Polinomi di Taylor

Dati: f definita in $I(x_0)$, f derivabile fino all'ordine n in x_0, $n \in \mathbb{N}$ dato.

pol di Taylor $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ di grado n

il resto n-simo è $r_n(x) = f(x) - p_n(x) = o\left((x-x_0)^n\right)$

$\Rightarrow f(x) = p_n(x) + r_n(x)$

Es $f(x) = e^x \quad x_0 = 1 \quad n = 3$

$f(x) = e^x \qquad f(x_0) = e^1$
$f'(x) = e^x \qquad f'(x_0) = e$
$f''(x) = e^x \qquad f''(x_0) = e$
$f'''(x) = e^x \qquad f'''(x_0) = e$

$$P_3(x) = \frac{f(x_0)}{0!}(x-x_0)^0 + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$P_3(x) = \frac{e}{1}\underbrace{(x-1)^0}_{1} + \frac{e}{1}(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3$$

$$= e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3$$

$r_3(x) = o\left((x-1)^3\right)$ quando $x \to 1$

$f(x) = e^x \qquad x_0 = 0$

$$p_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} \qquad r_n(x) = o(x^n)$$

$$\left.\begin{aligned} n=0: \quad & e^x = 1 + o(1) \\ n=1 \quad & e^x = 1 + x + o(x) \\ n=2 \quad & e^x = 1 + x + \frac{x^2}{2} + o(x^2) \end{aligned}\right\} \quad e^x = p_n(x) + \underbrace{r_n(x)}_{o(x^n)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin x = x + o(x) = x + o(x^2)$$

$$\sin x = x - \frac{x^3}{3!} + o(x^4)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6)$$

$\cos x$? pol di Taylor con $x_0 = 0$

$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= +\sin x & f'''(0) &= 0 \\ f^{IV}(x) &= \cos x & f^{IV}(0) &= 1 \end{aligned}$$

$$P_4(x) = \frac{1}{1} \cdot x^0 + \cancel{\frac{0}{1!} x} + \frac{-1}{2!} x^2 + \cancel{\frac{0}{3!} x^3} + \frac{1}{4!} x^4$$

$$= 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4$$

$$\cos x = 1 + o(x)$$
$$\cos x = 1 - \frac{1}{2}x^2 + o(x^3)$$
$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o(x^5)$$
$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + o(x^7)$$

ES $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - \sin x}{1-\cos x}$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$
$$\Downarrow$$
$$1-\cos x \sim \frac{1}{2}x^2 \quad x\to 0$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$
$$\sin x = x - \frac{x^3}{3!} + o(x^4)$$
$$N = \cancel{1} + \cancel{x} + \frac{x^2}{2} + o(x^2) - \cancel{1} - \cancel{x} + \frac{x^3}{3!} + o(x^4) =$$
$$= \frac{x^2}{2} + \frac{x^3}{3!} + o(x^2) + o(x^4) =$$
$$= \underbrace{\frac{x^2}{2}}_{\text{parte principale}} + o(x^2)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3) \rightarrow 1-\cos x = \frac{x^2}{2} + o(x^3$$
$$D = 1 - \cos x = \cancel{1} - \cancel{1} + \frac{x^2}{2} + o(x^3) = \frac{x^2}{2} + o(x^3)$$

$$\lim_{x\to 0} \frac{e^x - 1 - \sin x}{1-\cos x} = \lim_{x\to 0} \frac{\frac{x^2}{2} + o(x^2)}{\frac{x^2}{2} + o(x^3)} = \lim_{x\to 0} \frac{x^2/2}{x^2/2} = 1$$

scrivere $1-\cos x \sim \frac{1}{2}x^2 \iff 1-\cos x = \frac{1}{2}x^2 + o(x^2)$

Come non lavorare

non ho una parte principale $\Rightarrow$ il lavoro fatto non dà info utile

$$\lim_{x\to 0} \frac{\overbrace{e^x-1}^{\sim x} - \overbrace{\sin x}^{\sim x}}{1-\cos x} = \lim_{x\to 0} \frac{\cancel{x}+o(x) - \cancel{x} + o(x)}{1-\cos x}$$

$e^x-1 \sim x$ per $x \to 0$ si ottiene anche da Taylor

utilizzando uno sviluppo di grado 1

$$e^x = 1+x+o(x) \Rightarrow e^x-1 = x+o(x)$$

$$\sin x = x + o(x)$$

$$\lim_{x\to 0} \frac{(e^x-1)\cdot \sin x}{1-\cos x} = \lim_{x\to 0} \frac{x \cdot x}{\frac{x^2}{2}+o(x^2)} =$$

$$\lim_{x\to 0} \frac{e^x-1-\sin x}{1-\cos x}$$

$e^x-1 = x+o(x) \iff e^x-1 \sim x$ per $x\to 0$

$\sin x = x+o(x) \iff \sin x \sim x$ per $x \to 0$

$$= \lim_{x\to 0} \frac{x^2}{\frac{x^2}{2}} = 2$$

$$\lim_{x\to 0} \frac{\underbrace{\frac{e^x-1}{x}}_{\to 1} - \underbrace{\cos x}_{\to 1}}{\underbrace{x+\sin x}_{\to 0}} = \frac{0}{0}$$

io so che $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

$$= \lim_{x\to 0} \frac{\frac{e^x - 1 - x\cos x}{x}}{x + \sin x} = \lim_{x\to 0} \frac{e^x - 1 - x\cos x}{x} \cdot \frac{1}{x+\sin x} =$$

$$= \lim_{x\to 0} \frac{e^x - 1 - x\cos x}{x^2 + x\sin x} = (*)$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3)$$

$$N = \cancel{1} + \cancel{x} + \frac{x^2}{2} + o(x^2) - \cancel{1} - \cancel{x} + \frac{x^3}{2} + o(x^4) = \frac{x^2}{2} + o(x^2)$$

$$\sin x = x - \underbrace{\frac{x^3}{3!}}_{\text{non serve}} + o(x^4)$$

$$D = x^2 + x^2 - \underbrace{\frac{x^4}{3!}}_{\text{non serve}} + o(x^5) = 2x^2 + o(x^3)$$

$$(*) \lim_{x\to 0} \frac{\frac{x^2}{2} + o(x^2)}{2x^2 + o(x^3)} = \lim_{x\to 0} \frac{x^2/2}{2x^2} = \frac{1}{4}$$

Sviluppo di $f(x) = \log(x+1)$ $\quad x_0 = 0 \quad n = 3$

$f(x) = \log(x+1)$ $\qquad f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2} = -(1+x)^{-2}$ $\qquad f''(0) = -1$

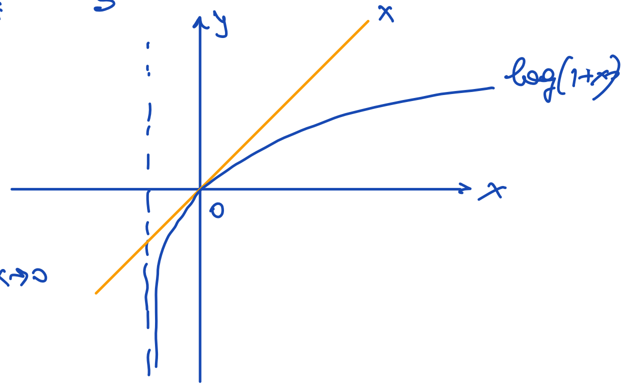
$f'''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3}$ $\qquad f'''(0) = 2$

$$P_3(x) = \cancel{\frac{0}{1} x^0} + \frac{1}{1}x + \frac{(-1)}{2!}x^2 + \frac{2}{3!}x^3$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots$$



$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

$\log(1+x) \sim x$ per $x \to 0$

$$\log(1+x) = x + o(x)$$

$$\lim_{x\to 0} \frac{\log(\cosh x)}{(\sin x)^2}$$

$$D = (\sin x)^2 = \left(x + o(x)\right)^2 = x^2 + o(x^2) + x \cdot o(x)$$
$$= x^2 + o(x^2)$$

$\sin x \sim x$ per $x \to 0$

$(\sin x)^2 \sim x^2$ per $x \to 0$

$$D = x^2 + o(x^2)$$

N: parto da $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) = x + o(x)$$

$$\log(\cosh(x)) = \log\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+o(x^5)\right)$$

semplifico

$$= \log\left(1+\frac{x^2}{2!}+o(x^3)\right)$$

$$= \left(\frac{x^2}{2}+o(x^3)\right)+o\left(\frac{x^2}{2}+o(x^3)\right) =$$

$$= \frac{x^2}{2}+\underbrace{o(x^3)+o(x^2)}= \frac{x^2}{2}+o(x^2)$$

$$\lim_{x\to 0}\frac{\log(\cosh x)}{(\sin x)^2} = \lim_{x\to 0}\frac{\frac{x^2}{2}+o(x^2)}{x^2+o(x^2)} = \frac{1}{2}$$

$$\log(\sinh x) = \log(\underbrace{x+o(x)}_{t})$$

$\sinh x = x+o(x)$

$\log(1+t)$

$$\lim_{n\to\infty}\frac{n^n\left[(n+2)!-2^n\right]\cdot\log\left(1+\frac{1}{(n+1)!}\right)}{(n+6)^n\cdot\left(\frac{1}{n}-\sin\frac{1}{n}\right)\sqrt{n^8+3}} \quad (*)$$

- n^n
- $\left[(n+2)!-2^n\right] \sim (n+2)!$ per $n\to\infty$

  succ geom

- $\log\left(1+\frac{1}{(n+1)!}\right) \sim \frac{1}{(n+1)!}$ per $n\to\infty$

  x

$\log(1+x) = x+o(x)$

$\log(1+x) \sim x$ per $x\to 0$

- $(n+6)^n$

- $\left(\frac{1}{n} - \sin\frac{1}{n}\right)$ $\quad n \to \infty$

$\underset{x}{\frac{1}{n}} = x$, $\quad \underset{x}{\frac{1}{n}} = x$

$= x - \sin x$ con $x \to 0$

$\sin x = x + o(x) \Rightarrow x - \sin x = \cancel{x} - \cancel{x} + o(x)$

non va bene

$\sin x = x - \frac{x^3}{3!} + o(x^3) \Rightarrow x - \sin x = \cancel{x} - \cancel{x} + \frac{x^3}{3!} + o(x^3)$

$$x - \sin x = \frac{x^3}{3!} + o(x^3) \quad \text{per } x \to 0$$

$$\frac{1}{n} - \sin\frac{1}{n} = \frac{1}{3!}\left(\frac{1}{n^3}\right) + o\left(\frac{1}{n^3}\right)$$

$$\frac{1}{n} - \sin\frac{1}{n} \sim \frac{1}{3!\, n^3} \quad \text{per } n \to \infty$$

- $\sqrt{n^8+3} \sim \sqrt{n^8} = n^4$ per $n \to \infty$

$$(*) = \lim_{n\to\infty} \frac{n^n \cdot \overbrace{(n+2)!}^{(n+2)\cancel{(n+1)!}} \frac{1}{\cancel{(n+1)!}}}{(n+6)^n \frac{1}{3!\,\cancel{n^3}} \cdot n^{\cancel{4}}} = \lim_{n\to\infty} \frac{n^n \cancel{(n+2)}}{(n+6)^n \frac{1}{6} \cancel{n}} =$$

$$= 6 \cdot \lim_{n\to\infty} \frac{n^n}{(n+6)^n} = 6 \cdot \lim_{n\to\infty} \frac{\cancel{n^n}}{e^6 \cancel{n^n}} = \frac{6}{e^6}$$

non è vero che $(n+6)^n \sim n^n$

perché $\lim_{n\to\infty} \frac{(n+6)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+6}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{6}{n}\right)^n = e^6$

$$\Rightarrow (n+6)^n \sim e^6 \cdot n^n$$

dire che $a_n \sim b_n$ per $n\to\infty$ $\equiv \lim_{n\to\infty} \frac{a_n}{b_n} = 1$

$a_n \sim l\cdot b_n$ per $n\to\infty$ $\equiv \lim_{n\to\infty} \frac{a_n}{b_n} = l$

$$\lim_{x\to 0} \frac{x\sin(3x) - 4\cosh(\sqrt{3}\,x) + 4 + 3x^2}{\log(1+2x^4)}$$

D: $\log(1+\underbrace{2x^4}_{\to 0}) \sim 2x^4$ per $x\to 0$

$$\log(1+\underbrace{2x^4}_{=t}) = 2x^4 + o(x^4) \qquad \log(1+t) = t + o(t)$$

- $\sin(3x) = 3x - \frac{(3x)^3}{6} + o\left((3x)^4\right)$ $\qquad \sin(t) = t - \frac{t^3}{6} + o(t^4)$

$$= 3x - \frac{\cancel{27}^{\,9}}{\cancel{6}_{\,2}}x^3 + o(x^4)$$

- $\cosh(\sqrt{3}\cdot x) = 1 + \frac{(\sqrt{3}\cdot x)^2}{2} + \frac{(\sqrt{3}\cdot x)^4}{4!} + o(x^5)$ $\qquad \cosh t = 1 + \frac{t^2}{2} + \frac{t^4}{4!} + o(t^5)$

$$= 1 + \frac{3}{2}x^2 + \frac{\cancel{9}^{\,3}}{\cancel{24}_{\,8}}x^4 + o(x^5)$$

$$\lim_{x\to 0} \frac{x\sin(3x) - 4\cosh(\sqrt{3}\,x) + 4 + 3x^2}{\log(1+2x^4)} =$$

$$= \lim_{x\to 0} \frac{x\left(3x - \frac{9}{2}x^3 + o(x^4)\right) - 4\left(1 + \frac{3}{2}x^2 + \frac{3}{8}x^4 + o(x^5)\right) + 4 + 3x^2}{2x^4}$$

$$= \lim_{x\to 0} \frac{\cancel{3x^2} - \frac{9}{2}x^4 + o(x^5) - \cancel{4} - \cancel{6x^2} - \frac{3}{2}x^4 + o(x^5) + \cancel{4} + \cancel{3x^2}}{2x^4} =$$

$$= \lim_{x\to 0} \frac{-6x^4 + \cancel{o(x^5)}}{2x^4} = -3$$

$$\lim_{x\to +\infty} \frac{3\cos\left(\frac{1}{x}\right)\cdot\left(\sin\left(\frac{1}{x}\right) - \sinh\left(\frac{1}{x}\right)\right)}{4\left(e^{1/x}-1\right)^4} = \frac{0}{0}$$

$t = \frac{1}{x}$

$$\lim_{t\to 0^+} \frac{3\cdot\underbrace{\cos(t)}_{\to 1}\cdot\underbrace{(\sin t - \sinh t)}_{\to 0}}{4\underbrace{(e^t-1)^4}_{\to 0}} = \underbrace{\lim_{t\to 0^+} 3\cos t}_{=3} \cdot \lim_{t\to 0^+} \frac{\sin t - \sinh t}{4(e^t-1)^4} \overset{=}{\circledast}$$

D: $e^t - 1 \sim t$ per $t\to 0$ $\Rightarrow$ $4(e^t-1)^4 \sim 4t^4$ per $t\to 0$

N: $\sin t = t - \frac{t^3}{3!} + o(t^4)$ $\qquad \sinh t = t + \frac{t^3}{3!} + o(t^4)$

$$\sin t - \sinh t = \cancel{t} - \frac{t^3}{3!} + o(t^4) - \cancel{t} - \frac{t^3}{3!} + o(t^4) =$$

$$= -\frac{1}{3}t^3 + o(t^4)$$

$$\overset{\circledast}{=} 3\cdot\lim_{t\to 0^+} \frac{-\frac{1}{3}t^3 + \cancel{o(t^4)}}{4t^4} = 3\cdot\lim_{t\to 0^+} \frac{-\frac{1}{3}\cancel{t^3}}{t^{\cancel{4}}} = -\lim_{t\to 0^+}\frac{1}{t} = -\infty$$