$f(x) = \cos x$ sviluppo di Maclaurin = Taylor con $x_0 = 0$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$n = 4$, $x_0 = 0$

$$P_4(x) = \frac{f^{(0)}(0)}{0!}x^0 + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$f(x) = \cos x$ $\quad f(0) = 1$

$f'(x) = -\sin x$ $\quad f'(0) = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f'''(x) = +\sin x$ $\quad f'''(0) = 0$

$f^{IV}(x) = \cos x$ $\quad f^{IV}(0) = 1$

$$P_4(x) = \frac{1}{1!}\cdot 1 + 0\cdot x - \frac{1}{2!}x^2 + 0\cdot x^3 + \frac{1}{4!}x^4$$

$$P_4(x) = 1 - \frac{x^2}{2} + \frac{1}{4!}x^4$$

$$\boxed{\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)}$$

$+ o(x^5)$

funzione pari

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \iff \boxed{1-\cos x \sim \frac{1}{2}x^2,\ x\to 0}$$

$$\frac{x^2}{2} - \frac{x^4}{4!} + o(x^5) = 1-\cos x$$

$$1-\cos x = \frac{x^2}{2} + o(x^3)$$

$$\frac{1-\cos x}{x^2} = \frac{1}{2} + \underbrace{o(x)}_{\to 0 \text{ per } x \to 0}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{e^x - 1 - \sin x}{1-\cos x} =$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + o(x^3)$$

$$\sin x = x - \frac{x^3}{3!} + o(x^4)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^5)$$

$$= \lim_{x\to 0} \frac{\cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{3!} + o(x^3) - \cancel{1} - \cancel{(}\cancel{x} \overset{+}{-} \frac{x^3}{3!} + o(x^4)\cancel{)}}{\cancel{1} - \cancel{(}\cancel{1} \overset{+}{-} \frac{x^2}{2} \overset{-}{+} \frac{x^4}{4!} + o(x^5)\cancel{)}}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2} + o(x^2)}{\frac{x^2}{2} + o(x^3)} =$$

$$= \lim_{x\to 0} \frac{x^2/2}{x^2/2} = 1$$

$$\lim_{x\to 0} \frac{(e^x - 1)\cdot \sin x}{1-\cos x} =$$

$$\underline{e^x - 1} = 1 + x + \frac{x^2}{2} + o(x^2) - 1 = \underline{x} + o(x)$$

$$\sin x = x + o(x^2)$$

$$1 - \cos x = \frac{x^2}{2} + o(x^3)$$

$$= \lim_{x \to 0} \frac{(x + o(x))(x + o(x^2))}{\frac{x^2}{2} + o(x^3)} = \lim_{x \to 0} \frac{x^2 + o(x^2) + o(x^3)}{\frac{x^2}{2} + o(x^3)} =$$

$$= \lim_{x \to 0} \frac{x^2}{x^2/2} = 2$$

Risolvo con i limiti fondam.

$$\lim_{x \to 0} \frac{\underbrace{(e^x - 1)}_{\sim x} \underbrace{\sin x}_{\sim x}}{\underbrace{1 - \cos x}_{\sim \frac{x^2}{2}}} = \lim_{x \to 0} \frac{x^2}{\frac{x^2}{2}} = 2$$

Torno al $\lim_{x \to 0} \frac{e^x - 1 - \sin x}{1 - \cos x}$

questo non si può risolvere con i limiti notevoli al num.

Se utilizzassi i limiti notevoli avrei

$$e^x - 1 \sim x \qquad \sin x \sim x$$

$$\Rightarrow e^x - 1 - \sin x \sim x - x = 0$$ che non dice nulla

Al numeratore non uso limiti notevoli.

Al denominatore posso usare limite notevole

$1-\cos x \sim \frac{x^2}{2}$ perché è da solo

$$\lim_{x\to 0} \frac{\frac{e^x-1}{x} - \cos x}{x+\sin x} = \frac{0}{0}$$

$$\frac{e^x-1}{x} \to 1 \qquad \cos x \to 1$$

$$= \lim_{x\to 0} \frac{\frac{e^x-1-x\cos x}{x}}{x+\sin x} = \lim_{x\to 0} \frac{e^x-1-x\cos x}{x(x+\sin x)}$$

$$= \lim_{x\to 0} \frac{\cancel{1}+x+\frac{x^2}{2}+o(x^2)-\cancel{1}-x\left(1-\frac{x^2}{2}+o(x^3)\right)}{x\left(x+x+o(x^2)\right)} =$$

$$= \lim_{x\to 0} \frac{\cancel{x}+\frac{x^2}{2}+o(x^2)-\cancel{x}+x^3/2+o(x^4)}{2x^2+o(x^3)} =$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2}+o(x^2)}{2x^2+o(x^3)} = \lim_{x\to 0} \frac{\frac{x^2}{2}}{2x^2} = \frac{1}{4}$$

Calcolare lo sviluppo di Maclaurin di $\sinh x$ e $\cosh x$

$x_0 = 0 \qquad n = 3$

$f(x) = \sinh x \qquad f(0) = 0$

$f'(x) = \cosh x \qquad f'(0) = 1$

$f''(x) = \sinh x \qquad f''(0) = 0$

$f'''(x) = \cosh x \qquad f'''(0) = 1$

$$P_3(x) = \frac{f^{(0)}(x_0)}{0!}(x-x_0)^0 + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$P_3(x) = 0 + 1 \cdot x + 0 + \frac{1}{3!}x^3$$

$$\sinh(x) = x + \frac{x^3}{3!} + o(x^4)$$ la derivata quarta è zero

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + o(x^8)$$ dispari

lavorando analogamente con $\cosh(x)$ si ha

$$\cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + o(x^7)$$ pari

$$\sinh(x) + \cosh(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots = e^x$$

$f(x) = \log(x+1)$ $\quad x_0 = 0 \quad n = 3$

$$f(x) = \log(x+1) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{x+1} \cdot 1 = (x+1)^{-1} \qquad f'(0) = 1$$

$$f''(x) = -1 \cdot (x+1)^{-2} = -\frac{1}{(x+1)^2} \qquad f''(0) = -1$$

$$f'''(x) = (-1)\cdot(-2)(x+1)^{-3} = \frac{2}{(x+1)^3} \qquad f'''(0) = 2$$

$$P_3(x) = \frac{0}{1} \cdot x^0 + \frac{1}{1!}x^1 + \frac{(-1)}{2!}x^2 + \frac{2}{3!}x^3$$

$$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots$$

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1 \qquad \log(1+x) \sim x \quad \text{per } x\to 0$$

$$\lim_{x\to 0} \frac{\log(\cosh x)}{(\sin x)^2}$$

num: $\log(\cosh x) = \log\Big(1 + \overbrace{\frac{x^2}{2} + o(x^3)}^{t}\Big) \overset{(*)}{=}$

$$\cosh x = 1 + \frac{x^2}{2} + o(x^3)$$

se $x \to 0 \Rightarrow$ anche $t \to 0$

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3)$$

$$\overset{(*)}{=} \frac{x^2}{2} + o(x^3) - \frac{1}{2}\left(\frac{x^2}{2} + o(x^3)\right)^2 + \frac{1}{3}\left(\frac{x^2}{2} + o(x^3)\right)^3 + o\left(\left(\frac{x^2}{2} + o(x^3)\right)^3\right)$$

$$= \frac{x^2}{2} + o(x^3)$$

den: $(\sin x)^2 \sim x^2$

$$= x^2 + o(x^2)$$

$$\lim_{x\to 0} \frac{\log(\cosh x)}{(\sin x)^2} = \lim_{x\to 0} \frac{\frac{x^2}{2}+o(x^3)}{x^2+o(x^2)} = \lim_{x\to 0} \frac{x^2/2}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{x\cdot\sin(3x)-4\cosh(\sqrt{3}x)+4+3x^2}{\log(1+2x^4)} = (*)$$

DEN: $t=2x^4 \to 0$ per $x\to 0$ $\qquad \log(1+t)=t-\frac{t^2}{2}+o(t^2)$

$$\log(1+2x^4) = \underbrace{2x^4}_{t} + \underbrace{o(x^4)}_{+\,o(t)}$$

NUM: $\sin(3x) = 3x - \frac{(3x)^3}{3!} + o(x^4)$ $\qquad \sin(t)=t-\frac{t^3}{3!}+o(t^4)$

$$= 3x - \frac{9}{2}x^3 + o(x^4) \qquad \frac{3^3}{3!} = \frac{\cancel{3}\cdot 3^2}{\cancel{3}\cdot 2}$$

$$\cosh(\sqrt{3}\cdot x) = 1+\frac{(\sqrt{3}\cdot x)^2}{2}+\frac{(\sqrt{3}\cdot x)^4}{4!}+o(x^4)$$

$\cosh(t)=1+\frac{t^2}{2}+\frac{t^4}{4!}+o(t^5)$

$$= 1+\frac{3}{2}x^2+\frac{3}{8}x^4+o(x^4) \qquad \frac{\cancel{9}\,3}{4\cdot\cancel{3}\cdot 2} = \frac{3}{8}$$

$$\text{NUM} = x\cdot\sin(3x)-4\cosh(\sqrt{3}x)+4+3x^2 =$$

$$= x\left(3x-\frac{9}{2}x^3+o(x^4)\right)-4\left(1+\frac{3}{2}x^2+\frac{3}{8}x^4+o(x^4)\right)+4+3x^2 =$$

$$= \cancel{3x^2}-\frac{9}{2}x^4+o(x^5)\cancel{-4}-\cancel{6x^2}-\frac{3}{2}x^4+o(x^4)\cancel{+4}\cancel{+3x^2}$$

$$= -\frac{\cancel{12}^{\,6}}{\cancel{2}}x^4+o(x^4)$$

$$(*) = \lim_{x\to 0}\frac{-6x^4+o(x^4)}{2x^4+o(x^4)} = -3$$

$f(x) = \operatorname{arctg}(x) \qquad x_0 = 0 \qquad n = 3$

$f(x) = \operatorname{arctg} x$

$f'(x) = \dfrac{1}{1+x^2}$

$f''(x) = \dfrac{0 - 1 \cdot 2x}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$

$$f'''(x) = \frac{-2\cdot(1+x^2)^2 + 2x \cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4} = \frac{2\cancel{(1+x^2)}\overbrace{\left[-(1+x^2)+4x^2\right]}^{3x^2-1}}{(1+x^2)^{\cancel{4}\,3}}$$

$f(0) = 0 \qquad f'(0) = 1 \qquad f''(0) = 0 \qquad f'''(0) = -2$

$$P_3(x) = 0\cdot x^0 + \frac{1}{1!}x + 0\cdot x^2 - \frac{2}{3!}x^3 = x - \frac{x^3}{3}$$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + o(x^4)$$

- $\displaystyle\lim_{x\to 0} \frac{x(e^x - 1 - x\cos x)}{3x - \operatorname{arctg}(3x)} = \frac{1}{18}$

- $\displaystyle\lim_{x\to 0} \frac{e^{x^2/2} - \cos x - x^2}{x^4} = \frac{1}{12}$