Sviluppi di Maclaurin notevoli

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + o(x^n) = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2k+1}}{(2k+1)!} + o(x^{2k+2})$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2k}}{(2k)!} + o(x^{2k+1})$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2k+2})$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2k+1})$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^5)$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2k+1})$$

$$\rightarrow \quad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + \underline{(-1)^n x^n} + o(x^n)$$

$\boxed{\alpha = -1}$

$$\rightarrow \quad \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + o(x^3)$$

$\sqrt{1+x} = (1+x)^{1/2}$ $\quad \alpha = \frac{1}{2}$

$$\rightarrow \quad \boxed{(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \binom{\alpha}{n} x^n + o(x^n)}$$

dove $\quad D(x^\alpha) = \alpha x^{\alpha-1}$

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \forall \alpha \in \mathbb{R}, \ n \geq 1$$

$f(x) = \frac{1}{1+x}$ $\quad n = 3 \quad x_0 = 0$

? pol di Taylor $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

quando $x_0 = 0$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

$f(x) = \frac{1}{1+x} = (1+x)^{-1}$ $\qquad f(0) = 1$

$f'(x) = -1 \cdot (1+x)^{-2}$ $\qquad f'(0) = -1$

$f''(x) = (-1)\cdot(-2)(1+x)^{-3}$ $\qquad f''(0) = +2$

$f'''(x) = (-1)(-2)\cdot(-3)(1+x)^{-4}$ $\qquad f'''(0) = -6$

$$P_3(x) = 1 + (-1)x + \frac{\cancel{2}^{\,1}}{\cancel{2}}x^2 + \frac{-\cancel{6}}{\cancel{6}}x^3 = 1 - x + x^2 - x^3$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 +$$

$(\alpha-4+1)$

$$+ \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}x^4 + \dots\dots$$

$$+ \frac{\alpha(\alpha-1)\dots\dots(\alpha-n+1)}{n!}x^n$$

- $\lim_{x\to 0} \dfrac{x\left(e^x - 1 - x\cos x\right)}{3x - \operatorname{arctg}(3x)} \overset{(*)}{=} \dfrac{0}{0}$

DEN: $\operatorname{arctg}(3x) = 3x - \dfrac{(3x)^3}{3} + o(x^4)$ | $\operatorname{arctg} t = t - \dfrac{t^3}{3} + o(t^4)$

$$3x - \operatorname{arctg}(3x) = \cancel{3x} - \cancel{3x} + \underline{\underline{9x^3}} + o(x^4)$$

$$= \lim_{x\to 0} \frac{\cancel{x}\left(e^x - 1 - x.\cos x\right)}{9x^{\cancel{3}\,2}} =$$

NUM: $e^x = 1 + x + \dfrac{x^2}{2} + o(x^2)$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3)$$

$$\text{NUM} = \cancel{1} + \cancel{x} + \frac{x^2}{2} + o(x^2) - \cancel{1} - \cancel{x} + \frac{x^3}{2} + o(x^4)$$

$$= \frac{x^2}{2} + o(x^2)$$

$$= \lim_{x\to 0} \frac{x^2/2}{9x^2} = \frac{1}{18}$$

- $$\lim_{x\to 0} \frac{e^{x^2/2} - \cos x - x^2}{x^4} =$$

- $$e^{x^2/2} = 1 + \frac{x^2}{2} + \frac{1}{2}\left(\frac{x^2}{2}\right)^2 + o(x^4) \qquad e^t = 1 + t + \frac{t^2}{2} + o(t^2)$$

- $$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$$

$$= \lim_{x\to 0} \frac{\cancel{1} + \cancel{\frac{x^2}{2}} + \frac{1}{8}x^4 + o(x^4) - \cancel{1} + \cancel{\frac{x^2}{2}} - \frac{x^4}{24} + o(x^4) - \cancel{x^2}}{x^4}$$

$$= \lim_{x\to 0} \frac{\left(\frac{1}{8} - \frac{1}{24}\right)x^4 + o(x^4)}{x^4} = \frac{1}{8} - \frac{1}{24} = \frac{2}{24} = \frac{1}{12}$$

- $$\lim_{x\to +\infty} \frac{\overset{\to 1}{3\cos\left(\frac{1}{x}\right)}\left(\overset{\to 0}{\sin\frac{1}{x}} - \overset{\to 0}{\sinh\frac{1}{x}}\right)}{4\left(e^{\overset{\to 0}{1/x}} - 1\right)^4} = \frac{0}{0}$$

$t = \frac{1}{x}$ se $x \to +\infty \Rightarrow t \to 0^+$

$$= \lim_{t\to 0^+} \frac{3\,\overset{\to 1}{\boxed{\cos t}}\,(\sin t - \sinh t)}{4(e^t - 1)^4} =$$

$$= \boxed{\lim_{t\to 0^+} \frac{3\cos t}{4}} \cdot \lim_{t\to 0^+} \frac{\sin t - \sinh t}{(e^t-1)^4} =$$

$$\frac{3}{4}$$

$$= \frac{3}{4} \cdot \lim_{t\to 0^+} \frac{\sin t - \sinh t}{(e^t-1)^4}$$

DEN: $e^t - 1 \sim t$ quando $t \to 0$

$(e^t-1)^4 \sim t^4$ $t \to 0$

$$= \frac{3}{4} \lim_{t\to 0^+} \frac{\sin t - \sinh t}{t^4} =$$

$$\sin t = t - \frac{t^3}{3!} + o(t^4)$$

$$\sinh t = t + \frac{t^3}{3!} + o(t^4)$$

$$\sin t - \sinh t = \cancel{t} - \frac{t^3}{3!} + o(t^4) - \cancel{t} - \frac{t^3}{3!} + o(t^4)$$

$$= -\frac{\cancel{2}^1}{\cancel{6}_3} t^3 + o(t^4)$$

$$= \frac{3}{4} \cdot \lim_{t\to 0^+} \frac{-\frac{1}{3}t^3}{t^4} =$$

$$= -\frac{1}{4} \lim_{t\to 0^+} \frac{1}{t} = -\frac{1}{4} \cdot +\infty = -\infty$$

- $$\lim_{x\to 2^+} \frac{\sqrt[3]{x-2} - 3\log^2\left(1+\sqrt[3]{x-2}\right)}{\left(\sinh(x-2) - \sin(x-2)\right)^\beta} = \qquad \text{con } \beta\in\mathbb{R}$$

$t = x-2$ se $x\to 2^+ \Rightarrow t\to 0^+$

$$= \lim_{t\to 0^+} \frac{\sqrt[3]{t} - 3\log^2\left(1+\sqrt[3]{t}\right)}{\left(\sinh t - \sin t\right)^\beta} \qquad \sqrt[3]{t} = t^{1/3}$$

$\underline{\text{NUM}} = t^{1/3} - 3\left[\log\left(1+t^{1/3}\right)\right]^2 \overset{(*)}{=} \qquad \log(1+x) = x - \frac{x^2}{2} + o(x^2)$

$$\log\left(1+t^{1/3}\right) = t^{1/3} - \frac{t^{2/3}}{2} + o\left(t^{2/3}\right)$$

$$= t^{1/3} + o\left(t^{1/3}\right)$$

$$\left[\log\left(1+t^{1/3}\right)\right]^2 = \left[t^{1/3} + o\left(t^{1/3}\right)\right]^2 = t^{2/3} + \underline{o\left(t^{2/3}\right)}$$

doppio prodotto e quadrato di o

$$\overset{(*)}{=} t^{1/3} - 3t^{2/3} + o\left(t^{2/3}\right)$$

$$= t^{1/3} + o\left(t^{1/3}\right)$$

$$\text{DEN} = \left(\sinh t - \sin t\right) = \left(\cancel{t} + \frac{t^3}{3!} + o(t^4) - \cancel{t} + \frac{t^3}{3!} + o(t^4)\right)$$

$$= \frac{1}{3}t^3 + o\left(t^4\right)$$

$$\sim \frac{1}{3}t^3 \qquad \text{per } t\to 0$$

$$\left(\sinh t - \sin t\right)^{\beta} \sim \left(\frac{1}{3}t^{3}\right)^{\beta} \quad \text{per } t \to 0$$

$$= \frac{1}{3^{\beta}} \cdot t^{3\beta}$$

$$\text{limite dato} = \lim_{t \to 0^+} \frac{t^{1/3}}{\frac{1}{3^{\beta}} \cdot t^{3\beta}}$$

$$= 3^{\beta} \cdot \lim_{t \to 0^+} t^{\frac{1}{3} - 3\beta} = \begin{cases} 0 & \frac{1}{3} - 3\beta > 0 \iff \beta < \frac{1}{9} \\ 3^{1/9} \cdot \cancel{1} & \frac{1}{3} - 3\beta = 0 \iff \beta = \frac{1}{9} \\ +\infty & \frac{1}{3} - 3\beta < 0 \iff \beta > \frac{1}{9} \end{cases}$$

- $$\lim_{n \to \infty} \frac{\cos\left(\frac{1}{n}\right) - \cosh\left(\frac{1}{n}\right)}{n^{\beta}} \overset{(*)}{=} \qquad \text{con } \beta \in \mathbb{R}$$

se $n \to \infty \Rightarrow \frac{1}{n} \to 0 \Rightarrow \cos\left(\frac{1}{n}\right)$ può essere sviluppato

ricordando $\cos x = 1 - \frac{x^2}{2} + o(x^3)$, dove $x = \frac{1}{n}$

$$\cos\left(\frac{1}{n}\right) = 1 - \frac{1}{2}\left(\frac{1}{n}\right)^2 + o\left(\frac{1}{n^3}\right)$$

Idem per $\cosh\left(\frac{1}{n}\right)$:

$$\cosh x = 1 + \frac{x^2}{2} + o(x^3)$$

$$\cosh\left(\frac{1}{n}\right) = 1 + \frac{1}{2}\left(\frac{1}{n}\right)^2 + o\left(\frac{1}{n^3}\right) \qquad \left(\frac{1}{n}\right)^2 = \frac{1}{n^2}$$

• $\cos\left(\frac{1}{n}\right) - \cosh\left(\frac{1}{n}\right) = \cancel{1} - \frac{1}{2n^2} + o\left(\frac{1}{n^3}\right) - \cancel{1} - \frac{1}{2n^2} + o\left(\frac{1}{n^3}\right)$

$= -\frac{1}{n^2} + o\left(\frac{1}{n^3}\right)$

$$\stackrel{(*)}{=} \lim_{n\to\infty} \frac{-\frac{1}{n^2}}{n^\beta}$$

$$= -\lim_{n\to\infty} \frac{1}{n^{\beta+2}} = \begin{cases} 0 & \text{se } \beta+2>0 \iff \beta>-2 \\ -1 & \text{se } \beta+2=0 \iff \beta=-2 \\ -\infty & \text{se } \beta+2<0 \iff \beta<-2 \end{cases}$$

• $$\lim_{x\to 0^+} \frac{\log\left(1-\underbrace{\cos(3x^2)}_{\to 1} - \frac{9}{2}\underbrace{x^4}_{\to 0}\right)}{\underbrace{\log x}_{\to -\infty}} = \frac{-\infty}{-\infty}$$

? de l'Hopital

$$\stackrel{(H)}{=} \lim_{x\to 0^+} \frac{\frac{1}{1-\cos(3x^2)-\frac{9}{2}x^4} \cdot \left(\sin(3x^2)\cdot 6x - \frac{9}{\cancel{2}} \cdot \cancel{4}^{2} x^3\right)}{\frac{1}{x}} =$$

$$= \lim_{x\to 0^+} \frac{6x \cdot x\left(\sin(3x^2)\cdot\cancel{6x} - \cancel{18}^{3}\, x^{\cancel{3}\,2}\right)}{1-\cos(3x^2)-\frac{9}{2}x^4}$$

$$= \lim_{x\to 0^+} \frac{6x^2\left(\sin(3x^2) - 3x^2\right)}{1-\cos(3x^2) - \frac{9}{2}x^4}$$

NUM: $\sin(3x^2) = 3x^2 - \frac{(3x^2)^3}{6} + o(x^6)$

$$\sin(3x^2) - 3x^2 = -\frac{\cancel{27}^{\,9}}{\cancel{6}_{\,2}}x^6 + o(x^6)$$

$$6x^2\left(\sin(3x^2) - 3x^2\right) = -27x^8 + o(x^8)$$

DEN : $\cos(3x^2) = 1 - \frac{(3x^2)^2}{2} + o(x^4)$ sviluppo è troppo grossolano

$$1-\cos(3x^2) - \frac{9}{2}x^4 = \cancel{1} - \cancel{1} + \cancel{\frac{9}{2}x^4} + o(x^4) - \cancel{\frac{9}{2}x^4}$$

devo considerare un termine in più

$$\cos(3x^2) = 1 - \frac{(3x^2)^2}{2} + \frac{(3x^2)^4}{24} + o(x^8)$$

$$1-\cos(3x^2) - \frac{9}{2}x^4 = \cancel{1} - \cancel{1} + \cancel{\frac{9x^4}{2}} - \frac{3^{\cancel{4}\,3}}{\cancel{3}\cdot 8}x^8 + o(x^8) - \cancel{\frac{9}{2}x^4}$$

$$= -\frac{27}{8}x^8 + o(x^8)$$

$$\text{lim dato} = \lim_{x\to 0^+} \frac{-27x^8}{-\frac{27}{8}x^8} = 8$$

• $\lim_{n\to\infty} \sin\left(\frac{2^n}{n}\right) \cdot \underbrace{\frac{\sqrt{1+\frac{1}{n^2}}-1}{n^\alpha\cdot\left(\sinh\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n}\right)\right)}}_{b_n}$ $\forall\alpha\in\mathbb{R}$

$\frac{2^n}{n} \to +\infty$

indeterminata
non limitata

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^2}}-1}{n^\alpha\left(\underbrace{\sinh\left(\frac{1}{n}\right)-\sin\frac{1}{n}}_{\to 0}\right)} = \frac{0}{?}$$

n^α ?

NUM: $\sqrt{1+\frac{1}{n^2}}-1=$

$= \cancel{1}+\frac{1}{2}\left(\frac{1}{n^2}\right)+o\left(\frac{1}{n^2}\right)-\cancel{1}$

$= \frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)$

$\sqrt{1+x} = (1+x)^{1/2} =$

$= 1+\frac{1}{2}x+o(x)$

$x=\frac{1}{n^2}$

DEN $\sinh\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n}\right)=$

$= \cancel{\frac{1}{n}}+\frac{1}{6}\left(\frac{1}{n}\right)^3+o\left(\frac{1}{n^3}\right)-\cancel{\frac{1}{n}}+\frac{1}{6}\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)$

$= \frac{1}{3n^3}+o\left(\frac{1}{n^3}\right)$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\frac{1}{2n^2}}{n^\alpha\cdot\frac{1}{3n^3}} = \frac{3}{2}\lim_{n\to\infty}\frac{1}{n^{\alpha+2-3}} =$$

$$= \frac{3}{2} \lim_{n\to\infty} \frac{1}{n^{\alpha-1}} = \begin{cases} 0 & \alpha-1>0 \iff \alpha>1 \\ \frac{3}{2} & \alpha-1=0 \iff \alpha=1 \\ +\infty & \alpha-1<0 \iff \alpha<1 \end{cases}$$

Il limite dato $\exists$ ed è $=0$ solo se $\alpha > 1$ –

$\sin\left(\frac{2^n}{n}\right) \cdot b_n$ infinitesima = succ. α-sima

(corollario al 2° thm del confronto).

Quando $\alpha \leq 1 \Rightarrow$ il limite della succ data non esiste