) = \int_0^{\pi/2} \sqrt{1-\sin^2 x}\cdot \cos x\, dx =$$

$$1-\sin^2 x = \cos^2 x$$

$$\sqrt{1-\sin^2 x} = \cos x$$



$$= \int_0^{\pi/2} \underset{f'}{\cos x} \cdot \underset{g}{\cos x}\, dx$$

$$\int f'g = fg - \int fg'$$

$$\int \underset{f'}{\cos x} \cdot \underset{g}{\cos x}\, dx = \sin x \cdot \cos x + \int \overbrace{\sin x \cdot \sin x}^{1-\cos^2 x}\, dx =$$

$$f(x) = \sin x \qquad g'(x) = -\sin x$$

$$= \sin x \cdot \cos x + \int 1 - \cos^2 x\, dx$$

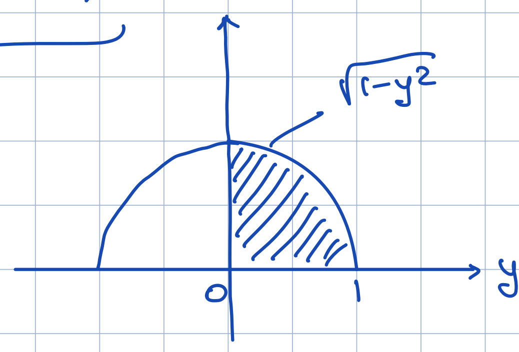
$$= \sin x \cdot \cos x + \underbrace{\int 1\, dx}_{x} - \int \cos^2 x\, dx$$

$$2\int \cos^2 x\, dx = \sin x \cdot \cos x + x$$

$$\int \cos^2 x\, dx = \frac{1}{2}\left(\sin x \cdot \cos x + x\right) + C = G(x)$$

$$\Rightarrow \int_0^{\pi/2} \cos^2 x\, dx = \left[\frac{1}{2}\left(\sin x \cdot \cos x + x\right)\right]_0^{\pi/2} =$$

$$= \frac{1}{2}\left(\sin\frac{\pi}{2} \cdot \underbrace{\cos\frac{\pi}{2}}_{0} + \frac{\pi}{2}\right) - \underbrace{\frac{1}{2}\left(\sin 0 \cdot \cos 0 + 0\right)}_{0} = \frac{\pi}{4}$$



Regole di integrazione

Integrazione per parti

Se $f, g \in C^1([a,b])$, allora

$$\int_a^b f(x)'g(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx$$

Integrazione per sostituzione

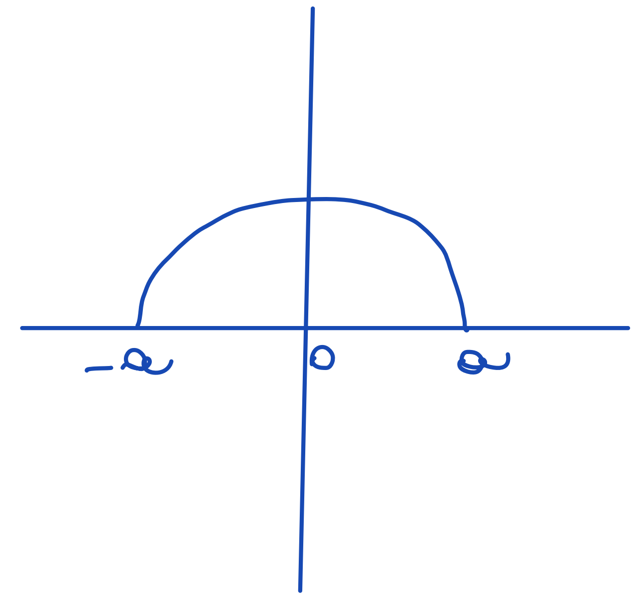
Sia $\varphi : [\alpha, \beta] \to [a,b]$, $\varphi \in C^1([\alpha, \beta])$. Sia $f(y)$ continua su $[a,b]$ e sia $F(y)$ una sua primitiva. Allora

$$\int_\alpha^\beta f(\varphi(x))\varphi'(x)dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(y)dy$$

Se φ è biettiva, allora si ha anche

$$\int_a^b f(y)dy = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(x))\varphi'(x)dx$$

Integrazione di funzioni con simmetria

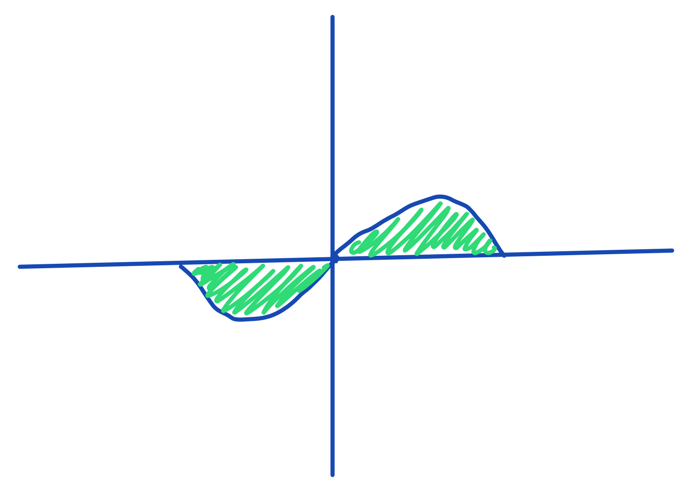


Teorema (utilissimo per evitare conti inutili)

Sia $a \in \mathbb{R}^+$ e sia f integrabile su $[-a, a]$.

Se f è pari, allora $\displaystyle\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

Se f è dispari, allora $\displaystyle\int_{-a}^{a} f(x)dx = 0.$



Funzioni integrabili non elementarmente

Alcune funzioni sono integrabili in senso indefinito, cioè sono la derivata di una primitiva, ma la loro primitiva non è esprimibile in termini di funzioni elementari. Diciamo allora che queste funzioni sono integrabili ma non elementarmente.

Esempi:

$$\frac{\sin(x)}{x},\ \frac{\cos(x)}{x},\ \frac{e^x}{x},\ e^{x^2},\ e^{-x^2},\ \frac{\log x}{1+x},\ \cos(x^2),\ \sin(x^2),\ \frac{\cos(x)}{x^2},\ \frac{\sin(x)}{x^2}$$

sono tutte funzioni continue sul loro dominio e quindi integrabili (grazie al teorema fondamentale del calcolo integrale), ma non possiamo scrivere le loro primitive in termini di funzioni elementari.

Tuttavia la primitiva è sempre esprimibile mediante una funzione integrale.

Sia $I \subset \mathbb{R}$ un intervallo contenuto nel dominio di f. Scelto $x_0 \in I$, una primitiva di f è

$$F(x) = \mathcal{F}_{x_0}(x) = \int_{x_0}^{x} f(t)dt \qquad \forall x \in I.$$

- $$\int_{-\frac{\pi}{2}}^{+\pi/2} \Big(\underbrace{x^{10}\cdot \sin x}_{\text{disp}} + \underbrace{\cos^3 x}_{\text{pari}}\Big)dx \overset{\text{lin}}{=} \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^{10}\sin x\, dx}_{0 \text{ perché } f_1 \text{ è disp}} + \underbrace{\int_{-\frac{\pi}{2}}^{\pi/2} \cos^3 x\, dx}_{2\int_0^{\pi/2}\cos^3 x\, dx \quad f_2 \text{ è pari}} \overset{*}{=}$$

$$f_1(x) = x^{10}\cdot \sin x \qquad f_2(x) = \cos^3 x = (\cos x)^3$$

$$f_1(-x) = (-x)^{10}\cdot \sin(-x) = -x^{10}\cdot \sin(x) = -f_1(x) \quad \text{dispari}$$

$$f_2(-x) = (\cos(-x))^3 = (\cos x)^3 = f_2(x)$$

$$\overset{*}{=} 2\int_0^{\pi/2} \cos^3 x\, dx = 2\int_0^{\pi/2} \overbrace{\cos x \cdot \underbrace{(1-\sin^2 x)}_{\cos^2 x}\, dx}^{dy} =$$

$y = \sin x$

$dy = \cos x\, dx$

se $x = 0 \Rightarrow y = \sin 0 = 0$

se $x = \frac{\pi}{2} \Rightarrow y = \sin\frac{\pi}{2} = 1$

$$= 2\int_0^1 1 - y^2\, dy =$$

$$= 2\int_0^1 1\, dy - 2\int_0^1 y^2\, dy$$

$$= 2\big[y\big]_0^1 - 2\left[\frac{1}{3}y^3\right]_0^1 = 2(1-0) - 2\left(\frac{1}{3} - 0\right) =$$

$$= 2 - \frac{2}{3} = \frac{4}{3}$$