Esercizi

- Calcolare la media integrale di

$$f(x) = \sqrt{x}\cdot\arctan\left(\sqrt{x^3}\right) \quad \text{in } [1, 3]$$

$$\text{media integrale} = \fint_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\fint_1^3 \sqrt{x}\cdot\arctan\left(\sqrt{x^3}\right)dx = \frac{1}{2}\int_1^3 \underbrace{\sqrt{x}\cdot\arctan\left(\sqrt{x^3}\right)}_{f(x)}dx$$

posso calcolare una primitiva $G(x)$ di $f(x)$ e applicare il 2° thm del calcolo:

↑ continua

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

$$\text{Calcolo } G(x) = \int \underbrace{\sqrt{x}}_{f'}\cdot\underbrace{\arctan\left(\sqrt{x^3}\right)}_{g}dx = \circledast$$

$$\left(\int f'g = fg - \int fg'\right)$$

$$f'(x) = \sqrt{x} = x^{1/2} \qquad f(x) = \frac{1}{\frac{1}{2}+1}\cdot x^{1/2+1}$$

$$\int x^{\alpha}dx = \frac{1}{\alpha+1}x^{\alpha+1}$$

$$f(x) = \frac{2}{3}x^{3/2}$$

$$g(x) = \arctan\left(\sqrt{x^3}\right) \qquad g'(x) = \frac{1}{1+x^3}\cdot\frac{3}{2}\cdot x^{1/2}$$

$$D\left(\sqrt{x^3}\right) = D\left(x^{3/2}\right) = \frac{3}{2}x^{1/2}$$

$$\circledast = \frac{2}{3}x^{3/2}\cdot\arctan\left(\sqrt{x^3}\right) - \underbrace{\int \cancel{\frac{2}{3}}\,\overbrace{x^{3/2}\,\frac{1}{1+x^3}\cdot\cancel{\frac{3}{2}}\,x^{1/2}}^{x^2}dx}_{I_1} = \circledast\circledast$$

$$I_1 = \int \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int \frac{\overset{dy}{\boxed{3x^2\,dx}}}{\underset{y}{\boxed{1+x^3}}} = \frac{1}{3}\int \frac{dy}{y} = \frac{1}{3}\log|y| = \frac{1}{3}\log|1+x^3|$$

$$y = 1+x^3$$
$$dy = 3x^2\,dx$$

$$\boxed{**} = \frac{2}{3}x^{3/2}\cdot \operatorname{atan}\left(x^{3/2}\right) - \frac{1}{3}\log|1+x^3| = G(x)$$

$$\Rightarrow \int_1^3 \sqrt{x}\cdot \operatorname{atan}\left(\sqrt{x^3}\right)dx = \frac{1}{2}\int_1^3 \underbrace{\sqrt{x}\cdot \operatorname{atan}\left(\sqrt{x^3}\right)dx}_{f(x)} =$$

$$= \frac{1}{2}\left[\frac{2}{3}x^{3/2}\operatorname{atan}\left(x^{3/2}\right) - \frac{1}{3}\log|1+x^3|\right]_1^3 =$$

$$= \frac{1}{2}\left[\frac{2}{\cancel{3}}\cdot\cancel{3}\sqrt{3}\,\operatorname{atan}\left(3\sqrt{3}\right) - \frac{1}{3}\log(28) - \frac{2}{3}\cdot\underbrace{\operatorname{atan}(1)}_{\frac{\pi}{4}} + \frac{1}{3}\log 2\right]$$

$$= \sqrt{3}\,\operatorname{atan}\left(3\sqrt{3}\right) + \frac{1}{6}\cdot\log\frac{1}{14} - \frac{1}{3}\cdot\frac{\pi}{4} \quad -$$

$$\log a - \log b = \log\frac{a}{b}$$

$$\int \frac{3x^2\,dx}{1+x^3} = \int \frac{\overset{dy}{\boxed{\varphi'(x)\,dx}}}{\underset{y}{\boxed{\varphi(x)}}} = \log\left|\overset{y}{\boxed{\varphi(x)}}\right| + c$$

$$\int \frac{\overset{\varphi'(x)}{\boxed{\cos x}}}{\underset{\varphi(x)}{\boxed{\sin x}}}\,dx = \log|\sin x| + c$$

$$\int \frac{\overset{\varphi'(x)}{\boxed{1}}}{\underset{\varphi(x)}{\boxed{x+3}}}\,dx = \log|x+3| + c$$

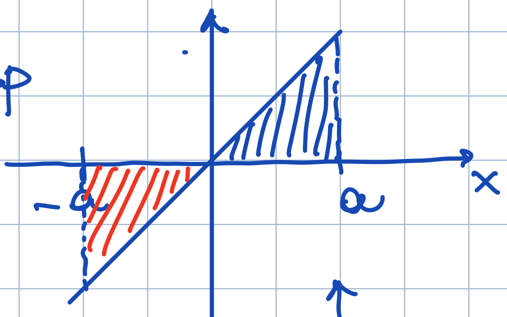
$$\frac{1}{2}\int \frac{1\cdot 2}{\underset{\varphi(x)}{\boxed{2x+3}}}\,dx = \frac{1}{2}\int \frac{2dx}{2x+3} = \frac{1}{2}\log|2x+3| + c$$

- Calcolare $I = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \underbrace{(\sin x + x^3)}_{f(x)\text{ è dispari}}\,dx$

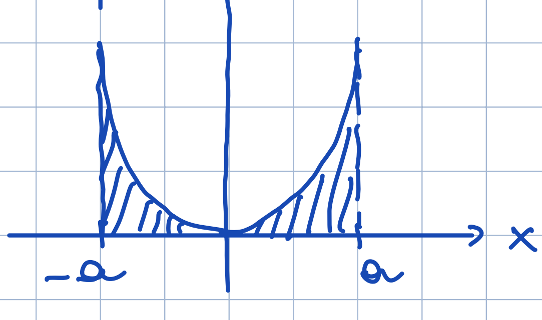
Proprietà se $f(x)$ è dispari $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

Se $f(x)$ è pari $\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Es : f disp



f pari



Rigorosamente : f dispari

$$\int_{-a}^{a} f(x)\,dx \underset{\substack{\uparrow\\ \text{add.}}}{=} \underbrace{\int_{-a}^{0} f(x)\,dx}_{I_-} + \underbrace{\int_{0}^{a} f(x)\,dx}_{I_+} = 0$$

$$I_- = \int_{-a}^{0} f(x)\,dx \underset{\substack{\uparrow\\ f(x)=-f(-x)}}{=} \int_{-a}^{0} \overbrace{-f(\underset{\overset{\|}{t}}{-x})\,dx}^{dt} \underset{\substack{\uparrow\\ t=-x\\ dt=-dx}}{=} \int_{a}^{0} f(t)\,dt =$$

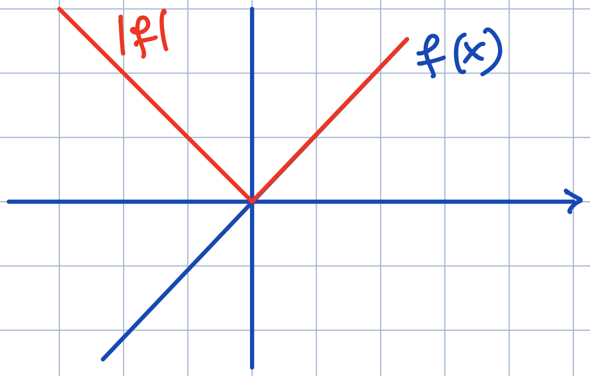
se $x=0 \Rightarrow t=0$

se $x=-a \Rightarrow t=+a$

$$= -\int_{0}^{a} f(\underset{x}{t})\,d\underset{x}{t} = -\int_{0}^{a} f(x)\,dx$$

$$= -I_+$$

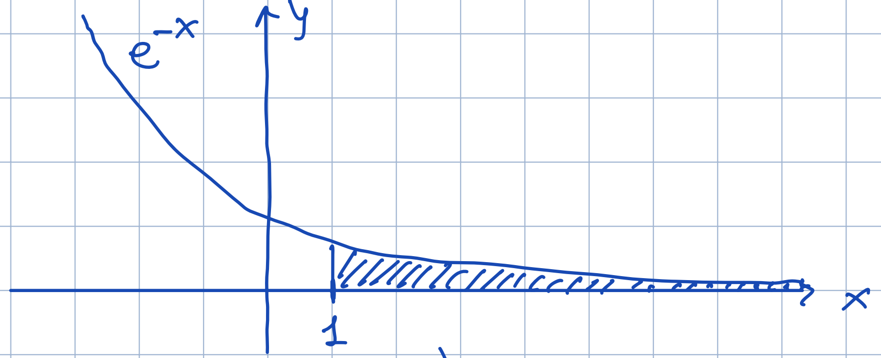
Oss : Se f è dispari $\Rightarrow$ $|f|$ è pari



- $$\int_1^{+\infty} e^{-x}\,dx = \lim_{x\to+\infty} \boxed{\int_1^x e^{-t}\,dt} = \lim_{x\to+\infty}\Big(\underbrace{-e^{-x}}_{\downarrow\, 0} + e^{-1}\Big) = \frac{1}{e}$$

$$\int_1^x e^{-t}\,dt = \Big[-e^{-t}\Big]_1^x = \overbrace{-e^{-x} + e^{-1}}$$

se $f(t) = e^{-t}$ $\quad G(t) = -e^{-t}$



l'integrale dato converge

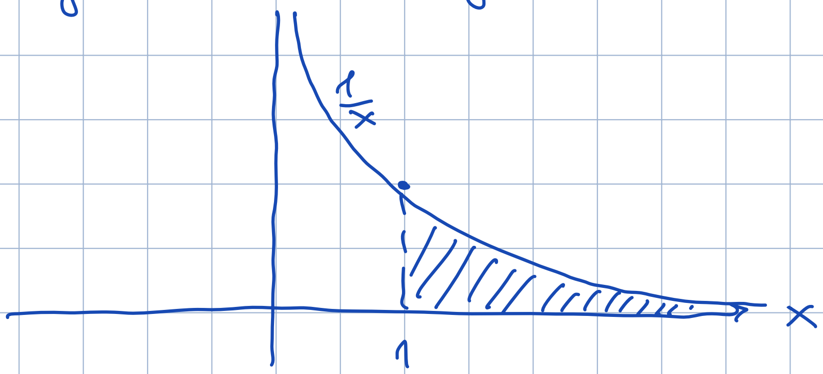
- $$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{x\to+\infty}\int_1^x \frac{1}{t}\,dt = \lim_{x\to+\infty} \log|x| = +\infty$$

$$\int_1^x \frac{1}{t}\,dt = \int_1^x t^{-1}\,dt = \Big[\log|t|\Big]_1^x = \log|x| - \overset{0}{\boxed{\log 1}}$$

$$\int t^{\alpha}\,dt = \frac{1}{\alpha+1}\,t^{\alpha+1} + c \quad \text{se } \alpha \neq -1$$

$$\int \frac{1}{t}\,dt = \log|t| + c$$

Questo integrale è divergente



$$\int_1^{+\infty} \frac{1}{x}\,dx = +\infty \qquad \sum_{n=1}^{\infty} \frac{1}{n} \text{ è divergente}$$

- $$\int_0^{+\infty} \sin x\,dx = \lim_{x\to+\infty} \int_0^x \sin t\,dt = \lim_{x\to+\infty} \left[-\cos t\right]_0^x =$$

$$= \lim_{x\to+\infty} \left(-\cos x + \cos 0\right) = \nexists$$

l'integrale dato è oscillante

? Area del trapezoide sotteso a $f(x) = \dfrac{e^{2x}}{4e^{2x}+4e^x+1}$ sull'intervallo $[0,2]$

$$A = \int_0^2 |f(x)|\,dx$$

Osservo che
$f(x) > 0 \quad \forall x \in dom(f)$ perché sia a numeratore che a denom ho esp > 0 $\Rightarrow$

$$|f(x)| = f(x) \quad \forall x \in dom(f) \equiv \mathbb{R}$$

$$\Rightarrow A = \int_0^2 f(x)\,dx = \int_0^2 \frac{e^{2x}\,dx}{4e^{2x}+4e^x+1} \underset{\substack{\uparrow \\ y=e^x}}{=} \int_1^{e^2} \frac{y}{4y^2+4y+1}\,dy$$

$e^{2x} = e^x \cdot e^x$ $\qquad$ $e^x\,dx = dy$

$$e^{2x} = (e^x)^2 = y^2$$

$y = e^x$
$dy = e^x\,dx$
se $x = 0 \Rightarrow y = e^0 = 1$
se $x = 2 \Rightarrow y = e^2$

ho una funz. razionale a fratta

Cerco $\int \overset{\textcircled{+}}{\frac{y}{4y^2+4y+1}}\,dy = \frac{1}{8}\int \frac{(8y+4)\overset{\downarrow}{-4}}{4y^2+4y+1}\,dx =$

$$D(4y^2+4y+1) = 8y+4$$

$$= \underbrace{\frac{1}{8}\int \frac{\overset{\varphi'(y)}{\boxed{8y+4}}}{\underset{\varphi(y)}{\boxed{4y^2+4y+1}}}\,dy}_{G_1(y)} - \underbrace{\frac{1}{\underset{2}{\cancel{8}}}\cdot \cancel{4}\int \frac{dy}{4y^2+4y+1}}_{G_2(y)} = \textcircled{*}$$

$$G_1(y) = \frac{1}{8}\int \frac{8y+4}{4y^2+4y^2+1}\,dy = \frac{1}{8}\log(4y^2+4y+1)$$

$$G_2(y) = \frac{1}{2}\cdot\frac{1}{2}\int \frac{2\,dy}{(2y+1)^2} = \frac{1}{4}\int \frac{dt}{t^2} = -\frac{1}{4t} = -\frac{1}{4(2y+1)}$$

$t = 2y+1$

$dt = 2\,dy$

$$\int t^{-2} = \frac{1}{-2+1}t^{-2+1} = -\frac{1}{t}$$

$$\textcircled{*} = \frac{1}{8}\log\overbrace{(4y^2+4y+1)}^{(2y+1)^{\textcircled{2}}} + \frac{1}{4(2y+1)}$$

$$= \frac{1}{4}\log|2y+1| + \frac{1}{4(2y+1)} = \text{primitiva di } \textcircled{+}$$

$$\Rightarrow A = \int_1^{e^2} \frac{y\,dy}{4y^2+4y+1} = \left[\frac{1}{4}\log|2y+1| + \frac{1}{4(2y+1)}\right]_1^{e^2} =$$

$$= \frac{1}{4}\log(2e^2+1) + \frac{1}{4(2e^2+1)} - \frac{1}{4}\log 3 - \frac{1}{12}$$

- Calcolare la primitiva $G(x)$ di $f(x) = \frac{1}{2x^2+3}$ t.c. $G(0)=1$

$$\int \frac{1}{2x^2+3}\,dx =$$

$$t = \sqrt{\frac{2}{3}}\cdot x$$

$$dt = \sqrt{\frac{2}{3}}\,dx$$

$$= \sqrt{\frac{3}{2}}\,\frac{1}{3}\int \frac{1}{\left(\underbrace{\sqrt{\frac{2}{3}}x}_{t}\right)^2+1}\,\underbrace{\sqrt{\frac{2}{3}}\,dx}_{dt} = \frac{1}{\sqrt{6}}\int \frac{dt}{t^2+1} = \frac{1}{\sqrt{6}}\operatorname{arctg}(t) + c$$

$$= \underbrace{\frac{1}{\sqrt{6}}\operatorname{arctg}\left(\sqrt{\frac{2}{3}}x\right) + c}_{}$$

$$= G(x)$$

$$\frac{1}{2x^2+3} = \frac{1}{3}\,\frac{1}{\frac{2}{3}x^2+1} = \frac{1}{3}\,\frac{1}{\left(\sqrt{\frac{2}{3}}\cdot x\right)^2+1}$$

$$\frac{2}{3}x^2 = t^2 \qquad t = \sqrt{\frac{2}{3}}\cdot x$$

$?c:\quad G(0) = 1 \qquad G(0) = \frac{1}{\sqrt{6}}\cdot\underbrace{\operatorname{arctg}(0)}_{0} + c = 1$

$$\Rightarrow c = 1$$

$$G(x) = \frac{1}{\sqrt{6}}\operatorname{arcTg}\left(\sqrt{\frac{2}{3}}x\right) + 1$$

Sia $f(x) = 2x\sqrt{x^2+1}$

1) calcolare $\int_0^1 f(x)\,dx$

2) Area del trapezoide sotteso a $f(x)$ su $[-1,1]$

1) $\int_0^1 \overbrace{\boxed{2x}\sqrt{x^2+1}\,\boxed{dx}}^{dy} =$ $\int_1^2 \sqrt{y}\,dy = \int_1^2 y^{1/2}\,dy = \left[\frac{2}{3}y^{3/2}\right]_1^2$

$y = x^2+1$

$dy = 2x\,dx$

se $x = 0 \Rightarrow y = 1$

se $x = 1 \Rightarrow y = 2$

$\int y^{1/2} = \frac{2}{3}y^{3/2}$

$= \frac{2}{3}\sqrt{8} - \frac{2}{3} = \frac{2}{3}\left(\sqrt{8} - 1\right)$

2) $A = \int_{-1}^{1} |f(x)|\,dx$

$f(x) = 2x\sqrt{x^2+1}$

$f(-x) = -2x\sqrt{x^2+1} = -f(x)$ f è dispari

$\Rightarrow |f(x)|$ è pari

$= 2\int_0^1 |f(x)|\,dx = 2\int_0^1 f(x)\,dx = \frac{4}{3}\left(\sqrt{8} - 1\right)$

se $x \geq 0$

$|f(x)| = f(x)$

$$\bullet \int_0^4 \frac{x-1}{\sqrt{x}+2}\,dx = \int_0^4 \frac{(\overset{y^2}{\boxed{x}}-1)}{\underset{y}{\boxed{\sqrt{x}}}+2}\, 2\overset{y}{\boxed{\sqrt{x}}}\, \underset{dy}{\boxed{\frac{1}{2\sqrt{x}}\,dx}} = \int_0^2 \frac{(y^2-1)\,\boxed{2}y\,dy}{y+2}$$

$$y=\sqrt{x} \rightarrow x = y^2$$

$$dy = \frac{1}{2\sqrt{x}}\,dx$$

se $x=0 \Rightarrow y=0$

se $x=4 \Rightarrow y=2$

$$= 2\int_0^2 \frac{y^3-y}{y+2}\,dy$$

funz fratta con grado num > grado den.

devo dividere y^3-y per $y+2$

$$\begin{array}{rrrr|l}
y^3 & +0y^2 & -y & +0 & y^2-2y+3 \\
\cline{5-5}
-y^3 & -2y^2 & & & y+2 \\
\cline{1-4}
// & -2y^2 & -y & & \uparrow \\
 & +2y^2 & +4y & & \\
\cline{2-4}
 & // & 3y & & \\
 & & -3y & -6 & \\
\cline{2-4}
 & & // & -6 \nwarrow &
\end{array}$$

$$\frac{y^3-y}{y+2} = y^2-2y+3 \quad - \frac{6}{y+2}$$

$$\int \frac{y^3-y}{y+2}\,dy = \int \left(y^2-2y+3 - \frac{6}{y+2}\right) dy =$$

$$= \frac{1}{3}y^3 - y^2 + 3y - 6\log|y+2|$$

$$\int \frac{6}{y+2}\,dy = 6\int \frac{1}{y+2}\,dy = 6\log|y+2|$$

$$\mathbf{II} = 2\left[\frac{1}{3}y^3 - y^2 + 3y - 6\log|y+2|\right]_0^2$$

$$= 2\left(\frac{1}{3}8 - 4 + 6 - 6\log 4 + 6\log 2\right)$$

$$= \frac{28}{3} - 6\log 2$$