Esercizi su successioni 28/10/24

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$$

- $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n =$

con $a \in \mathbb{R}^+ \setminus \{0\}$

$$= \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{a}}\right)^{\frac{n}{a}\cdot a} = \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{n}{a}}\right)^{\frac{n}{a}}\right]^a =$$

$t = \frac{n}{a}$

se $n \to \infty$ $\Rightarrow t \to +\infty$

$$= \lim_{t\to+\infty}\underbrace{\left[\left(1+\frac{1}{t}\right)^t\right]^a}_{g(f(t))} =$$

$y = \left(1+\frac{1}{t}\right)^t = f(t)$

se $t \to +\infty \Rightarrow y \to e$

$g(y) = y^a$ è continua in $y = e$

per il thm di sost (2° caso)

$$= \lim_{y\to e} y^a = e^a$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$$

vale anche con x reale

$$\left(1+\frac{a}{n}\right)^n = \left(\frac{n+a}{n}\right)^n = \frac{(n+a)^n}{n^n}$$

$$\to \lim_{n\to\infty}\frac{(n+a)^n}{n^n} = e^a$$

$\Downarrow$

$(n+a)^n \sim e^a n^n$ per $n\to\infty$

$(n+2)^n \sim e^2 n^n$

se $\lim_{n\to\infty}\frac{a_n}{b_n} = \ell \in \mathbb{R}\setminus\{0\}$

diciamo che

$a_n \sim \ell \cdot b_n$

quando $n \to \infty$

Se $a < 0$ $\quad \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{n}{a}}\right)^{\frac{n}{a}}\right]^a =$

$$= \lim_{n\to\infty}\frac{1}{\left[\left(1+\frac{1}{\frac{n}{a}}\right)^{\frac{n}{a}}\right]^{-a}} \underset{t=\frac{n}{a}}{=} \lim_{t\to-\infty}\frac{1}{\left[\left(1+\frac{1}{t}\right)^t\right]^{-a}}$$

se $n\to\infty$, $t\to-\infty$

ora $-a>0$

$$= \lim_{y\to e}\frac{1}{e^{-a}} = e^a$$

sostituzione

$y=\left(1+\frac{1}{t}\right)^t$ se $t\to-\infty \Rightarrow y\to e$

Quindi:

$$\lim_{n\to+\infty}\left(1+\frac{a}{n}\right)^n = e^a \qquad \forall a\in\mathbb{R}\setminus\{0\}$$

- $\displaystyle\lim_{n\to\infty} \frac{7n^2+n}{n+\sin(n!)} \cdot \frac{n^n \cdot n!}{(n+2)^n \cdot (n+1)!} =$

- $7n^2+n \sim 7n^2$ per $n \to +\infty$ $\qquad \displaystyle\lim_{n\to\infty} \frac{7n^2+n}{7n^2} = 1$
- $n+\sin(n!) \sim n$ per $n \to \infty$ $\qquad \displaystyle\lim_{n\to\infty} \frac{n+\sin(n!)}{n} = 1$
- $n^n \cdot n!$ resta così (non è $n^n + n!$)
- $(n+2)^n \cdot (n+1)! = (n+2)^n (n+1)\, n!$

$$= \lim_{n\to\infty} \frac{7n^{\cancel{2}}}{\cancel{n}} \cdot \frac{n^n \cdot \cancel{n!}}{(n+2)^n \underbrace{(n+1)}_{\xrightarrow{n\to\infty} 1} \cancel{n!}} = \lim_{n\to\infty} 7 \frac{n^n}{(n+2)^n} =$$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

$$= 7 \lim_{n\to\infty} \frac{1}{\frac{(n+2)^n}{n^n}} = 7 \lim_{n\to\infty} \frac{1}{\underbrace{\left(1+\frac{2}{n}\right)^n}_{\to e^2}} = \frac{7}{e^2}$$

N.B. $(n+2)^n$ non si comporta come n^n, $n \to \infty$

anche se $(n+2) \sim n$ per $n \to \infty$

ma $(n\ 2)^n \sim e^2 n^n$ per $n \to \infty$

- $$\lim_{n\to\infty} \frac{\log(n+a)}{\log n} = 1$$

$a \in \mathbb{R}$

$\log(n+a) \sim 1 \cdot \log n$

per $n \to \infty$

$$\lim_{n\to\infty} \frac{\log(n+a)}{\log n} = \lim_{n\to\infty} \frac{\log\left(n \cdot \left(1+\frac{a}{n}\right)\right)}{\log n} =$$

$\log(a \cdot b) = \log a + \log b$

$$= \lim_{n\to\infty} \frac{\log n + \log\left(1+\frac{a}{n}\right)}{\log n} =$$

$\frac{a}{n} \to 0$

$\to \log 1 = 0$

$$= \underbrace{\lim_{n\to\infty} \frac{\log n}{\log n}}_{1} + \underbrace{\lim_{n\to\infty} \frac{\log\left(1+\frac{a}{n}\right)}{\log n}}_{\frac{0}{\infty} = 0} = 1$$

$\log n \to +\infty$

- $$\lim_{n\to\infty} \frac{n^{1/n^2} - 1}{2\cdot n^{-2}\cdot \log(n+7)} \overset{(*)}{=}$$

$$a_n^{b_n} = e^{\log a_n^{b_n}} = e^{b_n \cdot \log a_n}$$

- $$n^{1/n^2} - 1 = e^{\frac{1}{n^2}\cdot \log n} - 1 \sim \frac{1}{n^2}\log n \text{ per } n\to\infty$$

$\frac{1}{n^2}\cdot \log n = x \to 0$

osservo che quando $n\to\infty$, $\frac{1}{n^2}\log n \to 0$

$$\left(\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \equiv e^x - 1 \sim x \text{ per } x \to 0 \right)$$

$$\overset{(*)}{=} \lim_{n\to\infty} \frac{\cancel{\frac{1}{n^2}}\cdot \log n}{\frac{2}{\cancel{n^2}}\log(n+7)} = \frac{1}{2}\underbrace{\lim_{n\to\infty} \frac{\log n}{\log(n+7)}}_{=1} = \frac{1}{2}$$

$\log(n+7) \sim \log n$ per $n\to\infty$

$$\lim_{n\to\infty} \frac{\log(n+2)}{n} = \lim_{n\to\infty}\frac{\log n}{n} = 0$$

$$\lim_{n\to\infty} \frac{\log(e^n+2)}{n} =$$

$e^n + 2 \sim e^n$ per $n\to\infty$

$$= \lim_{n\to\infty}\frac{\log e^n}{n} = \lim_{n\to\infty}\frac{n}{n} = 1$$