Analysis of the INTERNODES method for non-conforming discretizations of elliptic equations[1]

Paola Gervasio[a], Alfio Quarteroni[b,c]

[a] *DICATAM, Università degli Studi di Brescia, via Branze 38, 25123 Brescia, Italy*
[b] *MOX, Department of Mathematics, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy*
[c] *École Polytechnique Fédérale de Lausanne (EPFL), (honorary professor)*

Abstract

INTERNODES is a general method to deal with non-conforming discretizations of second order partial differential equations on regions partitioned into two or several subdomains. It exploits two intergrid interpolation operators, one for transferring the Dirichlet trace across the interface, the others for the Neumann trace. In every subdomain the original problem is discretized by the finite element method, using a priori non-matching grids and piece-wise polynomials of different degree. In this paper we provide several interpretations of the method and we carry out its stability and convergence analysis, showing that INTERNODES exhibits optimal convergence rate with respect to the finite element sizes. Finally we propose an efficient algorithm for the solution of the corresponding algebraic system.

Keywords: domain decomposition, non-conforming approximation, non-conforming grids, interpolation, finite element method, *hp* finite element method

1. Introduction

The INTERNODES (INTERpolation for NOnconforming DEcompositionS) method was introduced in [16] for the *non-conforming* numerical approximation of second order elliptic boundary-value problems. By non-conforming we mean that the computational domain is partitioned into subdomains with non-matching grids at subdomain interfaces or/and different polynomial subspaces are used on the subdomains.

The most distinguishing feature of INTERNODES is that it is built on two independent *interpolation* operators at the subdomain interfaces that allow to exchange information between adjoining subdomains on the problem solution and on its normal fluxes, respectively.

The continuity of the trace of the solution is enforced on the interface by means of one of the two interpolation operators.

In order to impose the continuity of the fluxes, first we compute independently and on each side of the interface the residuals of the weak local discrete subproblems (the computed values are in fact the degrees of freedom of the discrete fluxes with respect to the dual basis of the Lagrange one);

[1] Please cite this article as: P. Gervasio, A. Quarteroni, Analysis of the INTERNODES method for non-conforming discretizations of elliptic equations, Comput. Methods Appl. Mech. Engrg. (2018), vol 334, pp. 138-166. https://doi.org/10.1016/j.cma.2018.02.004

* Corresponding author. E-mail: paola.gervasio@unibs.it

then we transform the dual degrees of freedom to Lagrange degrees of freedom by means of the inverse of the local interface mass matrices. Once the Lagrange degrees of freedom of the residuals are obtained, the second interpolation operator is called into play to enforce the continuity of the fluxes.

Differently than in mortar methods, *no cross-mass matrix* involving basis functions living on different grids of the interface are required to build the intergrid operators. Instead, two separate interface mass matrices (separately on either interface) are used.

INTERNODES share some similarities with the so-called *unsymmetric mortar methods* [12, 24] (see Sect. 8), however the two approaches do not coincide. Moreover the well-posedness and the convergence analysis of the unsymmetric mortar method have not been proved to date, to the knowledge of the authors.

While Lagrange interpolation often represents a natural choice to build the intergrid operators of INTERNODES, other interpolation methods can be used as well. For instance in [17] interpolations based on Radial Basis Functions are employed in cases of non-matching (curved) interfaces. This makes the INTERNODES method very suitable in dealing with non-straight interfaces. INTERNODES was successfully applied also beyond elliptic problems, for instance for Navier-Stokes equations in domains with sliding grids and for nonlinear fluid-structure interaction problems in [16, 17].

The *interpolatory construction* represents the main difference between INTERNODES and the well known mortar method [7, 3, 4, 33, 34, 24, 12, 28, 29, 27], the latter being based on a single L^2- projection operator at subdomain interface. The analysis of INTERNODES, that is carried out for the first time in this paper, based on sharp interpolation estimates in fractional Sobolev spaces (see Sect. 9), substantially departs from that of mortar method.

In this paper we prove that, when regular quasi-uniform affine triangulations are used and the intergrid operators are based on Lagrange interpolation, INTERNODES exhibits optimal convergence rate with respect to the finite element mesh sizes, without downgrading the convergence order of the finite element discretizations employed to solve the local subproblems (see Theorem 12).

Our theoretical results are corroborated by numerical results for both 2D and 3D geometries. Further numerical results are presented in [16] where INTERNODES is systematically compared with the mortar method for $h-$ and $hp-$fem discretizations. Numerical results show that the two approaches exhibit *the same order of convergence*.

As observed above, even if the two interpolation operators of INTERNODES are built starting from the same set of data (the left and right nodes on the interface), they are two independent operators, in particular they are not one the transposition of the other; the latter choice would lead to the so-called pointwise matching method, that is sub-optimal (see [7, 3]).

In spite of featuring the same accuracy of mortar methods, INTERNODES is much simpler to implement from a programming point of view. First of all, only the coordinates of the interface nodes are needed to assemble the interpolation operators and the interface mass matrices, and only the interface degrees of freedom are required to pass information from one side to the other. Moreover, the implementation of INTERNODES for non-matching grids does not feature any additional difficulty with respect to the case of matching interfaces. Secondly, but not less important (as already mentioned above), INTERNODES does not require any cross-mass matrix involving basis functions from both sides of the interfaces, therefore no ad-hoc quadrature formula has to be devised in order to preserve the optimal accuracy. On the contrary, to build such cross-mass matrix in the case of non-straight interfaces, mortar methods require several steps such as projection, intersection, local meshing and ad-hoc numerical quadrature (see, e.g. [27, Sect.3.2.3]). We refer to [16, Sect. 6] for a detailed comparison of the implementation aspects of both INTERNODES and mortar methods.

In multiphysics problems, INTERNODES has an immediate physical interpretation in terms of interface continuity fulfillment for both the primal (displacement, velocity, etc.) and the dual (normal Cauchy stresses, normal fluxes, etc.) physical variables.

In the last decades, a rich family of approaches to deal with nonconforming discretization have been proposed and applied especially to solve contact problems in structural analysis. Far from being exhaustive, we cite PUFEM [25] and GFEM/XFEM [22, 19, 5]. The substantial difference between these methods and INTERNODES consists in the fact that the former ones use a partition of unity to enrich the finite element space, while the latter does not add any shape function to those of the local finite element subspaces.

In this paper first and above all we prove that the INTERNODES method yields a solution that is unique, stable, and convergent with an *optimal* rate of convergence (i.e., that of the best approximation error in every subdomain) in the case of Lagrange interpolation and regular, quasi-uniform and affine triangulations on each subdomain.

Then, we extend the INTERNODES method to the case of a computational domain split into several (more than two) subdomains with internal cross-points (i.e. boundary points shared by at least three subdomains). Finally, we propose an efficient solution algorithm for the INTERNODES problem after reformulating it as a Schur-complement system depending solely on the interface nodal variables.

An outline of the paper is as follows. In Section 2 we present the differential problem and its two-domain formulation. In Section 3 we recall the two-domain conforming finite element discretizations, while in Section 4 we present the intergrid operators and the INTERNODES method. Sections 5 and 6 are devoted to the algebraic form of INTERNODES: we present an efficient algorithm implementing INTERNODES and we extend the method to decompositions with more than three subdomains and internal cross points. In Section 7 some numerical results are shown for non-conforming *hp*-FEM approximation of second order elliptic boundary-value problems. In Section 8, we compare the algebraic formulation of INTERNODES and that of the unsymmetric mortar method ([12, 24]) and show that the two methods are actually different. Last but not least, in Section 9 we prove the well-posedness of the INTERNODES problem and carry out its convergence analysis.

2. Problem setting

Let $\Omega \subset \mathbb{R}^{d_\Omega}$, with $d_\Omega = 2,\ 3$, be an open domain with Lipschitz boundary $\partial\Omega$. $\partial\Omega_N$ and $\partial\Omega_D$ are suitable disjoint subsets of $\partial\Omega$ such that $\overline{\partial\Omega_D} \cup \overline{\partial\Omega_N} = \partial\Omega$. We make the following assumption, all along the paper.

Assumptions 1. Let f, α, γ and $\mathbf{b}$ be given functions such that $f \in L^2(\Omega)$, $\alpha \in L^\infty(\Omega)$, $\gamma \in L^\infty(\Omega)$, $\mathbf{b} \in W^{1,\infty}(\Omega)$. Moreover, $\exists \alpha_0 > 0$ such that $\alpha \geq \alpha_0$, $\gamma \geq 0$, $\gamma - \frac{1}{2}\nabla \cdot \mathbf{b} \geq 0$ a.e. in Ω, and $\mathbf{b} \cdot \mathbf{n} \geq 0$ on $\partial\Omega_N$. Finally, if $\partial\Omega_D = \emptyset$, we require that $\gamma - \frac{1}{2}\nabla \cdot \mathbf{b} > 0$ a.e. in Ω.

Then we look for the solution u of the second order elliptic equation

$$\begin{cases} Lu \equiv -\nabla \cdot (\alpha \nabla u) + \mathbf{b} \cdot \nabla u + \gamma u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega_D, \\ \partial_L u = 0 & \text{on } \partial\Omega_N, \end{cases} \tag{1}$$

where $\partial_L u = \alpha \dfrac{\partial u}{\partial \mathbf{n}}$ and $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$. We set

$$V = H^1_{\partial\Omega_D}(\Omega) = \{v \in H^1(\Omega) : \; v = 0 \text{ on } \partial\Omega_D\}. \tag{2}$$

The weak form of problem (1) is: find $u \in V$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)} \qquad \forall v \in V, \tag{3}$$

where

$$a(u, v) = \int_\Omega (\alpha \nabla u \cdot \nabla v + (\mathbf{b} \cdot \nabla u)v + \gamma uv) d\Omega, \tag{4}$$

while $(\cdot, \cdot)_{L^2(\Omega)}$ denotes the inner product in $L^2(\Omega)$. Under Assumption 1 there exists a unique solution of (3) (see, e.g., [30]).

We partition Ω into two non-overlapping subdomains Ω_1 and Ω_2 with Lipschitz boundary and such that $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$. $\Gamma (= \overline{\Gamma}) = \partial\Omega_1 \cap \partial\Omega_2$ is the common interface and, for $k = 1, 2$, we set $\partial\Omega_{D,k} = \partial\Omega_D \cap \partial\Omega_k$ and $\partial\Omega_{N,k} = \partial\Omega_N \cap \partial\Omega_k$.

For $k = 1, 2$ let us introduce the local spaces

$$V_k = \{v \in H^1(\Omega_k) \mid v = 0 \text{ on } \partial\Omega_D \cap \partial\Omega_k\}, \quad V_k^0 = \{v \in V_k \mid v = 0 \text{ on } \Gamma\}, \tag{5}$$

and the bilinear forms

$$a_k(u, v) = \int_{\Omega_k} (\alpha \nabla u \cdot \nabla v + (\mathbf{b} \cdot \nabla u)v + \gamma uv) d\Omega. \tag{6}$$

Finally, let Λ be the space of traces of the elements of V on the interface Γ:

$$\Lambda = \{\lambda \in H^{1/2}(\Gamma) : \; \exists v \in V : v|_\Gamma = \lambda\}. \tag{7}$$

When $\partial\Gamma \cap \partial\Omega \subset \partial\Omega_N$, $\Lambda = H^{1/2}(\Gamma)$, while when $\partial\Gamma \cap \partial\Omega \subset \partial\Omega_D$, $\Lambda = H^{1/2}_{00}(\Gamma)$; in these cases Λ is endowed with the canonical norm of either $H^{1/2}(\Gamma)$ or $H^{1/2}_{00}(\Gamma)$, respectively ([1]). Intermediate situations can be tackled by suitably defining Λ and its norm (see, e.g., [23, Remark 11.5]).

For $k = 1, 2$, let u_k be the restriction of the solution u of (3) to Ω_k, then u_1 and u_2 are the solution of the *transmission problem* (see [15, Ch. VII, Sect. 4])

$$\begin{cases} Lu_k = f & \text{in } \Omega_k, \quad k = 1, 2, \\ u_1 = u_2, \qquad \partial_{L_1} u_1 + \partial_{L_2} u_2 = 0 & \text{on } \Gamma, \end{cases} \tag{8}$$

where $\partial_{L_k} u_k = \alpha_k \frac{\partial u_k}{\partial \mathbf{n}_k}$ (with $\alpha_k = \alpha_{|\Omega_k}$) denotes the conormal derivative associated with the differential operator L, and $\mathbf{n}_k$ is the outward unit normal vector to $\partial\Omega$ (in particular on Γ, we have $\mathbf{n}_1 = -\mathbf{n}_2$). We denote by $\mathbf{n}_{\Gamma_k}$ the restriction of $\mathbf{n}_k$ to Γ. [2]

[2]In the entire paper we assume that Γ is sufficiently regular to allow the conormal derivative of u to be well defined. This is certainly the case if Γ is of class $C^{1,1}$ (see [21, Def. 1.2.1.2]).

More precisely, u_1 and u_2 satisfy the following weak form of the transmission problem (8) (see [31, Lemma 1.2.1]): find $u_1 \in V_1$ and $u_2 \in V_2$ such that

$$\begin{cases} a_k(u_k, v_k^0) = (f, v_k^0)_{L^2(\Omega_k)} & \forall v_k^0 \in V_k^0, \quad k = 1,2 \\ u_2 = u_1 & \text{on } \Gamma, \\ \displaystyle\sum_{k=1,2} a_k(u_k, \mathcal{R}_k\eta) = \sum_{k=1,2} (f, \mathcal{R}_k\eta)_{L^2(\Omega_k)} & \forall \eta \in \Lambda, \end{cases} \tag{9}$$

where

$$\mathcal{R}_k : \Lambda \to V_k, \quad s.t. \quad (\mathcal{R}_k\eta)|_\Gamma = \eta \qquad \forall \eta \in \Lambda \tag{10}$$

denotes any possible linear and continuous *lifting operator* from Γ to Ω_k.

Remark 1. Let $\langle \cdot, \cdot \rangle$ denote the duality between Λ and its dual Λ'. If homogeneous boundary conditions (of either Dirichlet and Neumann type) are given on $\partial\Omega$, by counter-integration by parts, the interface equation $(9)_3$ is equivalent to

$$\langle \partial_{L_1} u_1 + \partial_{L_2} u_2, \eta \rangle = 0 \qquad \forall \eta \in \Lambda, \tag{11}$$

and therefore to the transmission condition $(8)_3$.

3. Recall on conforming discretization

Let us consider a family of triangulations $\mathcal{T}_h$ of the global domain Ω, depending on a positive parameter (the grid size) $h > 0$. Following standard assumptions we require $\mathcal{T}_h$ to be affine, regular, and quasi-uniform (see [30, Ch. 3]). For any $T \in \mathcal{T}_h$, we assume that $\partial T \cap \partial\Omega$ fully belongs to either $\partial\Omega_D$ or $\partial\Omega_N$. We shall denote by $\mathbb{P}_p$, with p a positive integer, the usual space of algebraic polynomials of total degree less than or equal to p. Let

$$X_h = \{v \in C^0(\overline{\Omega}) : \; v_{|T} \in \mathbb{P}_p, \, \forall T \in \mathcal{T}_h\}, \quad V_h = \{v \in X_h : \; v = 0 \text{ on } \partial\Omega_D\} \tag{12}$$

be the usual finite element spaces associated with $\mathcal{T}_h$. The Galerkin finite element approximation of (3) reads: find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h)_{L^2(\Omega)} \qquad \forall v_h \in V_h. \tag{13}$$

Let us split Ω into two subdomains Ω_1 and Ω_2 and assume that the triangulations $\mathcal{T}_h$ are such that Γ does not cut any element $T \in \mathcal{T}_h$. The triangulations $\mathcal{T}_{1,h}$ and $\mathcal{T}_{2,h}$ induced by $\mathcal{T}_h$ on Ω_1 and Ω_2 are therefore compatible on Γ, that is they share the same edges (if $d = 2$) or faces (if $d = 3$).

In each Ω_k ($k = 1, 2$) we introduce the finite element approximation spaces

$$X_{k,h} = \{v \in C^0(\overline{\Omega}_k) \, : \, v_{|T} \in \mathbb{P}_p, \, \forall T \in \mathcal{T}_{k,h}\}, \tag{14}$$

and the finite dimensional subspaces of V_k and V_k^0

$$V_{k,h} = X_{k,h} \cap V_k, \qquad V_{k,h}^0 = X_{k,h} \cap V_k^0. \tag{15}$$

Moreover, we consider the space of finite dimensional traces on Γ

$$\Lambda_h = \{\lambda = v|_\Gamma, \; v \in V_{1,h} \cup V_{2,h}\} \subset \Lambda. \tag{16}$$

For $k = 1,2$ we define two linear and continuous *discrete lifting operators*

$$\mathcal{R}_{k,h} : \Lambda_h \to V_{k,h}, \quad s.t. \quad (\mathcal{R}_{k,h}\eta_h)|_\Gamma = \eta_h, \quad \forall \eta_h \in \Lambda_h. \tag{17}$$

The problem: find $u_{1,h} \in V_{1,h}$ and $u_{2,h} \in V_{2,h}$ such that

$$\begin{cases} a_k(u_{k,h}, v^0_{k,h}) = (f, v^0_{k,h})_{L^2(\Omega_k)} & \forall v^0_{k,h} \in V^0_{k,h}, \quad k = 1,2 \\ u_{2,h} = u_{1,h} & \text{on } \Gamma, \\ \displaystyle\sum_{k=1,2} a_k(u_{k,h}, \mathcal{R}_{k,h}\eta_h) = \sum_{k=1,2} (f, \mathcal{R}_{k,h}\eta_h)_{L^2(\Omega_k)} & \forall \eta_h \in \Lambda_h. \end{cases} \tag{18}$$

is actually equivalent to (13), in the sense that $u_{k,h} = u_{h|\Omega_k}$, for $k = 1,2$ (see [31, Sect. 2.1]). Note that (18) is the discrete counterpart of (9); in particular, $(18)_3$ is the discrete counterpart of $(9)_3$.

In practical implementation, $\mathcal{R}_{k,h_k}\eta_h$ can be chosen as the finite element interpolant that extends to zero (at any interior finite element node) the values of η_h at the nodes on Γ.

Defining the *discrete residual functionals* $r_{k,h} \in \Lambda'_h$ by the relations

$$\langle r_{k,h}, \eta_h \rangle = a_k(u_{k,h}, \mathcal{R}_{k,h}\eta_h) - (f, \mathcal{R}_{k,h}\eta_h)_{L^2(\Omega_k)} \quad \text{for any } \eta_h \in \Lambda_h, \tag{19}$$

the interface equation $(18)_3$ is equivalent to

$$\langle r_{1,h} + r_{2,h}, \eta_h \rangle = 0 \quad \text{for any } \eta_h \in \Lambda_h. \tag{20}$$

As seen in Remark 1, if homogeneous boundary conditions (of either Dirichlet and Neumann type) are prescribed on $\partial\Omega$, the finite dimensional functionals $r_{k,h}$ represent the approximations of the distributional derivatives $\partial_{L_k} u_k$ on Γ. Then (20) can be regarded as the discrete counterpart of (11).

4. Non-conforming discretization

Now we consider two a-priori *independent families of triangulations* $\mathcal{T}_{1,h_1}$ in Ω_1 and $\mathcal{T}_{2,h_2}$ in Ω_2, respectively. This means that the meshes in Ω_1 and in Ω_2 can be non-conforming on Γ and characterized by different mesh-sizes h_1 and h_2. Moreover, different polynomial degrees p_1 and p_2 can be used to define the finite element spaces. Inside each subdomain Ω_k we assume that the triangulations $\mathcal{T}_{k,h_k}$ are affine, regular and quasi-uniform ([30, Ch.3]).

From now on, the finite element approximation spaces are (for $k = 1,2$):

$$\begin{array}{l} X_{k,h_k} = \{v \in C^0(\overline{\Omega}_k) \, : \, v_{|T} \in \mathbb{P}_{p_k}, \, \forall T \in \mathcal{T}_{k,h_k}\}, \\ V_{k,h_k} = X_{k,h_k} \cap V_k, \qquad V^0_{k,h_k} = \{v \in V_{k,h_k}, v|_\Gamma = 0\}, \end{array} \tag{21}$$

while the spaces of traces on Γ are

$$Y_{k,h_k} = \{\lambda = v|_\Gamma, \ v \in X_{k,h_k}\}, \text{ and } \Lambda_{k,h_k} = \{\lambda = v|_\Gamma, \ v \in V_{k,h_k}\}. \tag{22}$$

We set $N_k = \dim(V_{k,h_k})$, $N^0_k = \dim(V^0_{k,h_k})$, $\overline{n}_k = \dim(Y_{k,h_k})$, and $n_k = \dim(\Lambda_{k,h_k})$.

The space Λ_{k,h_k} takes into account the essential boundary conditions, while Y_{k,h_k} does not. Thus, if $\partial\Omega \cap \partial\Gamma \subset \partial\Omega_N$, then $\Lambda_{k,h_k} = Y_{k,h_k}$ and $n_k = \overline{n}_k$, otherwise $n_k < \overline{n}_k$ because the degrees of freedom associated with the nodes in $\partial\Omega_D \cap \partial\Gamma$ are eliminated.

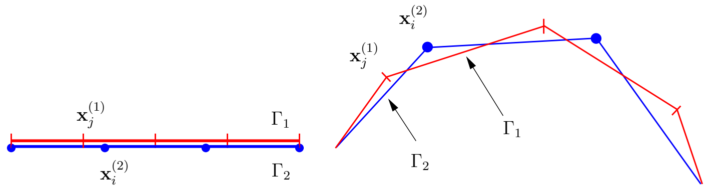

Figure 1: Γ_1 and Γ_2 induced by the triangulations $\mathcal{T}_{1,h_1}$ and $\mathcal{T}_{2,h_2}$

The Lagrange basis functions of V_{k,h_k} (for $k = 1, 2$) associated with the nodes $\mathbf{x}_i^{(k)}$ of the mesh $\mathcal{T}_{k,h_k}$ are denoted by $\{\varphi_i^{(k)}\}$ for $i = 1, \ldots, N_k$, and they are reordered so that the first $N_k^0 (\leq N_k)$ basis functions span V_{k,h_k}^0.

We denote by Γ_1 and Γ_2 the internal boundaries of Ω_1 and Ω_2, respectively, induced by the triangulations $\mathcal{T}_{1,h_1}$ and $\mathcal{T}_{2,h_2}$. If Γ is a straight segment, then $\Gamma_1 = \Gamma_2 = \Gamma$, otherwise Γ_1 and Γ_2 can be different (see Fig. 1).

For $k = 1, 2$, let $\{\mathbf{x}_1^{(\Gamma_k)}, \ldots, \mathbf{x}_{\overline{n}_k}^{(\Gamma_k)}\} \in \overline{\Gamma}_k$ be the nodes induced by the mesh $\mathcal{T}_{k,h_k}$.

The Lagrange basis functions of Y_{k,h_k} are denoted by $\{\mu_i^{(k)}\}$ for $i = 1, \ldots, \overline{n}_k$ and they are reordered so that the first $n_k (\leq \overline{n}_k)$ basis functions span Λ_{k,h_k}.

In formulating the INTERNODES method we will make use of the *interface mass matrices* M_{Γ_k}:

$$(M_{\Gamma_k})_{ij} = (\mu_j^{(k)}, \mu_i^{(k)})_{L^2(\Gamma)}, \qquad i, j = 1, \ldots, \overline{n}_k, \quad k = 1, 2. \tag{23}$$

We will also need the canonical *dual basis* $\{\Phi_i^{(k)}\}_{i=1}^{\overline{n}_k}$ of Y'_{k,h_k} (the dual space of Y_{k,h_k}) defined by

$$\langle \Phi_i^{(k)}, \mu_j^{(k)} \rangle = (\Phi_i^{(k)}, \mu_j^{(k)})_{L^2(\Gamma_k)} = \delta_{ij}, \quad i, j = 1, \ldots, \overline{n}_k. \tag{24}$$

It holds that (see, e.g., [10])

$$\Phi_i^{(k)} = \sum_{j=1}^{\overline{n}_k} (M_{\Gamma_k}^{-1})_{ji} \mu_j^{(k)}, \qquad i = 1, \ldots, \overline{n}_k, \tag{25}$$

meaning that Y'_{k,h_k} and Y_{k,h_k} are in fact the same (finite dimensional) linear space.

By expanding any element $r_{k,h_k} \in Y'_{k,h_k}$ with respect to the dual basis

$$r_{k,h_k}(\mathbf{x}) = \sum_{i=1}^{\overline{n}_k} r_i^{(k)} \Phi_i^{(k)}(\mathbf{x}) \qquad \forall \mathbf{x} \in \Gamma_k,$$

we note that, thanks to (25),

$$r_{k,h_k}(\mathbf{x}) = \sum_{j=1}^{\overline{n}_k} \underbrace{\left(\sum_{i=1}^{\overline{n}_k} (M_{\Gamma_k}^{-1})_{ji} r_i^{(k)} \right)}_{z_j^{(k)}} \mu_j^{(k)}(\mathbf{x}) = \sum_{j=1}^{\overline{n}_k} z_j^{(k)} \mu_j^{(k)}(\mathbf{x}) \qquad \forall \mathbf{x} \in \Gamma_k, \tag{26}$$

hence, (26) provides the expansion of r_{k,h_k} with respect to the Lagrange basis $\{\mu_i^{(k)}\}$.

Denoting by $\overline{\mathbf{z}}_k$, $\overline{\mathbf{r}}_k \in \mathbb{R}^{\overline{n}_k}$ the vectors whose entries are the values $z_j^{(k)}$ and $r_i^{(k)}$, respectively, it holds

$$\overline{\mathbf{z}}_k = M_{\Gamma_k}^{-1} \overline{\mathbf{r}}_k. \tag{27}$$

4.1. Interpolation and intergrid operators

We introduce two independent operators that exchange information between the two independent grids on the interface Γ.

If Γ is a straight interface, so that $\Gamma_1 = \Gamma_2$ as in Fig. 1, left, the first one $\Pi_{12} : Y_{2,h_2} \to Y_{1,h_1}$ is such that

$$(\Pi_{12}\mu_{2,h_2})(\mathbf{x}_i^{(\Gamma_1)}) = \mu_{2,h_2}(\mathbf{x}_i^{(\Gamma_1)}), \quad i = 1, \ldots, \overline{n}_1, \qquad \forall \mu_{2,h_2} \in Y_{2,h_2}, \tag{28}$$

while the second interpolation operator $\Pi_{21} : Y_{1,h_1} \to Y_{2,h_2}$ is such that

$$(\Pi_{21}\mu_{1,h_1})(\mathbf{x}_i^{(\Gamma_2)}) = \mu_{1,h_1}(\mathbf{x}_i^{(\Gamma_2)}), \quad i = 1, \ldots, \overline{n}_2, \qquad \forall \mu_{1,h_1} \in Y_{1,h_1}. \tag{29}$$

The operator Π_{12} is in fact the finite element interpolation operator

$$\mathcal{I}_1 : C^0(\overline{\Gamma}) \to Y_{1,h_1} : \ \forall \eta \in C^0(\overline{\Gamma}) \quad (\mathcal{I}_1\eta)(\mathbf{x}_i^{(\Gamma_1)}) = \eta(\mathbf{x}_i^{(\Gamma_1)}), \quad i = 1, \ldots, \overline{n}_1, \tag{30}$$

restricted to the functions of Y_{2,h_2} (rather than operating on the entire $C^0(\overline{\Gamma})$). Similarly Π_{21} is the restriction of

$$\mathcal{I}_2 : C^0(\overline{\Gamma}) \to Y_{2,h_2} : \ \forall \eta \in C^0(\overline{\Gamma}) \quad (\mathcal{I}_2\eta)(\mathbf{x}_i^{(\Gamma_2)}) = \eta(\mathbf{x}_i^{(\Gamma_2)}), \quad i = 1, \ldots, \overline{n}_2 \tag{31}$$

to the functions of Y_{1,h_1}.

Remark 2. Using only one intergrid interpolation operator would not guarantee an accurate non-conforming method; this would yield the so-called *pointwise matching* discussed, e.g., in [7, 3], where both trial and test functions satisfy the relation $(v_{|\Omega_2})_{|\Gamma} = \Pi_{21}((v_{|\Omega_1})_{|\Gamma})$. In our approach, the second operator (Π_{12} that maps Y_{2,h_2} on Y_{1,h_1}) matches, in a suitable way, the fluxes across the interface.

The (rectangular) matrices associated with Π_{21} and Π_{12} are, respectively, $R_{21} \in \mathbb{R}^{\overline{n}_2 \times \overline{n}_1}$ and $R_{12} \in \mathbb{R}^{\overline{n}_1 \times \overline{n}_2}$ and they are defined by

$$\begin{array}{ll} (R_{21})_{ij} & = \Pi_{21}\mu_j^{(1)}(\mathbf{x}_i^{(\Gamma_2)}) \quad i = 1, \ldots, \overline{n}_2, \ j = 1, \ldots, \overline{n}_1, \\ (R_{12})_{ij} & = \Pi_{12}\mu_j^{(2)}(\mathbf{x}_i^{(\Gamma_1)}) \quad i = 1, \ldots, \overline{n}_1, \ j = 1, \ldots, \overline{n}_2. \end{array} \tag{32}$$

Remark 3. When $\Gamma_1 \neq \Gamma_2$ (geometrical non-conformity) the Rescaled Localized Radial Basis Function (RL-RBF) interpolation (see [17]) represents a very effective alternative to Lagrange interpolation.

4.2. Formulation of INTERNODES

For $k = 1, 2$ we define two discrete linear and continuous *lifting operators*

$$\overline{\mathcal{R}}_k = \overline{\mathcal{R}}_{k,h_k} : Y_{k,h_k} \to X_{k,h_k}, \qquad s.t. \qquad (\overline{\mathcal{R}}_k \lambda_{k,h_k})_{|\Gamma} = \lambda_{k,h_k}, \tag{33}$$

such that, when restricted to Λ_{k,h_k}, $\overline{\mathcal{R}}_k$ coincides with the lifting $\mathcal{R}_{k,h_k}$ introduced in (17).

In practical implementation, we can define $\overline{\mathcal{R}}_k \lambda_{k,h_k}$ as the finite element interpolant that extends any $\lambda_{k,h_k} \in Y_{k,h_k}$ by setting to zero the values of $\overline{\mathcal{R}}_k \lambda_{k,h_k}$ at all nodes of $\mathcal{T}_{k,h_k}$ not belonging to Γ_k.

In particular, if $\lambda_{k,h_k} = \mu_j^{(k)}$ (the jth Lagrange basis function on Γ_k), then $\overline{\mathcal{R}}_k\mu_j^{(k)}$ is the Lagrange basis function of X_{k,h_k} whose restriction on Γ_k coincides with $\mu_j^{(k)}$.

The *weak form of INTERNODES* reads: find $u_{1,h_1} \in V_{1,h_1}$ and $u_{2,h_2} \in V_{2,h_2}$ such that

$$\begin{cases} a_k(u_{k,h_k}, v^0_{k,h_k}) = (f, v^0_{k,h_k})_{L^2(\Omega_k)} & \forall v^0_{k,h_k} \in V^0_{k,h_k}, \quad k = 1,2 \\ u_{2,h_2} = \Pi_{21} u_{1,h_1} & \text{on } \Gamma_2, \\ r_{1,h_1} + \Pi_{12} r_{2,h_2} = 0 & \text{in } \Lambda'_{1,h_1}, \end{cases} \tag{34}$$

where the *residuals* $r_{k,h_k} \in Y'_{k,h_k}$ are defined by

$$r_{k,h_k} = \sum_{i=1}^{\overline{n}_k} r_i^{(k)} \Phi_i^{(k)}, \tag{35}$$

whose coefficients $r_i^{(k)}$ are

$$r_i^{(k)} = \langle r_{k,h_k}, \mu_i^{(k)} \rangle = a_k(u_{k,h_k}, \overline{\mathcal{R}}_k \mu_i^{(k)}) - (f, \overline{\mathcal{R}}_k \mu_i^{(k)})_{L^2(\Omega_k)} \qquad \text{for } i = 1, \dots, \overline{n}_k. \tag{36}$$

Note the unsymmetrical role played by the domains Ω_1 and Ω_2 in (34). In particular the Dirichlet trace on Γ_1 is first interpolated and then transferred to Γ_2. For this reason, mimicking the mortar notation, Ω_1 is named *master* subdomain and Ω_2 *slave* subdomain.

Remark 4. Relation $(34)_2$ holds pointwise (at any $\mathbf{x}$) on Γ_2, whereas $(34)_3$ is an identity in the dual space Λ'_{1,h_1}. However, by expressing both r_{1,h_1} and $\Pi_{12} r_{2,h_2}$ with respect to the Lagrange basis (as done in (26)), also $(34)_3$ yields a pointwise relation on Γ_1. From a practical standpoint, both $(34)_2$ and $(34)_3$ will be expressed by simple matrix-vector algebraic relations, see (42) and (44).

Remark 5. If the discretizations in Ω_1 and Ω_2 are conforming on Γ, then Π_{21} and Π_{12} are the identity operators and problem (34)–(36) coincides with (18); (34)–(36) can therefore be regarded as the extension of (18) to the non-conforming case.

5. Algebraic form of INTERNODES

For ease of understanding, we first recall the algebraic form of the monodomain problem (13). Denoting by $\{\varphi_i\}$, for $i = 1, \dots, N$, the Lagrange basis functions of V_h associated with the nodes $\mathbf{x}_i$ of the mesh $\mathcal{T}_h$, and introducing the matrix $A_{ij} = a(\varphi_j, \varphi_i)$, for $i, j = 1, \dots, N$, and the vectors $\mathbf{f} = [(f, \varphi_i)]_{i=1}^N$, $\mathbf{u} = [u_h(\mathbf{x}_i)]_{i=1}^N$, the algebraic form of (13) reads

$$A\mathbf{u} = \mathbf{f}. \tag{37}$$

Now we derive the algebraic linear system associated with (18). For $k = 1, 2$, we define in a standard way the local stiffness matrices (see, e.g., [32, 31]), i.e. $(A_{kk})_{ij} = a_k(\varphi_j^{(k)}, \varphi_i^{(k)})$ for $i, j = 1, \dots, N_k^0$, $(A_{\Gamma_k,\Gamma_k})_{ij} = a_k(\overline{\mathcal{R}}_k\mu_j^{(k)}, \overline{\mathcal{R}}_k\mu_i^{(k)})$ for $i, j = 1, \dots, \overline{n}_k$, $A^0_{\Gamma_k,\Gamma_k}$ the submatrix of A_{Γ_k,Γ_k} of the first n_k rows and columns, $(A_{k,\Gamma_k})_{ij} = a_k(\overline{\mathcal{R}}_k\mu_j^{(k)}, \varphi_i^{(k)})$ for $i = 1, \dots, N_k^0$, $j =$

$1, \dots, \overline{n}_k$ and A^0_{k,Γ_k} the submatrix of A_{k,Γ_k} of the first n_k columns, $(A_{\Gamma_k,k})_{ij} = a_k(\varphi_j^{(k)}, \overline{\mathcal{R}}_k\mu_i^{(k)})$ for $i = 1, \dots, \overline{n}_k,\ j = 1, \dots, N_k^0$ and $A^0_{\Gamma_k,k}$ the submatrix of $A_{\Gamma_k,k}$ of the first n_k rows.

Then we set

$$\begin{array}{lll} \mathbf{f}_k = [(f, \varphi_i^{(k)})_{L^2(\Omega_k)}]_{i=1}^{N_k^0}, & \overline{\mathbf{f}}_{\Gamma_k} = [(f, \overline{\mathcal{R}}_k\mu_i^{(k)})_{L^2(\Omega_k)}]_{i=1}^{\overline{n}_k}, & \\ \mathbf{u}_k = [u_{k,h_k}(\mathbf{x}_j^{(k)})]_{j=1}^{N_k^0}, & \overline{\mathbf{u}}_{\Gamma_k} = [u_{k,h_k}(\mathbf{x}_j^{(\Gamma_k)})]_{j=1}^{\overline{n}_k}, & \overline{\mathbf{r}}_k = [r_i^{(k)}]_{i=1}^{\overline{n}_k}, \end{array} \tag{38}$$

while $\mathbf{f}_{\Gamma_k}$, $\mathbf{u}_{\Gamma_k}$ and $\mathbf{r}_k$ denote the subvectors of $\overline{\mathbf{f}}_{\Gamma_k}$, $\overline{\mathbf{u}}_{\Gamma_k}$ and $\overline{\mathbf{r}}_k$, respectively, of the first n_k components.

In the case that $\mathcal{T}_{1,h_1}$ and $\mathcal{T}_{2,h_2}$ are conforming on Γ (in which case $h_1 = h_2$ and $n_1 = n_2$), the algebraic counterpart of the conforming 2-domains problem (18) reads

$$\begin{bmatrix} A_{1,1} & A^0_{1,\Gamma_1} & 0 \\ A^0_{\Gamma_1,1} & A^0_{\Gamma_1,\Gamma_1} + A^0_{\Gamma_2,\Gamma_2} & A^0_{\Gamma_2,2} \\ 0 & A^0_{2,\Gamma_2} & A_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_{\Gamma_1} \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_{\Gamma_1} + \mathbf{f}_{\Gamma_2} \\ \mathbf{f}_2 \end{bmatrix}, \tag{39}$$

that is equivalent to (37), upon setting $\mathbf{u}_{\Gamma_1} = \mathbf{u}_{|\Gamma}$. Notice that we have eliminated the trace $\mathbf{u}_{\Gamma_2}$, since it coincides with $\mathbf{u}_{\Gamma_1}$.

The residual vectors $\mathbf{r}_k$, whose components are defined in (36), satisfy

$$\mathbf{r}_k = A^0_{\Gamma_k,k}\mathbf{u}_k + A^0_{\Gamma_k,\Gamma_k}\mathbf{u}_{\Gamma_k} - \mathbf{f}_{\Gamma_k}, \quad k = 1, 2; \tag{40}$$

hence the second row of (39) can be equivalently written as $\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{0}$, and it is the algebraic realization of (20).

We write now the algebraic form of the non-conforming problem (34)–(35).

To begin with, we define two *intergrid matrices*

$$Q_{21} = R_{21}, \qquad Q_{12} = M_{\Gamma_1} R_{12} M_{\Gamma_2}^{-1}. \tag{41}$$

The algebraic counterpart of $(34)_2$ reads

$$\overline{\mathbf{u}}_{\Gamma_2} = Q_{21}\overline{\mathbf{u}}_{\Gamma_1}. \tag{42}$$

The intergrid interpolation operator Π_{12} in $(34)_3$ applies on the Lagrange expansion (26) of r_{2,h_2}, i.e.,

$$\sum_{i=1}^{\overline{n}_1} z_i^{(1)}\mu_i^{(1)}(\mathbf{x}) + \Pi_{12}\left(\sum_{j=1}^{\overline{n}_2} z_j^{(2)}\mu_j^{(2)}(\mathbf{x})\right) = 0, \qquad \forall \mathbf{x} \in \Gamma_1 \tag{43}$$

and, thanks to (27) and (32), the algebraic form of $(34)_3$ reads

$$\overline{\mathbf{z}}_1 + R_{12}\overline{\mathbf{z}}_2 = \mathbf{0} \quad \text{ or, equivalently,} \quad \overline{\mathbf{r}}_1 + Q_{12}\overline{\mathbf{r}}_2 = \mathbf{0}. \tag{44}$$

Denoting by Q^0_{21} the restriction of Q_{21} to its first n_2 columns, by Q^0_{12} the restriction of Q_{12} to its first n_1 rows and by using (42), the algebraic form of (34) reads

$$\begin{bmatrix} A_{1,1} & A^0_{1,\Gamma_1} & 0 \\ A^0_{\Gamma_1,1} & A^0_{\Gamma_1,\Gamma_1} + Q^0_{12}A_{\Gamma_2,\Gamma_2}Q^0_{21} & Q^0_{12}A_{\Gamma_2,2} \\ 0 & A_{2,\Gamma_2}Q^0_{21} & A_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_{\Gamma_1} \\ \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_{\Gamma_1} + Q^0_{12}\overline{\mathbf{f}}_{\Gamma_2} \\ \mathbf{f}_2 \end{bmatrix}. \tag{45}$$

System (45) represents the algebraic form of INTERNODES implemented in practice. By taking $Q_{12} = Q_{21} = I$ we recover the algebraic system (39) of the conforming case.

Notice that, even though the residuals are defined up to the boundary of Γ_k, the algebraic counterpart of condition $(34)_3$ is imposed only on the internal nodes of Γ_1. In this way the number of equations and the number of unknowns in (45) do coincide.

In Section 6 we describe how to treat non-homogeneous Dirichlet boundary conditions and how to solve the algebraic system (45) by the Schur-complement approach; then we extend the INTERNODES method to decompositions with more than 2 subdomains.

6. Generalization and algorithmic aspects

6.1. Non-homogeneous Dirichlet conditions

When non-homogeneous Dirichlet boundary conditions are assigned on $\partial\Omega$, we can recover the homogeneous case by lifting the Dirichlet data, so that only the right hand side has to be modified (see, e.g., [30]). However, it is often common practice not to make use of lifting operators. In that case also the Dirichlet boundary nodes become degrees of freedom and the corresponding basis functions have to be extended. In this situation the INTERNODES algebraic form (45) has to undergo a slight modification yielding:

$$\begin{aligned}
(A_{\Gamma_k,\Gamma_k})_{ij} &= a_k(\overline{\mathcal{R}}_k\mu_j^{(k)}, \overline{\mathcal{R}}_k\mu_i^{(k)}) - \int_{\partial\Omega_{D,k}} \alpha \frac{\partial \overline{\mathcal{R}}_k\mu_j^{(k)}}{\partial \mathbf{n}_k} \overline{\mathcal{R}}_k\mu_i^{(k)}, \quad && j = 1, \ldots, \overline{n}_k, \\
(A_{\Gamma_k,k})_{ij} &= a_k(\varphi_j^{(k)}, \overline{\mathcal{R}}_k\mu_i^{(k)}) - \int_{\partial\Omega_{D,k}} \alpha \frac{\partial \varphi_j^{(k)}}{\partial \mathbf{n}_k} \overline{\mathcal{R}}_k\mu_i^{(k)}, && j = 1, \ldots, N_k^0,
\end{aligned} \tag{46}$$

where $\mu_i^{(k)}$ is any Lagrange basis function associated with $\mathbf{x}_i^{(\Gamma_k)} \in \partial\Gamma_k \cap \partial\Omega_D$. The subtraction of the boundary integrals in (46) is motivated by the fact that, for such $\mu_i^{(k)}$, $\overline{\mathcal{R}}_k\mu_i^{(k)}$ does not satisfy essential boundary conditions on $\partial\Omega_D$. With this change, the residuals (36) can still be regarded as being the approximations of the normal derivatives at the interface Γ.

6.2. An efficient solution algorithm for system (45)

After Gaussian elimination of the variables $\mathbf{u}_1$ and $\mathbf{u}_2$, the Schur complement form of (45) reads

$$S\mathbf{u}_{\Gamma_1} = \mathbf{b} \tag{47}$$

where

$$S = S_1^0 + Q_{12}^0 S_2 Q_{21}^0, \qquad \mathbf{b} = \mathbf{b}_1 + Q_{12}^0 \overline{\mathbf{b}}_2, \tag{48}$$

$$S_k = A_{\Gamma_k,\Gamma_k} - A_{\Gamma_k,k} A_{k,k}^{-1} A_{k,\Gamma_k}, \qquad k = 1,\ 2, \tag{49}$$

are the local Schur complement matrices, while

$$\overline{\mathbf{b}}_k = \overline{\mathbf{f}}_{\Gamma_k} - A_{\Gamma_k,k} A_{k,k}^{-1} \mathbf{f}_k \tag{50}$$

are the local right hand sides, $\mathbf{b}_1$ is the restriction of $\overline{\mathbf{b}}_1$ to its first n_1 components, and S_1^0 is the submatrix of the first n_1 rows of S_1.

System (47) can be solved, e.g., by the preconditioned Krylov method, with S_1^0 as preconditioner. (Notice that matrix $\widetilde{S}_2 = Q_{12}^0 S_2 Q_{21}^0$ is not a good candidate to play the role of preconditioner since it may be singular.)

The sketch of the algorithm is reported in Algorithm 1 for reader's convenience.

Algorithm 1 INTERNODES algorithm for 2 subdomains

for all $k = 1, 2$ **do**

build the local stiffness matrices $A_{k,k}$, A_{k,Γ_k}, $A_{\Gamma_k,k}$ and A_{Γ_k,Γ_k} (see Sect. 6.1 in the case of non-homogeneous Dirichlet conditions)

build the right hand sides $\mathbf{f}_k$ and $\overline{\mathbf{f}}_{\Gamma_k}$ (formula (38))

build the local interface mass matrices M_{Γ_k} (formula (23))

end for

build the interpolation matrices R_{21} and R_{12} (formulas (32))

build Q_{21} and Q_{12} (formula (41)) (only the nodes coordinates on the interfaces are needed in this step)

solve system (45) (or (47))

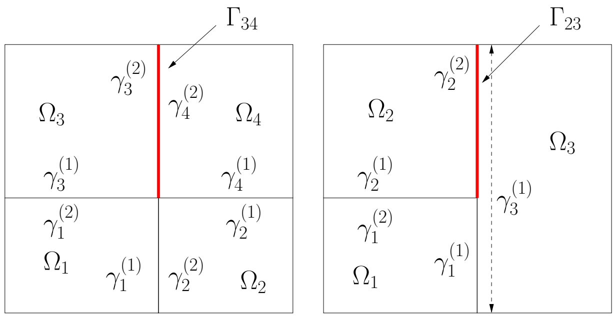

Figure 2: Two simple decompositions

6.3. Extension to more than 2 subdomains

INTERNODES can be extended to the case of $M > 2$ subdomains. Let us start with two simple decompositions as in Fig. 2, while an example of a more general decomposition is shown in Fig. 3, left.

Let us suppose that each Ω_k is convex with Lipschitz boundary $\partial\Omega_k$ (for $k = 1, \dots, M$), and that any angle between two consecutive edges is less than π. Let $\Gamma_k = \partial\Omega_k \setminus \partial\Omega$ be the part of the boundary of Ω_k internal to Ω, and $\gamma_k^{(i)} \subset \Gamma_k$ be the ith edge of Γ_k (the sub-index k identifies the domain, while i denotes the number of the internal edges of $\partial\Omega_k$),

Let $\Gamma_{k\ell} = \Gamma_{\ell k} = \partial\Omega_k \cap \partial\Omega_\ell$ be the interface between the two subdomains Ω_k and Ω_ℓ, and $\gamma_k^{(i)}$ and $\gamma_\ell^{(j)}$ be the two edges of Ω_k and Ω_ℓ, respectively, whose (non-empty) intersection is $\Gamma_{k\ell}$. Intersections reduced to a single point are considered empty.

In the example of Fig. 2, left, we have $\Gamma_{k\ell} = \gamma_k^{(i)} = \gamma_\ell^{(j)}$ for any interface $\Gamma_{k\ell}$ of the decomposition, while in the example depicted in Fig. 2, right, we have $\Gamma_{23} = \gamma_2^{(2)} \subset \gamma_3^{(1)}$ and $\Gamma_{13} = \gamma_1^{(1)} \subset \gamma_3^{(1)}$.

Between $\gamma_k^{(i)}$ and $\gamma_\ell^{(j)}$, one is tagged as *master* and the other as *slave*. Next, we mark each edge $\gamma_k^{(i)}$ with either the superscript "(m)" (if $\gamma_k^{(i)}$ is a master edge) or "(s)" (otherwise) and we define the skeleton

$$\Gamma^{(m)} = \bigcup_{k,i} \gamma_k^{(i),(m)}, \tag{51}$$

that in the mortar community is named *mortar interface.*

In the example of Fig. 2 right, we could tag as master the edge $\gamma_3^{(1)}$ (in which case $\gamma_1^{(1)}$ and $\gamma_2^{(2)}$ will be slave), or other way around.

Remark 6. Each cross-point (i.e. a vertex shared by more than two subdomains) belongs to the skeleton $\Gamma^{(m)}$. Cross-points shared by two (or more) master edges (like point P in Figs. 4–5) hold a single degree of freedom (that is, the finite element solution is continuous therein). Moreover, since a cross-point is always an interpolation node (as it is the endpoint of almost two edges), the value of the trace there is preserved when passing from the master edge to the slave one.

In the configurations of Fig. 4, the total number of points of $\Gamma^{(m)}$ is 9 and the point P (the number 3 of $\Gamma^{(m)}$) is shared by the three master edges $\gamma_1^{(1),(m)}$, $\gamma_2^{(1),(m)}$ and $\gamma_2^{(2),(m)}$. In the configurations of Fig. 5, the total number of points of $\Gamma^{(m)}$ is 10 and the point P (the number 4 of $\Gamma^{(m)}$) is shared by the two master edges $\gamma_2^{(1),(m)}$ and $\gamma_3^{(1),(m)}$.

If $\gamma_k^{(i)}$ and $\gamma_\ell^{(j)}$ are the master and the slave sides, respectively, whose intersection is $\Gamma_{k\ell}$, then $R_{(\ell,j),(k,i)}$ is the interpolation matrix that maps the master side to the slave one (it plays the role of matrix R_{21} defined in (32)), while $R_{(k,i),(\ell,j)}$ is the interpolation matrix from the slave to the master side (as R_{12} in (32)).

When the measure of $\gamma_\ell^{(j)}$ is larger than that of $\gamma_k^{(i)}$ (as, e.g., $\gamma_3^{(1)}$ and $\gamma_2^{(2)}$ in Fig. 2, right), all the basis functions of $\gamma_\ell^{(j)}$ whose support has non-empty intersection with $\gamma_k^{(i)}$ must be taken into account in building $R_{(k,i),(\ell,j)}$, included those basis functions associated with the nodes that do not belong to $\Gamma_{k\ell}$ (i.e. Γ_{23} in the case of Fig. 2, right). Alternatively, one can build the interface mass matrices and the interpolation matrices on the larger edge (as $\gamma_3^{(1)}$ in the case of Fig. 2, right), by assembling the contributions arising from the shorter edges of the opposite side of the interface (as $\gamma_1^{(1)}$ and $\gamma_2^{(2)}$ in the case of Fig. 2, right).

The modification presented in Sect. 6.1 for the case of two subdomains with non-homogeneous Dirichlet boundary conditions has to be implemented for the case of $M > 2$ subdomains. In particular, for any interface $\gamma_k^{(i)}$, the nodes of the boundary of $\gamma_k^{(i)}$ that are internal to Ω are treated as if they were "Dirichlet" boundary points with non-homogeneous boundary condition, thus in assembling the local stiffness matrices we use formulas (46) instead of $(A_{\Gamma_k,\Gamma_k})_{ij} = a_k(\overline{\mathcal{R}}_k\mu_j^{(k)}, \overline{\mathcal{R}}_k\mu_i^{(k)})$ and $(A_{\Gamma_k,k})_{ij} = a_k(\varphi_j^{(k)}, \overline{\mathcal{R}}_k\mu_i^{(k)})$.

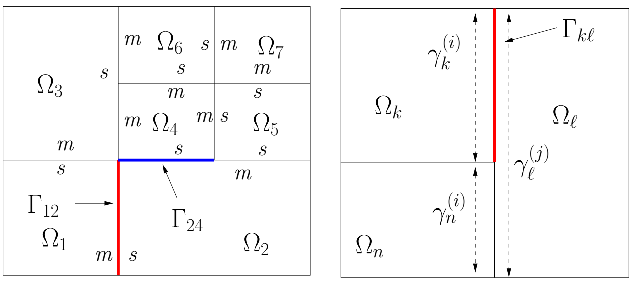

Figure 3: A partition of Ω into 7 subdomains (left figure). The letters m and s denote the choice made for the master and slave sides. Description of interfaces and edges (right)

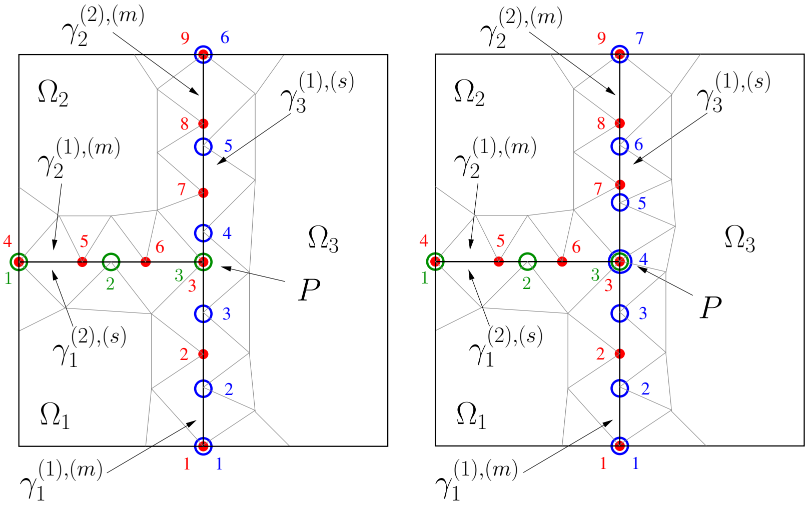

Figure 4: Each cross-point bears a single degree of freedom. The red full circles identify the nodes on the master edges. The empty blue circles identify the nodes on the slave edge $\gamma_3^{(1),(s)}$, while the empty green circles identify the nodes on the slave edge $\gamma_1^{(2),(s)}$. The cross-point P (i.e., the red point number 3) belongs to $\gamma_1^{(1),(m)}$, to $\gamma_2^{(1),(m)}$ and to $\gamma_2^{(2),(m)}$, moreover it coincides with one endpoint of the slave edge $\gamma_1^{(2),(s)}$ (in the left configuration), while it coincides with one endpoint of the slave edges $\gamma_1^{(2),(s)}$ and with one internal point of $\gamma_3^{(1),(s)}$ (in the right configuration).

The degrees of freedom of the global multidomain problem are the values of u_h at the nodes of $\Gamma^{(m)}$ jointly with the degrees of freedom internal to each Ω_k (as in (46)). As done in Section 6.2, we eliminate the degrees of freedom internal to the subdomains Ω_k and solve the Schur complement system (analogous to (47))

$$S\mathbf{u}_{\Gamma^{(m)}} = \mathbf{b} \tag{52}$$

by, e.g., a Krylov method. The matrix S is never assembled, the kernel subroutine to solve (52) (see Algorithm 2) computes the matrix-vector product $\mathbf{w} = S\boldsymbol{\lambda}$, for a given $\boldsymbol{\lambda}$ approximating $\mathbf{u}_{\Gamma^{(m)}}$.

6.4. Implementation

To better explain the construction of the intergrid matrices (41), we analyze the special configurations depicted in Figs. 4–5. Decompositions like that of Fig. 2 left, can be treated similarly, bearing in mind that each interface $\Gamma_{k\ell} = \partial\Omega_k \cap \partial\Omega_\ell$ of Fig. 2 is of the same nature of the interface $\Gamma_{12} = \gamma_1^{(2)} \cap \gamma_2^{(1)}$ in Figs. 4–5.

For simplicity, we consider $\mathbb{P}_1$ finite elements discretization in each subdomain.

First, we introduce two types of auxiliary matrices that are used to scatter and gather the degrees of freedom of the skeleton.

Let N and $n_{k,i}$ denote the total number of nodes in the skeleton $\Gamma^{(m)}$ and the number of nodes of the edge $\gamma_k^{(i)}$ (even master or slave), respectively.

For any master edge $\gamma_k^{(i),(m)}$ we define the operator $E_{k,i} : \Gamma^{(m)} \to \gamma_k^{(i),(m)}$ that extracts the degrees of freedom of $\gamma_k^{(i),(m)}$ from the array of the degrees of freedom on $\Gamma^{(m)}$. Its algebraic counterpart is a rectangular matrix of size $n_{k,i} \times N$ whose entries are 0 or 1 ($(E_{k,i})_{j\ell}) = 1$ only if the node $\mathbf{x}_j$ of $\gamma_k^{(i),(m)}$ is the node $\mathbf{x}_\ell$ of $\Gamma^{(m)}$).

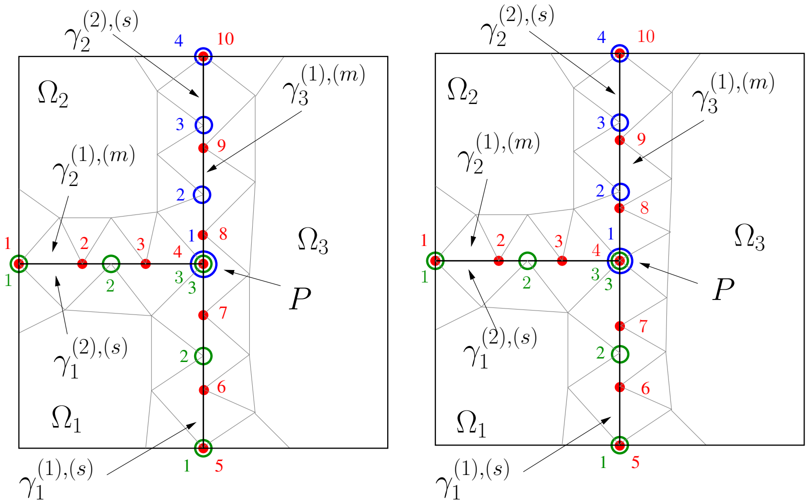

Figure 5: Each cross-point bears a single degree of freedom. The red full circles identify the nodes on the master edges. The empty blue circles identify the nodes on the slave edge $\gamma_2^{(2),(s)}$, while the empty green circles identify the nodes on the slave edges $\gamma_1^{(1),(s)}$ and $\gamma_1^{(2),(s)}$. At left, the cross-point P (i.e., the red point number 4) belongs to $\gamma_2^{(1),(m)}$ and it coincides with one endpoint of the slave edges $\gamma_1^{(1),(s)}$, $\gamma_1^{(2),(s)}$ and $\gamma_2^{(2),(s)}$. At right, the cross-point P (i.e., the red point number 4) belongs to $\gamma_2^{(1),(m)}$ and to $\gamma_3^{(1),(m)}$, moreover it coincides with one endpoint of the slave edges $\gamma_1^{(1),(s)}$, $\gamma_1^{(2),(s)}$ and $\gamma_2^{(2),(s)}$

For any edge $\gamma_k^{(i)}$ we define the diagonal matrix $D_{k,i}$ of size $n_{k,i}$ such that $(D_{k,i})_{jj} = 2$ if the point $\mathbf{x}_j$ of $\gamma_k^{(i)}$ is not an endpoint of $\gamma_k^{(i)}$ and at the same time it is a cross-point, otherwise $(D_{k,i})_{jj} = 1$. In the left configuration of both Fig. 4 and Fig. 5 the matrices $D_{k,i}$ are the identity matrices for any k and i; while in the right configurations of both Fig. 4 and Fig. 5, the entries $(D_{k,i})_{jj}$ are all equal to 1 with the exception of $(D_{3,1})_{44} = 2$ (P is the fourth node of $\gamma_3^{(1)}$).

By using formulas (32) we build the interpolation matrices $R_{(\ell,j),(k,i)}$ from $\gamma_k^{(i)}$ to $\gamma_\ell^{(j)}$ for any couple of edges $\gamma_k^{(i)}$ and $\gamma_\ell^{(j)}$ such that $\Gamma_{k\ell} = \gamma_k^{(i)} \cap \gamma_\ell^{(j)}$ is non-empty. Notice that some interpolation matrices have null rows and columns in view of the local support of the interface basis functions.

In the left configurations of Figs. 4 and 5, the interpolation matrices $R_{(l,j),(k,i)}$ from $\gamma_k^{(i)}$ to $\gamma_l^{(j)}$ are: $R_{(3,1),(1,1)} \in \mathbb{R}^{6\times 3}$ (rows 4,5,6 are null), $R_{(2,1),(1,2)} \in \mathbb{R}^{4\times 3}$, $R_{(1,2),(2,1)} \in \mathbb{R}^{3\times 4}$, $R_{(3,1),(2,2)} \in \mathbb{R}^{6\times 4}$ (rows 1,2,3 are null), $R_{(1,1),(3,1)} \in \mathbb{R}^{3\times 6}$ (columns 5,6 are null), $R_{(2,2),(3,1)} \in \mathbb{R}^{4\times 6}$ (columns 1,2 are null), while, in the right configurations of Figs. 4 and 5, the interpolation matrices $R_{(l,j),(k,i)}$ from $\gamma_k^{(i)}$ to $\gamma_l^{(j)}$ are: $R_{(3,1),(1,1)} \in \mathbb{R}^{7\times 3}$ (rows 5,6,7 are null), $R_{(2,1),(1,2)} \in \mathbb{R}^{4\times 3}$, $R_{(1,2),(2,1)} \in \mathbb{R}^{3\times 4}$, $R_{(3,1),(2,2)} \in \mathbb{R}^{7\times 4}$ (rows 1,2,3 are null), $R_{(1,1),(3,1)} \in \mathbb{R}^{3\times 7}$ (columns 5,6,7 are null), $R_{(2,2),(3,1)} \in \mathbb{R}^{4\times 7}$ (columns 1,2,3 are null).

If $\gamma_k^{(i),(m)}$ and $\gamma_\ell^{(j),(s)}$ are the master and the slave edge, respectively, such that $\Gamma_{k\ell} = \gamma_k^{(i),(m)} \cap \gamma_\ell^{(j),(s)}$ is non-empty, then the *master-to-slave intergrid matrices* are defined by

$$Q_{(\ell,j),(k,i)} = D_{\ell,j}^{-1} R_{(\ell,j),(k,i)}. \tag{53}$$

If $\boldsymbol{\lambda} = \mathbf{u}_{\Gamma^{(m)}}$ denotes the array of the degrees of freedom on the skeleton $\Gamma^{(m)}$, the Dirichlet datum $\mathbf{g}_\ell$ on the internal boundary $\Gamma_\ell = \partial\Omega_\ell \setminus \partial\Omega$ (for any ℓ) is computed as follows:

$$(\mathbf{g}_\ell)_{|\gamma_\ell^{(j)}} = \begin{cases} \sum_{\gamma_k^{(i)}} Q_{(\ell,j),(k,i)} E_{k,i}\boldsymbol{\lambda} & \text{if } \gamma_\ell^{(j)} \text{ is a slave edge,} \\ E_{\ell,j}\boldsymbol{\lambda} & \text{if } \gamma_\ell^{(j)} \text{ is a master edge,} \end{cases} \tag{54}$$

where the sum $\sum_{\gamma_k^{(i)}}$ has to be intended for all the master edges $\gamma_k^{(i)}$ such that $\Gamma_{k\ell} = \gamma_k^{(i)} \cap \gamma_\ell^{(j)}$ is non-empty. Matrices $D_{\ell,j}$ in (54) ensure that the interpolation process is consistent also at the cross-points shared by three subdomains.

As example, let us consider the configuration on the right of Fig. 4. We have chosen $\gamma_1^{(1)}$, $\gamma_2^{(1)}$ and $\gamma_2^{(2)}$ as master edges, thus the master-to-slave intergrid matrices are:

$$Q_{(3,1),(1,1)} = D_{3,1}^{-1} R_{(3,1),(1,1)}, \quad Q_{(1,2),(2,1)} = D_{1,2}^{-1} R_{(1,2),(2,1)}, \quad Q_{(3,1),(2,2)} = D_{3,1}^{-1} R_{(3,1),(2,2)},$$

while the Dirichlet data for the local subproblems are:

$$\begin{array}{l} (\mathbf{g}_1)_{|\gamma_1^{(2)}} = Q_{(1,2),(2,1)} E_{2,1}\boldsymbol{\lambda}, \qquad (\mathbf{g}_1)_{|\gamma_1^{(1)}} = E_{1,1}\boldsymbol{\lambda}, \\ (\mathbf{g}_2)_{|\gamma_2^{(1)}} = E_{2,1}\boldsymbol{\lambda}, \qquad (\mathbf{g}_2)_{|\gamma_2^{(2)}} = E_{2,2}\boldsymbol{\lambda}, \\ (\mathbf{g}_3)_{|\gamma_3^{(1)}} = Q_{(3,1),(1,1)} E_{1,1}\boldsymbol{\lambda} + Q_{(3,1),(2,2)} E_{2,2}\boldsymbol{\lambda}. \end{array}$$

Notice that if we did not premultiply the matrices $R_{(3,1),(1,1)}$ and $R_{(3,1),(2,2)}$ by $D_{3,1}^{-1}$, the value $\mathbf{g}_3$ at the cross-point P would be the double of the correct value. This because P is shared by the two consecutive edges $\gamma_1^{(1)}$ and $\gamma_2^{(2)}$, and the interpolation process returns the value of the interpolated function at P on each edge.

In assembling the Dirichlet datum on Γ_ℓ, we suggest to assemble the vector $\mathbf{g}_\ell$ (for any ℓ) in this way: first loop on the slave edges of Γ_ℓ and then loop on the master edges of Γ_ℓ. Since a cross-point always belongs to the skeleton $\Gamma^{(m)}$ (see Remark 6), no ambiguity occurs when we define the Dirichlet datum, even when the cross-point is the endpoint of two consecutive slave edges of a subdomain. For example, let us consider the left configuration of Fig. 5, where both $\gamma_1^{(1)}$ and $\gamma_1^{(2)}$ are slave edges in Ω_1. It is evident that the best value to be taken into account for $\mathbf{g}_1$ at the cross-point P is the one stored in $(\mathbf{g}_1)_{|\gamma_1^{(2)}}$ (since it is exactly the value of the master trace at P). Nevertheless, no problem occurs if in $(\mathbf{g}_1)(P)$ we store the value obtained by interpolation of the trace on $\gamma_3^{(1)}$.

Now, let us suppose that all the residual arrays $\mathbf{r}_{k,i}$ ($\mathbf{r}_{k,i}$ is the residual on the edge $\gamma_k^{(i)}$) have been computed (with formulas (46) to be taken into account when dealing with cross-points) and let $M_{k,i}$ denote the interface mass matrix on $\gamma_k^{(i)}$, defined as in (23).

If $\gamma_\ell^{(j),(s)}$ and $\gamma_k^{(i),(m)}$ are the slave and the master edges, respectively, such that $\Gamma_{k\ell} = \gamma_k^{(i),(m)} \cap \gamma_\ell^{(j),(s)}$ is non-empty, then we define the *slave-to-master intergrid matrices* by:

$$Q_{(k,i),(\ell,j)} = M_{k,i} D_{k,i}^{-1} R_{(k,i),(\ell,j)} M_{\ell,j}^{-1}. \tag{55}$$

Finally, the sum of the residuals at the nodes of the skeleton $\Gamma^{(m)}$ is given by:

$$\mathbf{w} = \sum_{\gamma_k^{(i),(m)}} E_{k,i}^T (\mathbf{r}_{k,i} + \sum_{\gamma_\ell^{(j),(s)}} Q_{(k,i),(\ell,j)} \mathbf{r}_{\ell,j}), \tag{56}$$

where $\sum_{\gamma_\ell^{(j),(s)}}$ has to be intended on all the slave edges $\gamma_\ell^{(j),(s)}$ such that the interface $\Gamma_{k\ell} = \gamma_k^{(i),(m)} \cap \gamma_\ell^{(j),(s)}$ is non-empty.

The matrices $D_{k,i}$ in (55) ensure that, when a cross-point P is internal to a master edge and, at the same time it is the common endpoint of two consecutive slave edges, the (arithmetic) average of the two slave residuals at P is consistent with the master residual computed at P.

An example is given by the right configuration of Fig. 5: the master edge is $\gamma_3^{(1)}$, the consecutive slave edges are $\gamma_1^{(1)}$ and $\gamma_2^{(2)}$, the cross-point is P. In order to analyze what happens at P we can think at the normal derivatives to the interfaces. The normal derivatives to Γ_{12} balance each other and no problem occurs. On the other hand, the normal derivative to $\gamma_3^{(1)}$ at P balances either the normal derivative to $\gamma_1^{(1)}$ and the normal derivative to $\gamma_2^{(2)}$, but not their sum. Thus, if in (55) we did not premultiply both the matrices $R_{(3,1),(1,1)}$ and $R_{(3,1),(2,2)}$ by $D_{3,1}^{-1}$, the sum at the cross-point P would be about the double of the correct value.

Notice that, in the left configuration of Fig. 5 the latter problem does not occur, since each node of $\gamma_3^{(1)}$ is internal to a single slave edge, either $\gamma_1^{(1)}$ or $\gamma_2^{(2)}$.

7. Numerical results

Let us consider the Laplace problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \\ u = g & \text{on } \partial\Omega. \end{cases}$$

When g is different from zero, by standard arguments we recast the problem into the form (1). See [30], Section 6.1 and Section 6.

2D test case. The data f and g are such that the exact solution is $u(x, y) = \sin(xy\pi) + 1$. A decomposition of $\Omega = (0, 2)^2$ in 10 subdomains as in Fig. 6 is considered, and independent triangulations in each Ω_k are designed so that on each interface both polynomial non-conformity and geometric non-conformity occur. Either $\mathbb{P}_1$ or quadrilateral *hp*-fem ($\mathbb{Q}_p$) are used to approximate the numerical solution. In order to guarantee full non-conformity on each interface, different polynomial degrees and different element sizes are used inside the subdomains, by setting the polynomial degree equal to either p or $p + 1$ on two adjacent domains and the number of elements equal to either N or $N + 1$, then we set $h = 1/N$. A non-conforming grid, obtained with $\mathbb{Q}_p$ discretizations in each subdomain, is shown in Fig. 6, left.

In Fig. 7, the errors in broken norm (see formula (87)) are shown, w.r.t. to both h and p (the polynomial degree in the bottom-left subdomain). The error behavior versus h (see Fig. 7 left) agrees with the theoretical estimate of Theorem 12, for which we expect $\|u-u_h\|_* \leq c(u)h^p$ (in this case $p = 1, 2, 3$, see (101)). The convergence rate vs p shown in Fig. 7, right, is more than algebraic, as typical in *hp*-fem. The interested reader can find in [16, 18] a wide collection of numerical results on INTERNODES, even applied to both Navier-Stokes equations and fluid-structure interaction problems.

3D test case. The computational domain $\Omega = (0, 2) \times (0, 1) \times (0, 1)$ is decomposed into two subdomains $\Omega_1 = (0, 1)^3$ and $\Omega_2 = (1, 2) \times (0, 1) \times (0, 1)$. The data f and g are set in such a way that the exact solution is $u(x, y, z) = (y^2 - y)(z^2 - z)\sin(xyz\pi)$. In Ω_1 (Ω_2, resp.) we consider a triangulation in $N \times N \times N$ elements ($(N - 2) \times (N - 2) \times (N - 2)$, resp.). Then we set $h_1 = 1/N$ and $h_2 = 1/(N - 2)$. When $p = 1$ the triangulation is composed by tetrahedra, thus classical $\mathbb{P}_1$

Algorithm 2 matrix vector product $\mathbf{w} = S\boldsymbol{\lambda}$

INPUT: $\boldsymbol{\lambda}$ (trace on the master interface $\Gamma^{(m)}$)
OUTPUT: $\mathbf{w}$ (sum of fluxes on the master interface $\Gamma^{(m)}$)

for all $k = 1, \ldots, M$ (loop on the subdomains) **do**
  % set the Dirichlet data for the local subproblems
  for all i s.t. $\gamma_k^{(i)} \subset \Gamma_k = \partial\Omega_k \setminus \partial\Omega$ is a slave edge **do**
    recover all the master sides $\gamma_\ell^{(j)}$ of Ω_l associated with $\gamma_k^{(i)}$ (s.t. $\gamma_\ell^{(j)} \cap \gamma_k^{(i)} = \Gamma_{kl}$ is non-empty)
    extract $\boldsymbol{\lambda}_{|\gamma_\ell^{(j)}}$ from $\boldsymbol{\lambda}$ and interpolate from master to slave:
    $(\mathbf{g}_k)_{\gamma_k^{(i)}} = \sum_{\gamma_\ell^{(j)}} Q_{(k,i),(\ell,j)} E_{\ell,j} \boldsymbol{\lambda}$
  end for
  for all i s.t. $\gamma_k^{(i)} \subset \Gamma_k$ is a master edge **do**
    $(\mathbf{g}_k)_{|\gamma_k^{(i)}} = E_{k,i}\boldsymbol{\lambda}$
  end for
  % solve the local problem in Ω_k
  solve $A_k \mathbf{u}_k = \mathbf{f}_k$, where $\mathbf{f}_k$ takes into account only the Dirichlet datum $\mathbf{g}_k$ on Γ_k, while the external data (f and boundary conditions) are null on Γ_k
  % compute the local residual on each internal edge of Γ_k
  for all i s.t. $\gamma_k^{(i)} \subset \Gamma_k$ (loop on all the edges of Γ_k) **do**
    $\mathbf{r}_{k,i} = (A_k \mathbf{u}_k)_{|\gamma_k^{(i)}}$ (*)
  end for
end for
% interpolate the residuals from the slave to the master edges and assemble them on $\Gamma^{(m)}$

$$\mathbf{w} = \sum_{\gamma_k^{(i),(m)}} E_{k,i}^T \Big(\mathbf{r}_{k,i} + \sum_{\gamma_\ell^{(j),(s)}} Q_{(k,i),(\ell,j)} \mathbf{r}_{\ell,j}\Big) \quad \textbf{(**)}$$

(*) if a vertex of Γ_k belongs to two consecutive edges, keep distinct the contributions of the residuals arising from the two edges, since each $\mathbf{r}_{k,i}$ should approximate the normal derivative to the edge $\gamma_k^{(i)}$.
(**) $\gamma_k^{(i),(m)}$ denotes any master edge, the sum $\sum_{\gamma_\ell^{(j),(s)}}$ has to be intended on all the edges $\gamma_\ell^{(j),(s)}$ such that $\Gamma_{k\ell} = \gamma_k^{(i),(m)} \cap \gamma_\ell^{(j),(s)}$ is non-empty.

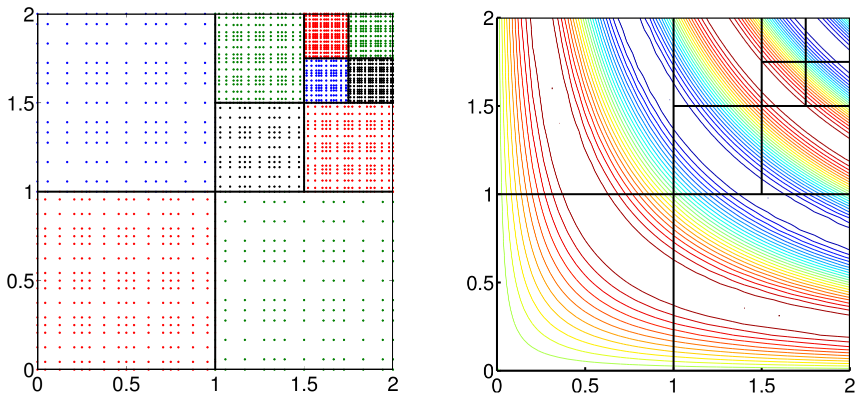

Figure 6: 2D test case. A partition into several subdomains (left picture); the dots are the nodes of the triangulations within the subdomains. The corresponding INTERNODES solution is reported on the right picture

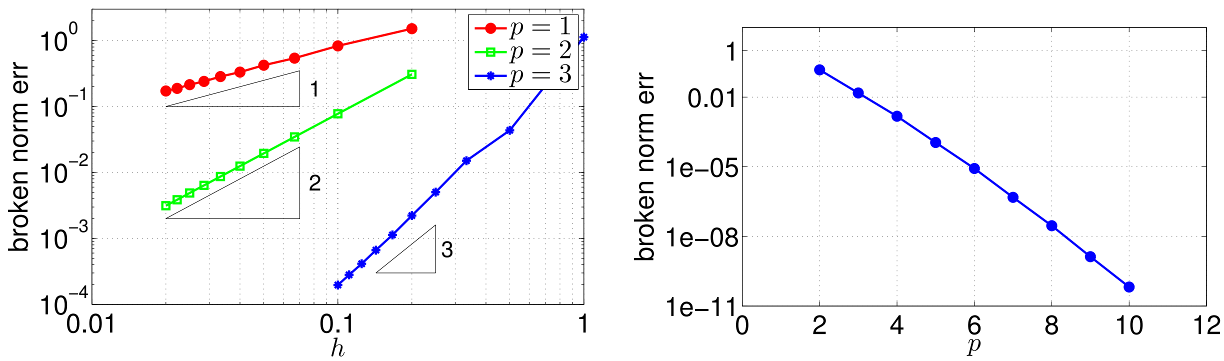

Figure 7: The broken norm error w.r.t. the mesh-size h with p fixed (left). The broken norm error w.r.t. p (it is the polynomial degree in the bottom-left subdomain), here the mesh size is fixed $h = 1/3$ (right)

fem are used, while when $p > 1$, the mesh is formed by hexahedra and hp-fem with $\mathbb{Q}_p$ local spaces are considered.

In Fig. 8, the errors in broken norm are shown, w.r.t. both the mesh size h_1 of Ω_1 and the local polynomial degree $p = p_1 = p_2$. Also in this case the numerical results agree with the theoretical estimate of Theorem 12.

8. A comparison between the algebraic form of INTERNODES and Mortar methods

We follow the notations of [7] for the classical mortar method and those of [24] for the unsymmetric mortar method, a special version of mortar method proposed in [12] in which the cross-domain mass matrices on the interface are computed by suitable quadrature formulas instead of (the computationally heavy) exact integration.

Let $\mu_i^{(k)}$ (for $k = 1, 2$) be the Lagrange basis functions on Γ, and $\psi_i^{(2)}$ the basis functions of the mortar space, being the latter associated with the slave domain Ω_2. Then we set the mortar mass

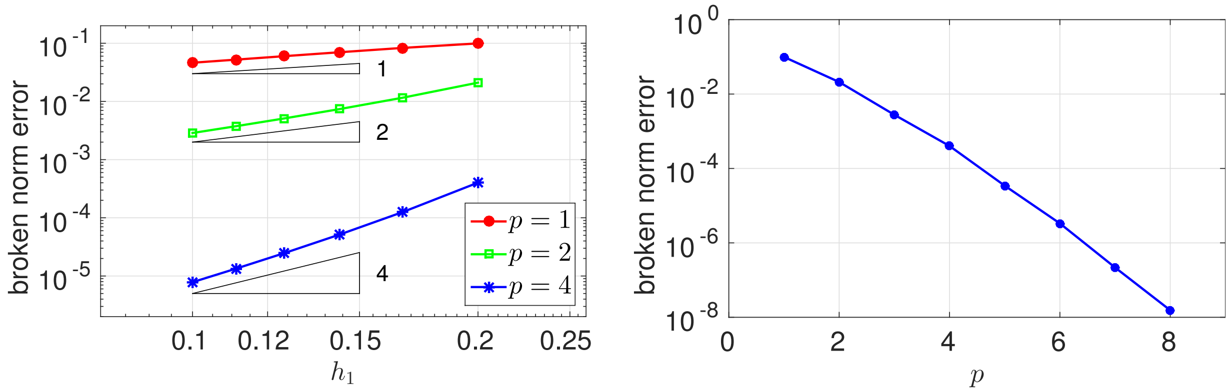

Figure 8: 3D test case. The broken norm error w.r.t. the mesh-size h_1 with $p = p_1 = p_2$ fixed (left) and w.r.t. the local polynomial degree $p = p_1 = p_2$, with fixed mesh size (right) $h_1 = 1/5$ and $h_2 = 1/3$.

matrices

$$\begin{aligned}
&\Xi = P^{-1}\Phi, && P_{ij} = \int_\Gamma \mu_j^{(2)}\psi_i^{(2)}, && \Phi_{ij} = \int_\Gamma \mu_j^{(1)}\psi_i^{(2)},\\
&\Xi^{--} = (P^-)^{-1}\Phi^-, && P^-_{ij} = \Sigma_-\mu_j^{(2)}\psi_i^{(2)}, && \Phi^-_{ij} = \Sigma_-\mu_j^{(1)}\psi_i^{(2)},\\
&\Xi^{-+} = (P^-)^{-1}\Phi^+, && && \Phi^+_{ij} = \Sigma_+\mu_j^{(1)}\psi_i^{(2)},
\end{aligned}$$

being Σ_- (Σ_+, resp.) the quadrature formula on the interface Γ induced by the discretization in the slave domain Ω_2 (master domain Ω_1, resp.).

Both classical and unsymmetric mortar methods can be recast in the form (45) provided that the matrices Q_{12} and Q_{21} are defined as follows:

	classical mortar	unsym. mortar	INTERNODES
Q_{21}	Ξ	Ξ^{--}	R_{21}
Q_{12}	Ξ^T	$(\Xi^{-+})^T$	$M_{\Gamma_1}R_{12}M_{\Gamma_2}^{-1}$

A similarity can however be established between INTERNODES and the unsymmetric mortar method presented in [12, 24]. As a matter of fact, in the case that the quadrature nodes used in Σ_- are a subset of the grid nodes induced on Γ by the discretization inside the slave domain Ω_2, then $\Xi^{--} = R_{21}$. Despite this, by choosing the basis functions $\psi_i^{(2)}$ of the mortar space as standard (see [7, 24]) we observed numerically that the matrix $M_{\Gamma_1}R_{12}M_{\Gamma_2}^{-1}$ does not coincide with $(\Xi^{-+})^T$: INTERNODES and unsymmetric mortar are indeed two different methods.

A more thorough comparison between the classical mortar method and INTERNODES can be found in [16]. In the same paper the implementation aspects and the computational complexity of the two approaches as well as their convergence rate with respect to the mesh sizes are discussed, concluding that in practice INTERNODES attains the same accuracy as the classical mortar method.

9. Analysis of INTERNODES

In order to analyze INTERNODES for the case of two subdomains, we write an interface formulation of transmission problem (8).

The analysis will be carried out in the case of straight interfaces and when the intergrid operators Π_{12} and Π_{21} are the classical Lagrange interpolation operators. (See also Remark 8.) Moreover, for sake of clearness (in fact to guarantee that $r_{k,h_k} \in \Lambda'_{k,h_k}$ and to identify Y'_{h,k_h} with Λ'_{k,h_k}), we make the following assumptions.

Assumptions 2. Let $dist(\overline{\Gamma}, \overline{\partial\Omega_D}) > 0$, that is Neumann boundary conditions are imposed on those parts of $\partial\Omega$ that contain the boundary of the interface Γ. Then, $\Lambda_{k,h_k} = Y_{k,h_k}$, $n_k = \overline{n}_k$ and $\mathcal{R}_{k,h_k} = \overline{\mathcal{R}}_k$ for $k = 1, 2$.

Notice that, Assumption 2 is redundant if the subdomains are either rectangles (when $d = 2$) or parallelepipedons (when $d = 3$) since, in such a case, $r_{k,h_k} \in \Lambda'_{k,h_k}$ when either Neumann or homogeneous Dirichlet conditions are assigned on those parts of $\partial\Omega$ that contain the boundary of the interface Γ.

In Section 9.3 we provide the weak formulation of INTERNODES for decompositions with internal cross-points, starting from the simple configuration of Fig. 2, right. As we will see, the analysis of INTERNODES for decompositions with internal cross-points does not introduce additional difficulties with respect to the case with only two subdomains.

Along the whole section, c will denote a generic positive constant independent of the mesh sizes h_1 and h_2, but not necessarily the same everywhere.

9.1. *Interface formulation of the continuous problem*

For $k = 1, 2$, given $\lambda \in \Lambda$ and $f \in L^2(\Omega)$, we consider the non-homogeneous Dirichlet problem

$$\text{find } u_k^{\lambda,f} \in V_k : \quad a_k(u_k^{\lambda,f}, v) = (f, v)_{L^2(\Omega_k)} \quad \forall v \in V_k^0, \quad u_k^{\lambda,f} = \lambda \quad \text{on } \Gamma. \tag{57}$$

Because of the linearity of $a_k(\cdot,\cdot)$, we have $u_k^{\lambda,f} = \widehat{u}_k + u_k^{\lambda}$, where:

$$\widehat{u}_k \in V_k^0 : \quad a_k(\widehat{u}_k, v) = (f, v)_{L^2(\Omega_k)} \qquad \forall v \in V_k^0 \tag{58}$$

and

$$u_k^{\lambda} \in V_k : \quad a_k(u_k^{\lambda}, v) = 0 \quad \forall v \in V_k^0, \qquad u_k^{\lambda} = \lambda \quad \text{on } \Gamma. \tag{59}$$

The following stability estimate holds (see, e.g., [30, Sect. 6.1.2])

$$\|u_k^{\lambda,f}\|_{H^1(\Omega_k)} \leq c(\|f\|_{L^2(\Omega_k)} + \|\lambda\|_{\Lambda}). \tag{60}$$

We consider the following *interface formulation* of equation (9) with four unknowns: find $\lambda_1 \in \Lambda$, $\lambda_2 \in \Lambda$, and $r_1 \in \Lambda'$, $r_2 \in \Lambda'$ such that

$$\begin{cases} a_k(u_k^{\lambda_k}, \mathcal{R}_k\mu_k) - \langle r_k, \mu_k\rangle = (f, \mathcal{R}_k\mu_k)_{L^2(\Omega_k)} - a_k(\widehat{u}_k, \mathcal{R}_k\mu_k) & \forall \mu_k \in \Lambda,\ k = 1, 2, \\ \langle t, \lambda_1 - \lambda_2\rangle = 0 & \forall t \in \Lambda' \\ \langle r_1 + r_2, \varphi\rangle = 0 & \forall \varphi \in \Lambda \end{cases} \tag{61}$$

where $\mathcal{R}_k$ are defined in (10). The multipliers r_k coincide with $\partial_{L_k} u_k$, see Remark 1. By eliminating $r_1 = -r_2$ from the last equation and summing up the first two equations we obtain another *interface formulation* of equation (9) with three unknowns: find $\lambda_1 \in \Lambda$, $\lambda_2 \in \Lambda$, and $r_2 \in \Lambda'$ s.t.

$$\begin{cases} \displaystyle\sum_{k=1,2} a_k(u_k^{\lambda_k}, \mathcal{R}_k\mu_k) + \langle r_2, \mu_1 - \mu_2\rangle = \sum_{k=1,2} [(f, \mathcal{R}_k\mu_k)_{L^2(\Omega_k)} - a_k(\widehat{u}_k, \mathcal{R}_k\mu_k)] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall(\mu_1,\mu_2) \in \Lambda \times \Lambda \\ \langle t, \lambda_1 - \lambda_2\rangle = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \forall t \in \Lambda'. \end{cases} \tag{62}$$

By setting $\boldsymbol{\mu} = (\mu_1, \mu_2)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, $\boldsymbol{\Lambda} = \Lambda \times \Lambda$ (endowed with the norm $\|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}} = (\|\lambda_1\|^2_\Lambda + \|\lambda_2\|^2_\Lambda)^{1/2}$), and $\forall \boldsymbol{\lambda},\ \boldsymbol{\mu} \in \boldsymbol{\Lambda}$, $\forall t \in \Lambda'$,

$$\begin{aligned} &\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{k=1,2} a_k(u_k^{\lambda_k}, \mathcal{R}_k\mu_k), \qquad \mathcal{B}(\boldsymbol{\mu}, t) = \langle t, \mu_1\rangle - \langle t, \mu_2\rangle \\ &\mathcal{F}(\boldsymbol{\mu}) = \sum_{k=1,2} [(f, \mathcal{R}_k\mu_k)_{L^2(\Omega_k)} - a_k(\widehat{u}_k, \mathcal{R}_k\mu_k)], \end{aligned} \tag{63}$$

problem (62) takes the saddle point form: find $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ and $r_2 \in \Lambda'$ s.t.

$$\begin{cases} \mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu}) + \mathcal{B}(\boldsymbol{\mu}, r_2) = \mathcal{F}(\boldsymbol{\mu}) & \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}, \\ \mathcal{B}(\boldsymbol{\lambda}, t) = 0 & \forall t \in \Lambda'. \end{cases} \tag{64}$$

Lemma 1. *The following properties hold:*

1. the bilinear form $\mathcal{A}$ is coercive and continuous on $\boldsymbol{\Lambda}$, i.e., there exist $\alpha_* > 0$ and $C_A > 0$ s.t.

$$\begin{aligned} &\mathcal{A}(\boldsymbol{\mu}, \boldsymbol{\mu}) \geq \alpha_* \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} && \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}, \\ &|\mathcal{A}(\boldsymbol{\lambda}, \boldsymbol{\mu})| \leq C_A \|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}} \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} && \forall \boldsymbol{\lambda}, \boldsymbol{\mu} \in \boldsymbol{\Lambda}; \end{aligned} \tag{65}$$

2. the bilinear form $\mathcal{B}$ is continuous and satisfies an inf-sup condition, i.e. there exist $C_B > 0$ s.t.

$$\begin{aligned} &|\mathcal{B}(\boldsymbol{\mu}, t)| \leq C_B \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} \|t\|_{\Lambda'} \qquad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda},\ t \in \Lambda', \\ &\inf_{t \in \Lambda'} \sup_{\boldsymbol{\mu} \in \boldsymbol{\Lambda}} \frac{\mathcal{B}(\boldsymbol{\mu}, t)}{\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} \|t\|_{\Lambda'}} \geq \sqrt{2}; \end{aligned} \tag{66}$$

3. the linear functional $\mathcal{F}$ is continuous, i.e. there exists $C_F > 0$ s.t.

$$|\mathcal{F}(\boldsymbol{\mu})| \leq C_F \|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} \qquad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}. \tag{67}$$

Proof. 1. By taking $\mathcal{R}_k\mu_k = u_k^{\lambda_k}$, continuity and coercivity of $\mathcal{A}$ are an immediate consequence of continuity and coercivity of the bilinear forms a_k (see [30, Sect. 1.2]).

2. The continuity of $\mathcal{B}$ follows from Cauchy-Schwarz inequality. To prove the inf-sup condition, we define the operators $B : \boldsymbol{\Lambda} \to \Lambda$ and $B^T : \Lambda' \to \boldsymbol{\Lambda}'$ such that

$$\mathcal{B}(\boldsymbol{\mu}, t) = \langle t, B\boldsymbol{\mu}\rangle = \langle B^T t, \boldsymbol{\mu}\rangle \qquad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda},\ \forall t \in \Lambda'.$$

Then, thanks to $(63)_1$, it holds $B\boldsymbol{\mu} = \mu_1 - \mu_2$ for any $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \boldsymbol{\Lambda}$, and $B^T t = [t,\ -t]^T$ for any $t \in \Lambda'$, thus $\|B^T t\|_{\boldsymbol{\Lambda}'} = \sqrt{2}\|t\|_{\Lambda'}$. It follows that

$$\inf_{t \in \Lambda'} \sup_{\boldsymbol{\mu} \in \boldsymbol{\Lambda}} \frac{\mathcal{B}(\boldsymbol{\mu}, t)}{\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} \|t\|_{\Lambda'}} = \inf_{t \in \Lambda'} \sup_{\boldsymbol{\mu} \in \boldsymbol{\Lambda}} \frac{\langle B^T t, \boldsymbol{\mu}\rangle}{\|\boldsymbol{\mu}\|_{\boldsymbol{\Lambda}} \|t\|_{\Lambda'}} = \inf_{t \in \Lambda'} \frac{\|B^T t\|_{\boldsymbol{\Lambda}'}}{\|t\|_{\Lambda'}} \geq \sqrt{2}.$$

3. (67) follows by Cauchy-Schwarz inequality and the continuity of the bilinear forms a_k.

Theorem 2. *Problem (62) is well posed. Moreover it is equivalent to (9), in the following sense: if* $\{\lambda_1, \lambda_2,\ r_2\}$ *solves (62), then* $u_1 = u_1^{\lambda,f}$ *and* $u_2 = u_2^{\lambda,f}$ *(with* $\lambda = \lambda_1 = \lambda_2$*) are the unique solutions of (9); conversely, if* $\{u_1,\ u_2\}$ *solves (9), then* λ_1*,* λ_2*, and* r_2 *solve (62), with*

$$\lambda_1 = (u_1)_{|\Gamma}, \quad \lambda_2 = (u_2)_{|\Gamma}, \quad \langle r_2, \mu\rangle = a_2(u_2, \mathcal{R}_2\mu) - (f, \mathcal{R}_2\mu)_{L^2(\Omega_2)} \qquad \forall\mu \in \Lambda.$$

PROOF. Thanks to Lemma 1, the well-posedness of problem (62) (existence, uniqueness and stability of the solution) follows by applying standard results for saddle point problems (see, e.g., [8, Cor. 4.2.1]) to (64).

The equivalence between (62) and (9) can be proved by standard arguments.

9.2. Interface formulation of the discrete non-conforming problem

Let Λ_{1,h_1} and Λ_{2,h_2} be induced by independent discretizations in Ω_1 and Ω_2 as in Sect. 4. Let $\mathbf{\Lambda}_h = (\Lambda_{1,h_1}, \Lambda_{2,h_2})$ be endowed with the norm of $\mathbf{\Lambda}$, and for $k = 1, 2$, let $\Lambda'_{k,h_k} = (\Lambda_{k,h_k}, \|\cdot\|_{\Lambda'})$ (Λ_{k,h_k} and Λ'_{k,h_k} are identical linear spaces, see Sect. 5 and [10]). Let the Assumptions 2 be satisfied.

By applying the conforming finite element approximation introduced in Sect. 3 in each subdomain Ω_k, we can write the finite dimensional counterparts of (57)–(59): given $f \in L^2(\Omega)$ and $\lambda_{k,h_k} \in \Lambda_{k,h_k}$, for $k = 1, 2$, we denote by $U_k = U_k(\lambda_{k,h_k}, f) \in V_{k,h_k}$ the solution of

$$a_k(U_k, v) = (f, v)_{L^2(\Omega_k)} \qquad \forall v \in V^0_{k,h_k}, \qquad U_k = \lambda_{k,h_k} \quad \text{on } \Gamma. \tag{68}$$

We note that $U_k = \overline{\mathcal{H}}_k\lambda_{k,h_k} + \widehat{U}_k$, where $\widehat{U}_k = \widehat{U}_k(f) \in V^0_{k,h_k}$ is the solution of

$$a_k(\widehat{U}_k, v) = (f, v)_{L^2(\Omega_k)} \qquad \forall v \in V^0_{k,h_k}, \tag{69}$$

and $\overline{\mathcal{H}}_k\lambda_{k,h_k} \in V_{k,h_k}$ is the solution of

$$a_k(\overline{\mathcal{H}}_k\lambda_{k,h_k}, v) = 0 \quad \forall v \in V^0_{k,h_k}, \qquad \overline{\mathcal{H}}_k\lambda_{k,h_k} = \lambda_{k,h_k} \quad \text{on } \Gamma. \tag{70}$$

Then similarly to the continuous case,

$$\|U_k\|_{H^1(\Omega_k)} \le c(\|f\|_{L^2(\Omega_k)} + \|\lambda_{k,h_k}\|_\Lambda). \tag{71}$$

We introduce the non-conforming counterpart of (62), i.e. an interface form of the non-conforming problem (34):

find $\lambda_{1,h_1} \in \Lambda_{1,h_1}$, $\lambda_{2,h_2} \in \Lambda_{2,h_2}$, and $r_{2,h_2} \in \Lambda'_{2,h_2}$ s.t.

$$\begin{cases} \displaystyle\sum_{k=1,2} a_k(\overline{\mathcal{H}}_k\lambda_{k,h_k}, \overline{\mathcal{R}}_k\mu_{k,h_k}) + \langle \Pi_{12} r_{2,h_2}, \mu_{1,h_1}\rangle - \langle r_{2,h_2}, \mu_{2,h_2}\rangle \\ \qquad = \displaystyle\sum_{k=1,2} \Big[(f, \overline{\mathcal{R}}_k\mu_{k,h_k})_{L^2(\Omega_k)} - a_k(\widehat{U}_k, \overline{\mathcal{R}}_k\mu_{k,h_k})\Big] \qquad \forall(\mu_{1,h_1}, \mu_{2,h_2}) \in \Lambda_{1,h_1} \times \Lambda_{2,h_2}, \\ \langle t_{2,h_2}, \lambda_{2,h_2} - \Pi_{21}\lambda_{1,h_1}\rangle = 0 \qquad\qquad \forall t_{2,h_2} \in \Lambda'_{2,h_2}, \end{cases} \tag{72}$$

where $\overline{\mathcal{R}}_k$ are the continuous discrete liftings defined in (33).

Theorem 3. *Problem (72) is equivalent to problem (34) – (35) in the following sense: if* $\{\lambda_{1,h_1}, \lambda_{2,h_2}, r_{2,h_2}\}$ *solves (72), then* $\{u_{1,h_1} = U_1,\ u_{2,h_2} = U_2,\ r_{2,h_2}\}$ *solves (34) – (35); conversely, if* $\{u_{1,h_1}, u_{2,h_2}, r_{2,h_2}\}$ *solves (34) – (35), then* $\{\lambda_{1,h_1} = (u_{1,h_1})_{|\Gamma}, \lambda_{2,h_2} = (u_{2,h_2})_{|\Gamma}, r_{2,h_2}\}$ *solves (72).*

Proof. Let $\{\lambda_{1,h_1}, \lambda_{2,h_2}, r_{2,h_2}\}$ solve (72). Then, $u_{1,h_1} = U_1$ and $u_{2,h_2} = U_2$ (solutions of (68)) solve $(34)_1$, while, from $(72)_2$ it follows $U_2 = \Pi_{21} U_1$ on Γ, i.e., $(34)_2$ holds. To prove $(34)_3$– (35), we set $\mu_{1,h_1} = 0$ and $\mu_{2,h_2} = \mu_i^{(2)}$ in $(72)_1$, then $\langle r_{2,h_2}, \mu_i^{(2)}\rangle = a_2(U_2, \overline{\mathcal{R}}_2 \mu_i^{(2)}) - (f, \overline{\mathcal{R}}_2 \mu_i^{(2)})_{L^2(\Omega_2)}$, i.e., r_{2,h_2} and U_2 satisfy (35) for $k = 2$. If we set $\mu_{2,h_2} = 0$ and $\mu_{1,h_1} = \mu_i^{(1)}$ in $(72)_1$, we have

$$\langle -\Pi_{12} r_{2,h_2}, \mu_i^{(1)}\rangle \quad = a_1(U_1, \overline{\mathcal{R}}_1 \mu_i^{(1)}) - (f, \overline{\mathcal{R}}_1 \mu_i^{(1)})_{L^2(\Omega_1)},$$

then, by setting $r_{1,h_1} = -\Pi_{12} r_{2,h_2}$, r_{1,h_1} and U_1 satisfy (35) for $k = 1$, and $(34)_3$ holds.

Conversely, let $\{u_{1,h_1}, u_{2,h_2}\}$ solve (34) – (35) and set $\lambda_{k,h_k} = (u_{k,h_k})_{|\Gamma}$ for $k = 1, 2$, thus $u_{k,h_k} = U_k$. We prove that $\{\lambda_{1,h_1}, \lambda_{2,h_2},\ r_{2,h_2}\}$ solves (72).

By using (35), (24) and (68) we have

$$\begin{aligned}\langle r_{k,h_k}, \mu_i^{(k)}\rangle \ &= a_k(U_k, \overline{\mathcal{R}}_k \mu_i^{(k)}) - (f, \overline{\mathcal{R}}_k \mu_i^{(k)})_{L^2(\Omega_k)} \\ &= a_k(\overline{\mathcal{H}}_k \lambda_{k,h_k}, \overline{\mathcal{R}}_k \mu_i^{(k)}) + a_k(\widehat{U}_k, \overline{\mathcal{R}}_k \mu_i^{(k)}) - (f, \overline{\mathcal{R}}_k \mu_i^{(k)})_{L^2(\Omega_k)}.\end{aligned} \tag{73}$$

Thus, by adding the two equations of (73) for $k = 1, 2$ and exploiting $(34)_3$, we obtain $(72)_1$. Equation $(72)_2$ follows from $(34)_2$.

The following result is a consequence of Theorem 3 and Remark 5.

Corollary 4. *If* $\Lambda_{1,h_1} = \Lambda_{2,h_2}$*, problem (72) is equivalent to problem (18).*

To study the well-posedness of problem (72) in the general case of $\Lambda_{1,h_1} \neq \Lambda_{2,h_2}$, we set $\boldsymbol{\mu}_h = (\mu_{1,h_1}, \mu_{2,h_2})$ for any $\mu_{1,h_1} \in \Lambda_{1,h_1}$ and $\mu_{2,h_2} \in \Lambda_{2,h_2}$ and define:

$$\begin{array}{ll}\mathcal{A}_h(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) = \displaystyle\sum_{k=1,2} a_k(\overline{\mathcal{H}}_k \lambda_{k,h_k}, \overline{\mathcal{R}}_k \mu_{k,h_k}) & \forall \boldsymbol{\lambda}_h,\ \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h, \\ \mathcal{B}_{1,h}(\boldsymbol{\mu}_h, t_{2,h_2}) = \langle \Pi_{12} t_{2,h_2}, \mu_{1,h_1}\rangle - \langle t_{2,h_2}, \mu_{2,h_2}\rangle & \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h,\ \forall t_{2,h_2} \in \Lambda'_{2,h_2} \\ \mathcal{B}_{2,h}(\boldsymbol{\mu}_h, t_{2,h_2}) = \langle t_{2,h_2}, \mu_{2,h_2} - \Pi_{21} \mu_{1,h_1}\rangle & \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h,\ \forall t_{2,h_2} \in \Lambda'_{2,h_2} \\ \mathcal{F}_h(\boldsymbol{\mu}_h) = \displaystyle\sum_{k=1,2} \left[(f, \overline{\mathcal{R}}_k \mu_{k,h_k})_{L^2(\Omega_k)} - a_k(U_k, \overline{\mathcal{R}}_k \mu_{k,h_k})\right] & \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h.\end{array} \tag{74}$$

$\mathcal{A}_h$, $\mathcal{B}_{1,h}$ and $\mathcal{B}_{2,h}$ are bilinear forms, $\mathcal{F}_h$ is a linear functional.

Problem (72) takes the following non-symmetric saddle point form (for its analysis in abstract form see [6]): find $\boldsymbol{\lambda}_h \in \boldsymbol{\Lambda}_h$, and $r_{2,h_2} \in \Lambda'_{2,h_2}$ s.t.

$$\left\{\begin{array}{ll}\mathcal{A}_h(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) + \mathcal{B}_{1,h}(\boldsymbol{\mu}_h, r_{2,h_2}) = \mathcal{F}_h(\boldsymbol{\mu}_h) & \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h \\ \mathcal{B}_{2,h}(\boldsymbol{\lambda}_h, t_{2,h_2}) = 0 & \forall t_{2,h_2} \in \Lambda'_{2,h_2}.\end{array}\right. \tag{75}$$

We define the operators B_{1h}, $B_{2h} : \boldsymbol{\Lambda}_h \to \Lambda_{2,h_2}$, and B_{1h}^T, $B_{2h}^T : \Lambda'_{2,h_2} \to \boldsymbol{\Lambda}'_h$ s.t.

$$\mathcal{B}_{k,h}(\boldsymbol{\mu}_h, t_{2,h_2}) = \langle t_{2,h_2}, B_{kh} \boldsymbol{\mu}_h\rangle = \langle B_{kh}^T t_{2,h_2}, \boldsymbol{\mu}_h\rangle, \quad k = 1, 2,$$

with

$$
\begin{aligned}
B_{1h}\boldsymbol{\mu}_h &= \Pi_{12}^*\mu_{1,h_1} - \mu_{2,h_2}, & B_{2h}\boldsymbol{\mu}_h &= \mu_{2,h_2} - \Pi_{21}\mu_{1,h_1},\\
B_{1h}^T t_{2,h_2} &= [\Pi_{12}t_{2,h_2},\ -t_{2,h_2}]^T, & B_{2h}^T t_{2,h_2} &= [-\Pi_{21}^* t_{2,h_2},\ t_{2,h_2}]^T,
\end{aligned}
$$

and, for $i,j = 1,2$, Π_{ij}^* is the adjoint operator of Π_{ij}, i.e.,

$$
\langle \Pi_{ij}^*\mu_i, \mu_j\rangle = \langle \mu_i, \Pi_{ij}\mu_j\rangle. \tag{76}
$$

In order to prove the continuity of the operators $\mathcal{B}_{k,h}$, the stability of the interpolation operators Π_{12} and Π_{21} is required. This is stated in the next Lemma.

We set $d_\Gamma = d_\Omega - 1$. The classical interpolation estimates used in the next theorems are stated in the Appendix.

Lemma 5. *There exist two positive constants c_{12} and c_{21} independent of h_1 and h_2 such that for any $q \in \left]\frac{d_\Gamma}{2}, \frac{3}{2}\right[$ it holds*

$$
\|\Pi_{k\ell}\lambda_\ell\|_{H^{1/2}(\Gamma)} \le c_{k\ell}\left(1 + \left(\frac{h_k}{h_\ell}\right)^q\right)^{1/2} \|\lambda_\ell\|_{H^{1/2}(\Gamma)} \qquad \forall \lambda_\ell \in Y_{\ell,h_\ell}, \tag{77}
$$

with $k = 1, \ell = 2$, or $k = 2, \ell = 1$.

Proof. We take $k = 2$ and $\ell = 1$ and we first prove that, for any real q such that $\frac{d_\Gamma}{2} < q \le \sigma < 3/2$,

$$
\|\Pi_{21}\lambda_1\|_{L^2(\Gamma)} \le c\,(1 + (h_2/h_1)^q)\,\|\lambda_1\|_{L^2(\Gamma)} \qquad \forall \lambda_1 \in Y_{1,h_1}. \tag{78}
$$

Since any $\lambda_1 \in Y_{1,h_1}$ belongs to $H^\sigma(\Gamma)$ for any $\sigma < 3/2$, in view of (114) with $s = q$ and by applying (113), we have

$$
\begin{aligned}
\|\Pi_{21}\lambda_1\|_{L^2(\Gamma)} &\le \left[\|\lambda_1 - \Pi_{21}\lambda_1\|^2_{L^2(\Gamma)} + \|\lambda_1\|_{L^2(\Gamma)}\right]\\
&\le c h_2^q \|\lambda_1\|_{H^q(\Gamma)} + \|\lambda_1\|_{L^2(\Gamma)} \le c\,(1 + (h_2/h_1)^q)\,\|\lambda_1\|_{L^2(\Gamma)}.
\end{aligned}
$$

The stability of Π_{21} in the H^1-norm follows from (114) with $s = r = 1$ when $d_\Gamma = 1$, and from (117) when $d_\Gamma = 2$. Thus we have $\|\Pi_{21}\lambda_1\|^2_{H^1(\Gamma)} \le c\|\lambda_1\|^2_{H^1(\Gamma)}$. Now (77) follows by interpolation of Sobolev spaces.

Lemma 6. *1. The bilinear form $\mathcal{A}_h$ is coercive and continuous on $\boldsymbol{\Lambda}_h$ i.e. there exist $\alpha_* > 0$ and $\widetilde{C}_A > 0$ independent of h_1 and h_2 such that*

$$
\mathcal{A}_h(\boldsymbol{\mu}_h, \boldsymbol{\mu}_h) \ge \alpha_* \|\boldsymbol{\mu}_h\|^2_{\boldsymbol{\Lambda}} \quad \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h, \tag{79}
$$

$$
\mathcal{A}_h(\boldsymbol{\mu}_h, \boldsymbol{\psi}_h) \le \widetilde{C}_A \|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}} \|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}} \quad \forall \boldsymbol{\mu}_h, \boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h; \tag{80}
$$

2. the bilinear forms $\mathcal{B}_{1,h}$ and $\mathcal{B}_{2,h} : \boldsymbol{\Lambda}_h \to \Lambda_{2,h_2}$ are continuous, i.e., there exist $C_{B1} > 0$ and $C_{B2} > 0$ (depending on the ratio h_1/h_2) such that for $k = 1,2$

$$
|\mathcal{B}_{k,h}(\boldsymbol{\mu}_h, t_{2,h_2})| \le C_{Bk} \|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}} \|t_{2,h_2}\|_{\Lambda'} \qquad \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h,\ \forall t_{2,h_2} \in \Lambda'_{2,h_2}; \tag{81}
$$

moreover, they satisfy the inf-sup conditions for arbitrary subspaces Λ_{1,h_1} and Λ_{2,h_2}, i.e.

$$
\inf_{t_{2,h_2} \in \Lambda'_{2,h_2}} \sup_{\boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h} \frac{\mathcal{B}_{k,h}(\boldsymbol{\mu}_h, t_{2,h_2})}{\|\boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}} \|t_{2,h_2}\|_{\Lambda'}} \ge 1 \qquad \text{for } k = 1,2; \tag{82}
$$

3. the linear functional $\mathcal{F}_h$ is continuous on $\mathbf{\Lambda}_h$.

PROOF. 1. To prove the continuity of $\mathcal{A}_h$ we use the following *finite element uniform extension theorem*: there exists a (discrete harmonic) lifting operator $\overline{\mathcal{R}}_k : \Lambda_{k,h_k} \to V_{k,h_k}$ s.t.

$$\|\overline{\mathcal{R}}_k \mu_{k,h_k}\|_{H^1(\Omega_k)} \leq c\|\mu_{k,h_k}\|_\Lambda \quad \forall \mu_{k,h_k} \in \Lambda_{k,h_k}, \tag{83}$$

with c independent of h_k (see, e.g. [31, Thm. 4.1.3]). The coercivity of $\mathcal{A}_h$ follows from the coercivity of the form (4) and the trace inequality (see [31, Sect. 2.2]).

2. Thanks to Lemma 5, for any $q \in]\frac{d_\Gamma}{2}, \frac{3}{2}[$, it holds

$$\begin{aligned} |\mathcal{B}_{2,h}(\boldsymbol{\mu}_h, t_{2,h_2})| &= |\langle t_{2,h_2}, B_{2h}\boldsymbol{\mu}_h\rangle| \leq \|t_{2,h_2}\|_{\Lambda'}(\|\mu_{2,h_2}\|_\Lambda + \|\Pi_{21}\mu_{1,h_1}\|_\Lambda) \\ &\leq c\|t_{2,h_2}\|_{\Lambda'} \left(\|\mu_{2,h_2}\|_\Lambda + (1+(h_2/h_1)^q)^{1/2}\|\mu_{1,h_1}\|_\Lambda\right). \end{aligned}$$

Estimate (81) for $k=2$ follows by setting $C_{B2} = 2c\,(1+(h_2/h_1)^q)^{1/2}$. Similarly,

$$\begin{aligned} |\mathcal{B}_{1,h}(\boldsymbol{\mu}_h, t_{2,h_2})| &= |\langle t_{2,h_2}, B_{1h}\boldsymbol{\mu}_h\rangle| \leq \|t_{2,h_2}\|_{\Lambda'}(\|\mu_{2,h_2}\|_\Lambda + \|\Pi_{12}^*\mu_{1,h_1}\|_\Lambda), \\ &\leq c\|t_{2,h_2}\|_{\Lambda'} \left(\|\mu_{2,h_2}\|_\Lambda + (1+(h_1/h_2)^q)^{1/2}\|\mu_{2,h_2}\|_\Lambda\right), \end{aligned}$$

where we have exploited the property $\|\Pi_{12}^*\| = \|\Pi_{12}\|$ and Lemma 5. We conclude that $B_{1,h}$ satisfies (81) with $C_{B1} = 2c\,(1+(h_1/h_2)^q)^{1/2}$.

For any $t_{2,h_2} \in \Lambda'_{2,h_2}$, B_{1h}^T and B_{2h}^T satisfy, respectively,

$$\begin{aligned} \|B_{1h}^T t_{2,h_2}\|_{\mathbf{\Lambda}'} &= \left(\|\Pi_{12}t_{2,h_2}\|^2_{\Lambda'} + \|t_{2,h_2}\|^2_{\Lambda'}\right)^{1/2} \geq \|t_{2,h_2}\|_{\Lambda'}, \\ \|B_{2h}^T t_{2,h_2}\|_{\mathbf{\Lambda}'} &= \left(\|\Pi_{21}^* t_{2,h_2}\|^2_{\Lambda'} + \|t_{2,h_2}\|^2_{\Lambda'}\right)^{1/2} \geq \|t_{2,h_2}\|_{\Lambda'}, \end{aligned}$$

thus (82) is fulfilled for both $k = 1, 2$.

3. $\mathcal{F}_h$ is continuous on $\mathbf{\Lambda}_h$ thanks to both the continuity of the bilinear form (4) and the finite element uniform extension theorem (see (83)).

Remark 7. The constants C_{Bk} do not affect the approximation errors, as we will see in Theorem 8.

Theorem 7. *There exists a unique solution $(\boldsymbol{\lambda}_h, r_{2,h_2}) \in \mathbf{\Lambda}_h \times \Lambda'_{2,h_2}$ of (75) and it satisfies*

$$\|\boldsymbol{\lambda}_h\|_{\mathbf{\Lambda}} \leq \frac{1}{\alpha_*}\|\mathcal{F}_h\|_{\mathbf{\Lambda}'}, \quad \|r_{2,h_2}\|_{\Lambda'} \leq \left(1 + \frac{C_A}{\alpha_*}\right)\|\mathcal{F}_h\|_{\mathbf{\Lambda}'}. \tag{84}$$

(The positive coercivity constant α_* was introduced in (65).) Moreover, by setting $K_1 = ker(B_{1h})$ and $K_2 = ker(B_{2h})$ there exists $\widetilde{\alpha} > 0$ such that

$$\inf_{\boldsymbol{\mu}_h \in K_2} \sup_{\boldsymbol{\psi}_h \in K_1} \frac{\mathcal{A}_h(\boldsymbol{\mu}_h, \boldsymbol{\psi}_h)}{\|\boldsymbol{\mu}_h\|_{\mathbf{\Lambda}}\|\boldsymbol{\psi}_h\|_{\mathbf{\Lambda}}} \geq \widetilde{\alpha}. \tag{85}$$

Proof. Thanks to Lemma 6, existence and uniqueness of the solution of problem (75), as well as inequality (84) follow by invoking Corollary 2.2 of [6].

The inequality (85) can now be obtained with the following arguments. First we prove that $dim(K_1) = dim(K_2)$. As a matter of fact, let I_{n_2} be the identity matrix of size n_2 and B_1, $\mathsf{B}_2 \in \mathbb{R}^{n_2\times(n_1+n_2)}$ the matrices associated with the operators B_{1h} and B_{2h}, i.e., $\mathsf{B}_1 = \left[R_{12}^T,\ -I_{n_2}\right]$, $\mathsf{B}_2 = \left[R_{21},\ -I_{n_2}\right]$. Then $rank(\mathsf{B}_1) = rank(\mathsf{B}_2) = n_2$; since $dim(ker(A)) + rank(A) = m$ for any $A \in \mathbb{R}^{n\times m}$, we obtain that $dim(ker(\mathsf{B}_1)) = dim(ker(\mathsf{B}_2)) = n_1$.

Now, thanks to [6, Cor. 2.2.] (see also [26, Sect. 4]) the properties (79)-(82) are sufficient conditions for the existence of a unique solution of problem (75); on the other hand, the inf-sup condition (85) jointly with (80)–(82), and the property that $dim(K_1) = dim(K_2)$, are necessary and sufficient conditions for proving the same result. This implicitly guarantees that (85) must be satisfied.

Theorem 8. *Let $(\boldsymbol{\lambda}, r_2) \in \boldsymbol{\Lambda} \times \Lambda_2'$ and $(\boldsymbol{\lambda}_h, r_{2,h_2}) \in \boldsymbol{\Lambda}_h \times \Lambda'_{2,h_2}$ be the solutions of (64) and (75), respectively. Then there exists $c = c(C_A, \widetilde{C}_A, C_B) >$ s.t.*

$$\begin{aligned}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\boldsymbol{\Lambda}} + \|r_2 - r_{2,h_2}\|_{\Lambda'} \leq\ & c\left(1 + \tfrac{1}{\tilde{\alpha}}\right)\Bigg\{ \inf_{\boldsymbol{\eta}_h\in K_2} \|\boldsymbol{\lambda} - \boldsymbol{\eta}_h\|_{\boldsymbol{\Lambda}} \\ & + \inf_{\boldsymbol{\mu}_h\in\boldsymbol{\Lambda}_h}\left[\|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}} + \sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h} \frac{|(\mathcal{A} - \mathcal{A}_h)(\boldsymbol{\mu}_h, \boldsymbol{\psi}_h)|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\right] \\ & + \inf_{t_{2,h_2}\in\Lambda'_{2,h_2}}\left[\|r_2 - t_{2,h_2}\|_{\Lambda'} + \sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h} \frac{|(\mathcal{B} - \mathcal{B}_{1,h})(\boldsymbol{\psi}_h, t_{2,h_2})|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\right] \\ & + \sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h} \frac{|(\mathcal{F} - \mathcal{F}_h)(\boldsymbol{\psi}_h)|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\Bigg\}.\end{aligned} \tag{86}$$

Proof. It is a direct application of Theorem 2.2 and Corollary 2.3 of [6], thanks to both Lemmas 1 and 6.

While the term $\inf\limits_{\boldsymbol{\eta}_h\in K_2} \|\boldsymbol{\lambda} - \boldsymbol{\eta}_h\|_{\boldsymbol{\Lambda}}$ depends on the interpolation error of the intergrid operator Π_{21}, the term involving $(\mathcal{B} - \mathcal{B}_{1,h})$ depends on that of Π_{12}. All the other terms only depend on the local finite element approximation in each subdomain.

For any $v \in L^2(\Omega) :\ v_{|\Omega_k} \in H^1(\Omega_k),\ k = 1, 2$, we define the H^1- broken norm

$$\|v\|_* = \sqrt{\sum_{k=1,2} \|v\|^2_{H^1(\Omega_k)}}. \tag{87}$$

If $(u_{1,h_1}, u_{2,h_2}) \in V_{1,h_1} \times V_{2,h_2}$ is the solution of the INTERNODES problem (34)-(35), we define

$$u_h = \begin{cases} u_{1,h_1} & \text{in } \Omega_1 \\ u_{2,h_2} & \text{in } \Omega_2. \end{cases} \tag{88}$$

In order to bound the error between the solution u of problem (3) and the INTERNODES solution u_h we need to estimate both the interpolation error due to a double approximation on the interface Γ (from Λ to Λ_{1,h_1} first and then from Λ_{1,h_1} to Λ_{2,h_2}; similarly, by exchanging Λ_{1,h_1} and Λ_{2,h_2}) and an inverse inequality.

Theorem 9. *There exist $c > 0$ and $q \in [1/2, 1[$ independent of h_1 and h_2 s.t.*

$$\|\lambda - \Pi_{21}\mathcal{I}_1\lambda\|_{H^{1/2}(\Gamma)} \leq c\left[h_2^{\varrho_2-1/2} + h_1^{\varrho_1-1/2}\left((h_2/h_1)^q + 1\right)\right]\|\lambda\|_{H^\sigma(\Gamma)} \quad \forall\lambda \in H^\sigma(\Gamma), \tag{89}$$

for any $\sigma > d_\Gamma/2$, where $\varrho_k = \min(\sigma, p_k + 1)$, for $k = 1, 2$.

Proof. Let $\lambda \in H^\sigma(\Gamma)$, with $\sigma > d_\Gamma/2$. We recall that $\Pi_{21}\eta = \mathcal{I}_2\eta$ for any $\eta \in Y_{1,h_1}$ and that $\mathcal{I}_1\lambda \in H^s(\Gamma)$ for any $s < 3/2$.[3] We denote by Id the identity operator, then

$$\|\lambda - \Pi_{21}(\mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)} \leq \sum_{k=1,2}\|\lambda - \mathcal{I}_k\lambda\|_{H^{1/2}(\Gamma)} + \|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)}.$$

If $d_\Gamma = 1$, in view of (114) it holds

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)} \leq ch_2^{1/2}\|\lambda - \mathcal{I}_1\lambda\|_{H^1(\Gamma)}$$

and by applying again (114) we have

$$\begin{aligned}\|\lambda - \Pi_{21}(\mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)} &\leq c(h_1^{\varrho_1-1/2} + h_2^{\varrho_2-1/2} + h_2^{1/2}h_1^{\sigma-1})\|\lambda\|_{H^\sigma(\Gamma)} \\ &\leq c\left(h_1^{\varrho_1-1/2}\left(1 + (h_2/h_1)^{1/2}\right) + h_2^{\varrho_2-1/2}\right)\|\lambda\|_{H^\sigma(\Gamma)}.\end{aligned}$$

To bound $\|(Id-\mathcal{I}_2)(\lambda-\mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)}$ when $d_\Gamma = 2$ we invoke the classical approximation results for general Sobolev spaces (see [14, Thms. 3.1.4, 3.1.5, and 3.1.6]).

Let us assume for now that $\lambda \in W^{t,2+\varepsilon}(\Gamma)$ for some $t \geq p_1 + 1$ and $\varepsilon > 0$. Let $\mathcal{E}_{h_k}$ be the triangulations on Γ induced by the meshes $\mathcal{T}_{k,h_k}$, for $k = 1, 2$. By applying Ciarlet's Theorem 3.1.6 of [14] on each $T \in \mathcal{E}_{h_2}$, thanks to the regularity assumptions on the meshes $\mathcal{T}_{k,h_k}$, for $m = 0, 1$ and any $\varepsilon > 0$ we have

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{W^{m,2}(\Gamma)} \leq ch_2^{\varepsilon/(2+\varepsilon)}h_2^{1-m}|\lambda - \mathcal{I}_1\lambda|_{W^{1,2+\varepsilon}(\Gamma)}. \tag{90}$$

(Notice that all the spaces inclusions required by Theorem 3.1.6 of [14] are satisfied.)

Now we apply Theorem 3.1.5 of [14] on each $T \in \mathcal{E}_{h_1}$, thus for any $p_1 \geq 1$,

$$|\lambda - \mathcal{I}_1\lambda|_{W^{1,2+\varepsilon}(\Gamma)} \leq ch_1^{p_1}\|\lambda\|_{W^{p_1+1,2+\varepsilon}(\Gamma)}, \tag{91}$$

and then

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{W^{m,2}(\Gamma)} \leq ch_2^{\varepsilon/(2+\varepsilon)}h_2^{1-m}h_1^{p_1}\|\lambda\|_{W^{p_1+1,2+\varepsilon}(\Gamma)}. \tag{92}$$

The generalization of Ciarlet's theorem provided in [20] for the case of lower regularity, i.e. when $t \in [p_1, p_1 + 1[$, yields, for $\tau = \min(t, p_1 + 1) > 1$,

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{W^{m,2}(\Gamma)} \leq ch_2^{\varepsilon/(2+\varepsilon)}h_2^{1-m}h_1^{\tau-1}\|\lambda\|_{W^{\tau,2+\varepsilon}(\Gamma)}. \tag{93}$$

[3] Notice that the finite element Lagrange interpolant $\mathcal{I}_1\lambda$ belongs to $H^1(\Gamma)$ and its partial derivatives are piecewise polynomial functions, not necessarily continuous, that belong to $H^{s'}(\Gamma)$ for any real $s' < 1/2$. The latter result is a consequence of Thm 11.4 in [23, Ch. 1].

Thanks to the Sobolev embedding theorems (see, e.g., [21, (1,4,4,5)]), it holds

$$\|\lambda\|_{W^{\tau,2+\varepsilon}(\Gamma)} \leq \|\lambda\|_{W^{\varrho_1,2}(\Gamma)} \qquad \forall \lambda \in W^{\tau,2+\varepsilon}(\Gamma) \cap W^{\varrho_1,2}(\Gamma), \tag{94}$$

for any $\varrho_1 \geq \tau$ s.t. $\varrho_1 - 1 = \tau - 2/(2+\varepsilon)$. It is sufficient to choose either $\varepsilon < 2(\varrho_1 - 1)/(2-\varrho_1)$ if $\varrho_1 \in]1, 2[$, or any $\varepsilon > 0$ if $\varrho_1 \geq 2$ and (94) follows.

Thus, by putting $\tau = \varrho_1 - \varepsilon/(2+\varepsilon)$ in (93), we conclude that

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{H^m(\Gamma)} \leq c(h_2/h_1)^{\varepsilon/(2+\varepsilon)} h_2^{1-m} h_1^{\varrho_1 - 1} \|\lambda\|_{H^{\varrho_1}(\Gamma)}. \tag{95}$$

Finally, by interpolation of Sobolev spaces (see, e.g., [11, Ch. 14])

$$\|(Id - \mathcal{I}_2)(\lambda - \mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)} \leq c(h_2/h_1)^{1/2+\varepsilon/(2+\varepsilon)} h_1^{\varrho_1 - 1/2} \|\lambda\|_{H^{\varrho_1}(\Gamma)} \tag{96}$$

and the thesis follows with $q = 1/2 + \varepsilon/(2+\varepsilon)$.

Theorem 10. *Let $\pi_{h_2} : L^2(\Gamma) \to Y_{2,h_2}$ denote the L^2- orthogonal projection operator. Then there exist $c > 0$ and $q \in [1, 3/2[$ independent of both h_1 and h_2 s.t. $\forall r \in H^\nu(\Gamma)$ with $\nu > 1$ and $\zeta_k = \min(\nu, p_k + 1)$ for $k = 1, 2$,*

$$\|\pi_{h_2} r - \Pi_{12}\pi_{h_2} r\|_{L^2(\Gamma)} \leq c\left[h_1^{\zeta_1} + h_2^{\zeta_2}(h_1/h_2)^q\right] \|r\|_{H^\nu(\Gamma)}. \tag{97}$$

Proof. Since $\nu > 1$ we can interpolate r on Γ and obtain (as $\Pi_{12}\eta_{2,h_2} = \mathcal{I}_1\eta_{2,h_2}$ for any $\eta_{2,h_2} \in Y_{2,h_2}$)

$$\|\pi_{h_2} r - \Pi_{12}(\pi_{h_2} r)\|_{L^2(\Gamma)} \leq \|(Id - \mathcal{I}_1)(r - \pi_{h_2} r)\|_{L^2(\Gamma)} + \|r - \mathcal{I}_1 r\|_{L^2(\Gamma)}.$$

By using (114), $\|r - \mathcal{I}_1 r\|_{L^2(\Gamma)} \leq c h_1^{\zeta_1} \|r\|_{H^\nu(\Gamma)}$. For the first term we proceed as follows. If $d_\Gamma = 1$, then

$$\begin{aligned}
\|(Id - \mathcal{I}_1)(r - \pi_{h_2} r)\|_{L^2(\Gamma)} &\leq c h_1 \|r - \pi_{h_2} r\|_{H^1(\Gamma)} && \text{(by (114))}\\
&\leq c h_1 (\|r - \mathcal{I}_2 r\|_{H^1(\Gamma)} + \|\mathcal{I}_2 r - \pi_{h_2} r\|_{H^1(\Gamma)})\\
&\leq c h_1 (h_2^{\zeta_2 - 1}\|r\|_{H^\nu(\Gamma)} + h_2^{-1}\|\mathcal{I}_2 r - \pi_{h_2} r\|_{L^2(\Gamma)}) && \text{(by (114) and (113))}\\
&\leq c(h_1/h_2) h_2^{\zeta_2}\|r\|_{H^\nu(\Gamma)} && \text{(by triangular inequality, (114) and (115))}
\end{aligned}$$

and the thesis follows with $q = 1$. If $d_\Gamma = 2$ we use the same arguments as before, but applying the interpolation estimates on general Sobolev spaces as in the proof of Theorem 9. Thus for a suitable $\varepsilon > 0$ we have

$$\begin{aligned}
\|(Id - \mathcal{I}_1)(r - \pi_{h_2} r)\|_{L^2(\Gamma)} &\leq c h_1^{1+\varepsilon/(2+\varepsilon)} |r - \pi_{h_2} r|_{W^{1,2+\varepsilon}(\Gamma)}\\
&\leq c h_1^{1+\varepsilon/(2+\varepsilon)} h_2^{\zeta_2 - 1 - \varepsilon/(2+\varepsilon)} \|r\|_{H^\nu(\Gamma)} \leq c(h_1/h_2)^{1+\varepsilon/(2+\varepsilon)} h_2^{\zeta_2}\|r\|_{H^\nu(\Gamma)}.
\end{aligned}$$

The thesis follows with $q = 1 + \varepsilon/(2+\varepsilon)$.

Lemma 11. *For any $i = 1, \ldots, n_1$ let ω_i be the support of the Lagrange basis function $\mu_i^{(1)}$ in Y_{1,h_1}. There exists $c > 0$ independent of h_1 such that*

$$\max_{1 \leq i \leq n_1} \|\psi_{1,h_1}\|_{L^\infty(\omega_i)} \leq c h_1^{(1-d_\Gamma)/2} \|\psi_{1,h_1}\|_{H^{1/2}(\Gamma)} \quad \forall \psi_{1,h_1} \in Y_{1,h_1}. \tag{98}$$

PROOF. If $d_\Gamma = 2$, it holds (see [32, Lemma 4.15], whose proof holds for any $p_1 \geq 1$)

$$\|\psi_{1,h_1}\|_{L^\infty(\omega_i)} \leq c\,(1+\log(diam(\omega_i)/h_1))^{1/2}\,\|\psi_{1,h_1}\|_{H^1(\omega_i)}, \quad \forall \psi_{1,h_1} \in Y_{1,h_1} \tag{99}$$

then by applying (113), we have

$$\|\psi_{1,h_1}\|_{L^\infty(\omega_i)} \leq c h_1^{-1/2}\,(1+\log(diam(\omega_i)/h_1))^{1/2}\,\|\psi_{1,h_1}\|_{H^{1/2}(\omega_i)}, \quad \forall \psi_{1,h_1} \in Y_{1,h_1}.$$

Since only a finite number of simplices is included in ω_i, the thesis follows for $d_\Gamma = 2$.

If $d_\Gamma = 1$, let us define $\widetilde{\psi}_{1,h_1} \in X_{1,h_1}$, such that $\widetilde{\psi}_{1,h_1} = \psi_{1,h_1}$ on ω_i and $\widetilde{\psi}_{1,h_1} = 0$ at all the mesh nodes in $\overline{\Omega}_1 \setminus \omega_i$, then let $\widetilde{\omega}_i$ be the support of $\widetilde{\psi}_{1,h_1}$. Thanks to the extension theorem for polynomials proved in [2], there exists $c > 0$ independent of h_1 such that

$$\|\widetilde{\psi}_{1,h_1}\|_{H^1(\widetilde{\omega}_i)} \leq c\|\psi_{1,h_1}\|_{H^{1/2}(\omega_i)}, \tag{100}$$

then, thanks to (99), it holds

$$\begin{aligned}\|\psi_{1,h_1}\|_{L^\infty(\omega_i)} &\leq \|\widetilde{\psi}_{1,h_1}\|_{L^\infty(\widetilde{\omega}_i)} \leq c\,(1+\log(diam(\widetilde{\omega}_i)/h_1))^{1/2}\,\|\widetilde{\psi}_{1,h_1}\|_{H^1(\widetilde{\omega}_i)} \\ &\leq c\,(1+\log(diam(\widetilde{\omega}_i)/h_1))^{1/2}\,\|\psi_{1,h_1}\|_{H^{1/2}(\omega_i)}.\end{aligned}$$

Since $diam(\widetilde{\omega}_i) \leq 2h_1$, the thesis for $d_\Gamma = 1$ follows.

We can prove now the main result of this section, i.e. the optimal error bound for the INTERNODES method.

Theorem 12. *Assume that the solution u of problem (3) belongs to $H^s(\Omega)$, for some $s > 3/2$, that $\lambda = u_{|\Gamma} \in H^\sigma(\Gamma)$ for some $\sigma > 1$ and that $r_2 = \partial_{L_2} u_2 \in H^\nu(\Gamma)$ for some $\nu > 0$. Then there exist $q \in [1/2, 1[$, $z \in [3/2, 2[$, and a constant $c > 0$ independent of both h_1 and h_2 s.t.*

$$\begin{aligned}\|u-u_h\|_* \leq\ & c\,\Big\{ \Big(h_1^{\varrho_1-1/2}\,(1+(h_2/h_1)^q) + h_2^{\varrho_2-1/2}\Big)\,\|\lambda\|_{H^\sigma(\Gamma)} \\ &+ \sum_{k=1,2} h_k^{\ell_k-1}(\|u_k\|_{H^s(\Omega_k)} + \|u_k^\lambda\|_{H^s(\Omega_k)} + \|\widehat{u}_k\|_{H^s(\Omega_k)}) \\ &+ \Big[\alpha h_1^{\zeta_1+1/2} + (1+(h_1/h_2)^z)\,h_2^{\zeta_2+1/2}\Big]\,\|r_2\|_{H^\nu(\Gamma)}\Big\},\end{aligned} \tag{101}$$

where $\ell_k = \min(s, p_k+1)$ for $k = 1, 2$, $\varrho_k = \min(\sigma, p_k+1)$, $\zeta_k = \min(\nu, p_k+1)$, $\alpha = 1$ if $\nu > 1$ and $\alpha = 0$ otherwise.

PROOF. For $k = 1, 2$ we set $u_k = u_{|\Omega_k}$. Let u_h be the INTERNODES solution defined in (88), $\lambda = u_{|\Gamma}$, $\lambda_k = (u_k)_{|\Gamma}$ (notice that $\lambda_1 = \lambda_2 = \lambda$) and $\lambda_{k,h_k} = (u_{k,h_k})_{|\Gamma}$. Then, in view of (57) and (68) we have $u_k = u_k^{\lambda,f} = u_k^\lambda + \widehat{u}_k$ and $u_{k,h_k} = U_k = \overline{\mathcal{H}}_k \lambda_{k,h_k} + \widehat{U}_k$, for $k = 1, 2$. Moreover, by standard Galerkin error analysis and (71) we have:

$$\begin{aligned}\|u-u_h\|_*^2 &= \sum_{k=1,2} \|u_k - u_{k,h_k}\|^2_{H^1(\Omega_k)} = \sum_{k=1,2} \|u_k^{\lambda,f} - U_k\|^2_{H^1(\Omega_k)} \\ &\leq c \sum_{k=1,2} \Big(\|\widehat{u}_k - \widehat{U}_k\|^2_{H^1(\Omega_k)} + \|\overline{\mathcal{H}}_k(\lambda - \lambda_{k,h_k})\|^2_{H^1(\Omega_k)}\Big) \\ &\leq c \sum_{k=1,2} \Big(h_k^{\ell_k-1}\|\widehat{u}_k\|_{H^s(\Omega_k)}\Big)^2 + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|^2_{\boldsymbol{\Lambda}}\end{aligned} \tag{102}$$

with $\ell_k = \min(s, p_k+1)$. In order to bound $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\boldsymbol{\Lambda}}$ we apply Theorem 8 and analyze each term on the right hand side of (86).

We have $K_2 = ker(B_{2h}) = \{\boldsymbol{\eta}_h = (\eta_{1,h_1}, \eta_{2,h_2}) \in \boldsymbol{\Lambda}_h : \ \eta_{2,h_2} = \Pi_{21}\eta_{1,h_1}\}$. If we choose $\boldsymbol{\eta}_h = (\mathcal{I}_1\lambda, \Pi_{21}\mathcal{I}_1\lambda) \in K_2$, using the interpolation error (114) and Theorem 9 we have

$$\begin{aligned} \|\boldsymbol{\lambda} - \boldsymbol{\eta}_h\|_{\boldsymbol{\Lambda}} &\le c\left(\|\lambda - \mathcal{I}_1\lambda\|_{H^{1/2}(\Gamma)} + \|\lambda - \Pi_{21}(\mathcal{I}_1\lambda)\|_{H^{1/2}(\Gamma)}\right) \\ &\le c\left(h_1^{\varrho_1-1/2}\left(1 + (h_2/h_1)^q\right) + h_2^{\varrho_2-1/2}\right)\|\lambda\|_{H^\sigma(\Gamma)}, \end{aligned}$$

with $\varrho_k = \min(\sigma, p_k+1)$ for $k = 1, 2$.

Taking now $\boldsymbol{\mu}_h = (\mathcal{I}_1\lambda, \mathcal{I}_2\lambda) \in \boldsymbol{\Lambda}_h$, still using (114) we have

$$\begin{aligned} \|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|_{\boldsymbol{\Lambda}} &\le c\left(\|\lambda - \mathcal{I}_1\lambda\|_{H^{1/2}(\Gamma)} + \|\lambda - \mathcal{I}_2\lambda\|_{H^{1/2}(\Gamma)}\right) \\ &\le c(h_1^{\varrho_1-1/2} + h_2^{\varrho_2-1/2})\|\lambda\|_{H^\sigma(\Gamma)}. \end{aligned} \tag{103}$$

With the same choice of $\boldsymbol{\mu}_h = (\mathcal{I}_1\lambda, \mathcal{I}_2\lambda)$ we can bound the error term involving $(\mathcal{A} - \mathcal{A}_h)$ in (86) as follows

$$\begin{aligned} |(\mathcal{A} - \mathcal{A}_h)(\boldsymbol{\mu}_h, \boldsymbol{\psi}_h)| &= |\sum_{k=1,2} a_k(u_k^{\mu_{k,h_k}} - \overline{\mathcal{H}}_k\mu_{k,h_k}, \overline{\mathcal{R}}_k\psi_{k,h_k})| \\ &\le C_A \sum_{k=1,2} \|u_k^{\mu_{k,h_k}} - \overline{\mathcal{H}}_k\mu_{k,h_k}\|_{H^1(\Omega_k)} \cdot \|\overline{\mathcal{R}}_k\psi_{k,h_k}\|_{H^1(\Omega_k)} \qquad \forall \boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h. \end{aligned}$$

Moreover, since $\mu_{k,h_k} = \mathcal{I}_k\lambda$, by triangular inequality we obtain

$$\begin{aligned} &\|u_k^{\mu_{k,h_k}} - \overline{\mathcal{H}}_k\mu_{k,h_k}\|_{H^1(\Omega_k)} \\ &\qquad \le \|u_k^\lambda - u_k^{\mathcal{I}_k\lambda}\|_{H^1(\Omega_k)} + \|u_k^\lambda - \overline{\mathcal{H}}_k\lambda\|_{H^1(\Omega_k)} + \|\overline{\mathcal{H}}_k\lambda - \overline{\mathcal{H}}_k(\mathcal{I}_k\lambda)\|_{H^1(\Omega_k)} \\ &\qquad\qquad \text{(by Céa's Lemma on the second term)} \\ &\qquad \le c(\|u_k^{\lambda-\mathcal{I}_k\lambda}\|_{H^1(\Omega_k)} + \|u_k^\lambda - \mathcal{I}_k^\Omega u_k^\lambda\|_{H^1(\Omega_k)} + \|\overline{\mathcal{H}}_k(\lambda - \mathcal{I}_k\lambda)\|_{H^1(\Omega_k)}) \\ &\qquad\qquad \text{(by (60), (114), and (71))} \\ &\qquad \le c\left(h_k^{\varrho_k-1/2}\|\lambda\|_{H^\sigma(\Gamma)} + h_k^{\ell_k-1}\|u_k^\lambda\|_{H^s(\Omega_k)}\right) \end{aligned}$$

where $\mathcal{I}_k^\Omega$ is the Lagrange interpolation operator on $\overline{\Omega}_k$. Thus, thanks to (83), we have

$$\sup_{\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h} \frac{|(\mathcal{A} - \mathcal{A}_h)(\boldsymbol{\mu}_h, \boldsymbol{\psi}_h)|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}} \le c \sum_{k=1,2} \left[h_k^{\varrho_k-1/2}\|\lambda\|_{H^\sigma(\Gamma)} + h_k^{\ell_k-1}\|u_k^\lambda\|_{H^s(\Omega_k)}\right]. \tag{104}$$

By using similar arguments,

$$\begin{aligned} |(\mathcal{F} - \mathcal{F}_h)(\boldsymbol{\psi}_h)| &= |\sum_{k=1,2} a_k(\widehat{u}_k - \widehat{U}_k, \overline{\mathcal{R}}_k\psi_{k,h_k})| \\ &\le C_A \sum_{k=1,2} \|\widehat{u}_k - \widehat{U}_k\|_{H^1(\Omega_k)} \cdot \|\overline{\mathcal{R}}_k\psi_{k,h_k}\|_{H^1(\Omega_k)} \\ &\le c \sum_{k=1,2} h_k^{\ell_k-1}\|\widehat{u}_k\|_{H^s(\Omega_k)} \cdot \|\psi_{k,h_k}\|_\Lambda, \end{aligned}$$

whence

$$\sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(\mathcal{F}-\mathcal{F}_h)(\boldsymbol{\psi}_h)|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\le c\sum_{k=1,2}h_k^{\ell_k-1}\|\widehat{u}_k\|_{H^s(\Omega_k)}. \tag{105}$$

We analyze now the error term

$$\mathsf{D}=\inf_{t_{2,h_2}\in\Lambda'_{2,h_2}}\left[\|r_2-t_{2,h_2}\|_{\Lambda'}+\sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(\mathcal{B}-\mathcal{B}_{1,h})(\boldsymbol{\psi}_h,t_{2,h_2})|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\right]. \tag{106}$$

We recall that Λ'_{2,h_2} is the dual space of Λ_{2,h_2}, that can be identified to Λ_{2,h_2} endowed with the norm $\|\cdot\|_{\Lambda'}$. Let ω_i denote the support of the Lagrange basis function $\mu_i^{(1)}$ of Y_{1,h_1}. By setting $t_2=t_{2,h_2}$, we have

$$\begin{aligned}|(\mathcal{B}-\mathcal{B}_{1,h})(\boldsymbol{\psi}_h,t_2)|&=|\langle t_2,\psi_{1,h_1}-\psi_{2,h_2}\rangle-\langle\Pi_{12}t_2,\psi_{1,h_1}\rangle+\langle t_2,\psi_{2,h_2}\rangle|\\&=|(t_2-\Pi_{12}t_2,\psi_{1,h_1})_{L^2(\Gamma)}|=\left|\sum_{i=1}^{n_1}\psi_{1,h_1}(\mathbf{x}_i^{(\Gamma_1)})(t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\Gamma)}\right|\\&=|\sum_{i=1}^{n_1}\psi_{1,h_1}(\mathbf{x}_i^{(\Gamma_1)})(t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\omega_i)}|\le M\left|\sum_{i=1}^{n_1}(t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\omega_i)}\right|,\end{aligned}$$

with $M=\max_j\|\psi_{1,h_1}\|_{L^\infty(\omega_j)}$.

Let π_{h_1} be the orthogonal projection operator from $L^2(\Gamma)$ onto its subspace Y_{1,h_1}. For any $i=1,\ldots,n_k$, by setting $\widetilde{\omega}_i=\cup_{j:\omega_j\cap\omega_i\neq\emptyset}\omega_j$, we find

$$\begin{aligned}|(t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\omega_i)}|&=|(\pi_{h_1}t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\omega_i)}|\\&=\left|\sum_{j:\ \omega_j\cap\omega_i\neq\emptyset}(\pi_{h_1}t_2-\Pi_{12}t_2)(x_j^{(\Gamma_1)}))(\mu_j^{(1)},\mu_i^{(1)})_{L^2(\omega_i)}\right|\\&\le c\|\pi_{h_1}t_2-\Pi_{12}t_2\|_{L^\infty(\widetilde{\omega}_i)}\sum_{j:\ \omega_j\cap\omega_i\neq\emptyset}(\mu_j^{(1)},\mu_i^{(1)})_{L^2(\omega_i)}\qquad\text{(by (113))}\\&\le ch_1^{d_\Gamma/2}\|\pi_{h_1}t_2-\Pi_{12}t_2\|_{L^2(\widetilde{\omega}_i)}.\end{aligned}$$

The Lagrange basis functions satisfy the estimate $\|\mu_i^{(1)}\|_{L^2(\omega_i)}\le ch_1^{d_\Gamma/2}$ and the number of elements in each $\widetilde{\omega}_i$ is finite and independent of h_1. Then

$$\begin{aligned}\left|\sum_{i=1}^{n_1}(t_2-\Pi_{12}t_2,\mu_i^{(1)})_{L^2(\omega_i)}\right|&\le ch_1^{d_\Gamma/2}\sum_{i=1}^{n_1}\|\pi_{h_1}t_2-\Pi_{12}t_2\|_{L^2(\widetilde{\omega}_i)}\\&=ch_1^{d_\Gamma/2}\|\pi_{h_1}(t_2-\Pi_{12}t_2)\|_{L^2(\Gamma)}\le ch_1^{d_\Gamma/2}\|t_2-\Pi_{12}t_2\|_{L^2(\Gamma)}.\end{aligned}$$

Thus, using the Cauchy-Schwarz inequality and Lemma 11 we obtain

$$\sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(\mathcal{B}-\mathcal{B}_{1,h})(\boldsymbol{\psi}_h,t_2)|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}=\sup_{\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}_h}\frac{|(t_2-\Pi_{12}t_2,\psi_{1,h_1})_{L^2(\Gamma)}|}{\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}}}\le ch_1^{1/2}\|t_2-\Pi_{12}t_2\|_{L^2(\Gamma)}.$$

We choose now $t_2=t_{2,h_2}=\pi_{h_2}r_2$, $\pi_{h_2}r_2$ being the L^2- orthogonal projection of r_2 on Y_{2,h_2}. Using (115) we obtain

$$\inf_{t_{2,h_2}\in\Lambda'_{2,h_2}}\|r_2-t_{2,h_2}\|_{\Lambda'}\le\|r_2-\pi_{h_2}r_2\|_{H^{-1/2}(\Gamma)}\le ch_2^{\nu+1/2}\|r_2\|_{H^\nu(\Gamma)}. \tag{107}$$

If $0 < \nu \leq 1$,

$$\begin{aligned}\|\pi_{h_2} r_2 - \Pi_{12}(\pi_{h_2} r_2)\|_{L^2(\Gamma)} &\leq c h_1 \|\pi_{h_2} r_2\|_{H^1(\Gamma)} && \text{(by (117))}\\ &\leq (h_1/h_2) h_2^\nu \|\pi_{h_2} r_2\|_{H^\nu(\Gamma)} && \text{(by (113))}\\ &\leq (h_1/h_2) h_2^\nu \|r_2\|_{H^\nu(\Gamma)} && \text{(by (116))}\end{aligned}$$

thus, from (106), we obtain that

$$\mathsf{D} \leq c \left(1 + (h_1/h_2)^{3/2}\right) h_2^{\nu+1/2} \|r_2\|_{H^\nu(\Gamma)}. \tag{108}$$

If $\nu > 1$, by using Theorem 10 and (107) in (106), we conclude that there exists $z \in [3/2, 2[$ such that

$$\mathsf{D} \leq c \left(h_1^{\zeta_1+1/2} + (1 + (h_1/h_2)^z)\, h_2^{\zeta_2+1/2}\right) \|r_2\|_{H^\nu(\Gamma)}. \tag{109}$$

By collecting all the intermediate estimates proved thus far we obtain

$$\begin{aligned}\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\boldsymbol{\Lambda}} \leq\ & c\Big[\left(h_1^{\varrho_1-1/2} (1 + (h_2/h_1)^q) + h_2^{\varrho_2-1/2}\right) \|\lambda\|_{H^\sigma(\Gamma)}\\ &+ \sum_{k=1,2} h_k^{\ell_k-1} (\|u_k^\lambda\|_{H^s(\Omega_k)} + \|\widehat{u}_k\|_{H^s(\Omega_k)})\\ &+ \left[\alpha h_1^{\zeta_1+1/2} + (1 + (h_1/h_2)^z)\, h_2^{\zeta_2+1/2}\right] \|r_2\|_{H^\nu(\Gamma)}\Big],\end{aligned} \tag{110}$$

with $\alpha = 1$ if $\nu > 1$ and $\alpha = 0$ otherwise. The thesis follows in view of (102).

Remark 8. In the case of curved interfaces we could consider regular isoparametric families of triangulations and, again, the Lagrange interpolation. Alternatively, if one chooses to work with regular affine triangulations, he has to turn to other types of interpolation such as, e.g., RBF (see Remark 3). In the former case, the theory we have developed for straight interfaces can be repeated by exploiting the approximation theory for curved elements (see, e.g., [14]) and provided that estimates similar to those given in the Appendix (precisely (115)–(117)) are satisfied. In the latter case, a more involved analysis of the interpolation operators Π_{12} and Π_{21} is needed; this work is in progress.

9.3. The case of decompositions with cross-points

In order to simplify the exposition, we formulate INTERNODES for the decomposition depicted in Fig. 2, right. We take into account the notations introduced in Sect. 6.3 and suppose that $\gamma_1^{(1)}, \gamma_2^{(1)}$, and $\gamma_2^{(2)}$ are of master type, while the others are of slave type.

Let $m = 3$ be the total number of subdomains of our decomposition. For $k = 1, \ldots, m$, let V_{k,h_k} be defined as in (21). Then we set:

$$\begin{aligned}&\Lambda_{k,h_k} = \{\lambda = v_{|\Gamma_k},\ v \in V_{k,h_k}\}, \qquad \Lambda_{k,h_k}^{(j)} = \{\lambda = v_{|\gamma_k^{(j)}},\ v \in V_{k,h_k}\},\\ &\boldsymbol{\Lambda}_h = \Lambda_{1,h_1}^{(1)} \times \Lambda_{1,h_1}^{(2)} \times \Lambda_{2,h_2}^{(1)} \times \Lambda_{2,h_2}^{(2)} \times \Lambda_{3,h_3}^{(1)}, \quad \boldsymbol{\Lambda}'_{s,h} = (\Lambda_{1,h_1}^{(2)})' \times (\Lambda_{3,h_3}^{(1)})'.\end{aligned}$$

More generally, $\boldsymbol{\Lambda}_h$ is the space of all the traces on $\Gamma = \cup_{k,i} \gamma_k^{(i)}$, while $\boldsymbol{\Lambda}'_{s,h}$ is the space of the discrete fluxes on the slave interfaces.

For any $\lambda_k \in \Lambda_{k,h_k}$, the function $\lambda_{k,j} = \lambda_{k_{|\gamma_k^{(j)}}}$ belongs to $\Lambda^{(j)}_{k,h_k}$ (for sake of clearness we omit the sub-index h_k), while $t_{k,i} \in (\Lambda^{(i)}_{k,h_k})'$. For the decomposition of Fig. 2, right, any element of $\mathbf{\Lambda}_h$ has the form $\boldsymbol{\lambda}_h = [\lambda_{1,1}, \lambda_{1,2}, \lambda_{2,1}, \lambda_{2,2}, \lambda_{3,1}]$, while any $\mathbf{t}_h \in \mathbf{\Lambda}'_{s,h}$ reads $\mathbf{t}_h = [t_{1,2}, t_{3,1}]$.

The generalization of the saddle point problem (75) reads: look for $\boldsymbol{\lambda}_h \in \mathbf{\Lambda}_h$, $\mathbf{r}_h \in \mathbf{\Lambda}'_{s,h}$ s.t.

$$\left\{ \begin{array}{ll} \mathcal{A}_h(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) + \mathcal{B}_{1,h}(\boldsymbol{\mu}_h, \mathbf{r}_h) = \mathcal{F}_h(\boldsymbol{\mu}_h) & \forall \boldsymbol{\mu}_h \in \mathbf{\Lambda}_h \\ \mathcal{B}_{2,h}(\boldsymbol{\lambda}_h, \mathbf{t}_h) = 0 & \forall \mathbf{t}_h \in \mathbf{\Lambda}'_{s,h}, \end{array} \right. \tag{111}$$

where $\mathcal{A}_h$, $\mathcal{B}_{1,h}$, $\mathcal{B}_{2,h}$ and $\mathcal{F}_h$ are the multidomain counterpart of the forms defined in (74). In our particular case they are:

$$\begin{array}{rll} \mathcal{A}_h(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) = & \displaystyle\sum_{k=1}^m a_k(\overline{\mathcal{H}}_k \lambda_{k,h_k}, \overline{\mathcal{R}}_k \mu_{k,h_k}) & \forall \boldsymbol{\lambda}_h,\ \boldsymbol{\mu}_h \in \mathbf{\Lambda}_h, \\ \mathcal{B}_{1,h}(\boldsymbol{\mu}_h, \mathbf{t}_h) = & \langle \Pi_{(1,1)(3,1)} t_{3,1}, \mu_{1,1}\rangle_{\Gamma_{13}} - \langle t_{3,1}, \mu_{3,1}\rangle_{\Gamma_{13}} & \\ & +\langle \Pi_{(2,2)(3,1)} t_{3,1}, \mu_{2,2}\rangle_{\Gamma_{23}} - \langle t_{3,1}, \mu_{3,1}\rangle_{\Gamma_{23}} & \\ & +\langle \Pi_{(2,1)(1,2)} t_{1,2}, \mu_{2,1}\rangle_{\Gamma_{12}} - \langle t_{1,2}, \mu_{1,2}\rangle_{\Gamma_{12}} & \forall \boldsymbol{\mu}_h \in \mathbf{\Lambda}_h,\ \forall \mathbf{t}_h \in \mathbf{\Lambda}'_{s,h} \\ \mathcal{B}_{2,h}(\boldsymbol{\mu}_h, t_{2,h_2}) = & \langle t_{3,1}, \mu_{3,1} - \Pi_{(3,1)(1,1)}\mu_{1,1}\rangle_{\Gamma_{13}} & \\ & +\langle t_{3,1}, \mu_{3,1} - \Pi_{(3,1)(2,2)}\mu_{2,2}\rangle_{\Gamma_{23}} & \\ & +\langle t_{1,2}, \mu_{1,2} - \Pi_{(1,2)(2,1)}\mu_{2,1}\rangle_{\Gamma_{12}} & \forall \boldsymbol{\mu}_h \in \mathbf{\Lambda}_h,\ \forall \mathbf{t}_h \in \mathbf{\Lambda}'_{s,h} \\ \mathcal{F}_h(\boldsymbol{\mu}_h) = & \displaystyle\sum_{k=1}^m \left[(f, \overline{\mathcal{R}}_k \mu_{k,h_k})_{L^2(\Omega_k)} - a_k(U_k, \overline{\mathcal{R}}_k \mu_{k,h_k}) \right] & \forall \boldsymbol{\mu}_h \in \mathbf{\Lambda}_h, \end{array} \tag{112}$$

where $\Pi_{(k,i)(\ell,j)}$ is the interpolation operator from $\Lambda^{(j)}_{\ell,h_\ell}$ to $\Lambda^{(i)}_{k,h_k}$, while the lifting operators $\overline{\mathcal{H}}_k$ and $\overline{\mathcal{R}}_k$ are defined in (70) and in (33), respectively.

The bilinear form $\mathcal{B}_{1,h}$ collects all the contributions that involve the interpolation of the discrete fluxes from the slave sides to the master ones. Each row of $\mathcal{B}_{1,h}$ replicates the definition of $\mathcal{B}_{1,h}$ in $(74)_2$ given for the 2-domain decomposition. Similarly, $\mathcal{B}_{2,h}$ collects all the contributions that involve the interpolation of the discrete traces from the master sides to the slave ones and each row replicates the definition of $\mathcal{B}_{2,h}$ in $(74)_3$.

When a more general decomposition is considered, the functional spaces $\mathbf{\Lambda}_h$ and $\mathbf{\Lambda}'_{s,h}$, as well as the bilinear forms $\mathcal{B}_{k,h}$, are defined coherently.

If we assume that all the subdomains Ω_k are convex with Lipschitz boundary and that any angle between two consecutive edges is less than π, the analysis of problem (111) can be carried out by exploiting the results of Sect. 9.2. Thus we conclude that problem (111) is well-posed (the analogous of Theorem 7 holds) and the convergence estimate (101) can be extended to decompositions with more than 2 subdomains and with internal cross-points. The optimal convergence rate of INTERNODES is confirmed by the numerical results shown in Fig. 7.

10. Appendix

Let $D \subset \mathbb{R}^{d_D}$ with $d_D = 1, 2, 3$ and $\mathcal{T}_h$ be a family of affine, regular and quasi-uniform triangulations in D. Let $X_h = \{v \in C^0(\overline{D}) :\ v_{|T} \in \mathbb{P}_p\ \forall T \in \mathcal{T}_h\}$, $\hat{T}$ the reference element, and $\hat{P}$ the polynomial space on $\hat{T}$ ([14]).

For any $q \in [1, +\infty]$ and $m \geq 0$ real, let $W^{m,q}(D)$ denote the generic Sobolev space ([1]); in particular $H^m(D) = W^{m,2}(D)$.

Inverse inequalities for piece-wise functions. Let there be given two pairs (ℓ, r) and (m, q) with $\ell, m \geq 0$ and $r, q \in [1, \infty]$ such that $\ell \leq m$ and $\hat{P} \subset W^{m,q}(\hat{T}) \cap W^{\ell,r}(\hat{T})$. There exists a positive constant c independent of h such that (see [14, Thm. 3.2.6])

$$\left(\sum_{T\in\mathcal{T}_h} |v|^q_{W^{m,q}(T)}\right)^{1/q} \leq ch^{\ell-m-d_D(1/r-1/q)} \left(\sum_{T\in\mathcal{T}_h} |v|^r_{W^{\ell,r}(T)}\right)^{1/r} \qquad \forall v \in X_h. \tag{113}$$

Lagrange interpolation error. Let $\mathcal{I}_h : C^0(\overline{D}) \to X_h$ be the Lagrange interpolation operator. For any $r,\ s \in \mathbb{R}$ with $0 \leq r \leq 1$, $s > d_D/2$, $\exists c > 0$ independent of h s.t.:

$$\|v - \mathcal{I}_h v\|_{H^r(D)} \leq ch^{\ell-r}\|v\|_{H^\ell(D)} \quad \forall v \in H^s(D), \tag{114}$$

where $\ell = \min(s, p+1)$ and p denotes the local polynomial degree. For the proof, see, e.g., [30, Thm 3.4.2] if $s \geq 2$ is an integer, and [20, Thm. 2.27] for $1 < s < 2$. The estimate with $d_D = 1$ and $1/2 < s < 2$ can be proved by following the same arguments used in the cited references.

Projection error. Let $\pi_h : L^2(D) \to X_h$ be the L^2-orthogonal projection operator. For any $r,\ s \in \mathbb{R}$, $\exists c > 0$ independent of h s.t.

$$\|v - \pi_h v\|_{(H^r(D))'} \leq ch^{r+\ell}\|v\|_{H^s(D)} \qquad \forall v \in H^s(D) \tag{115}$$

with $\ell = \min(s, p+1)$ (see [7, Lemma 2.4]). Moreover (see, e.g., [9])

$$\|\pi_h v\|_{H^s(D)} \leq c_s\|v\|_{H^s(D)} \qquad \forall v \in H^s(D),\ 0 \leq s \leq 1, \tag{116}$$

with $c_s > 0$ depending on s but independent of h.

By using the same arguments adopted to prove Lemma 1 in [13] for $p_k = 1$ we can prove for any $p_k \geq 1$,

$$|\mathcal{I}_\ell \eta_k|_{H^1(D)} \leq c|\eta_k|_{H^1(D)}, \qquad \|\eta_k - \mathcal{I}_\ell \eta_k\|_{L^2(D)} \leq ch_\ell|\eta_k|_{H^1(D)}, \qquad \forall \eta_k \in Y_{k,h_k}, \tag{117}$$

with $k = 2$ and $\ell = 1$, or $k = 1$ and $\ell = 2$.

Acknowledgments

We are grateful to Simone Deparis and Davide Forti for many useful discussions and for their advice.

References

[1] R.A. Adams and J.J.F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] I. Babuška, A. Craig, J. Mandel, and J. Pitkäranta. Efficient preconditioning for the p-version finite element method in two dimensions. *SIAM J. Numer. Anal.*, 28(3):624–661, 1991.

[3] C. Bègue, C. Bernardi, N. Debit, Y. Maday, G.E. Karniadakis, C. Mavriplis, and A.T. Patera. Nonconforming spectral element-finite element approximations for partial differential equations. In *Proceedings of the Eighth International Conference on Computing Methods in Applied Sciences and Engineering (Versailles, 1987)*, volume 75, pages 109–125, 1989.

[4] F. Ben Belgacem. The mortar finite element method with Lagrange multipliers. *Numer. Math.*, 84(2):173–197, 1999.

[5] T. Belytschko, R. Gracie, and G. Ventura. A review of extended/generalized finite element methods for material modeling. *Modelling Simul. Mater. Sci. Eng.*, 17, 2009.

[6] C. Bernardi, C. Canuto, and Y. Maday. Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem. *SIAM J. Numer. Anal.*, 25(6):1237–1271, 1988.

[7] C. Bernardi, Y. Maday, and A.T. Patera. A new nonconforming approach to domain decomposition: the mortar element method. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991)*, volume 299 of *Pitman Res. Notes Math. Ser.*, pages 13–51. Longman Sci. Tech., Harlow, 1994.

[8] D. Boffi, F. Brezzi, and M. Fortin. *Mixed Finite Element Methods and Applications*. Springer-Verlag, 2013.

[9] J. Bramble, J. Pasciak, and O. Steinbach. On the stability of the L^2 projection in $H^1(\Omega)$. *Math. Comp.*, 71(237):147–156, 2001.

[10] H. J. Brauchli and J. T. Oden. Conjugate approximation functions in finite-element analysis. *Quart. Appl. Math.*, 29:65–90, 1971.

[11] S.C. Brenner and L.R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[12] L. Cazabeau, C. Lacour, and Y. Maday. Numerical quadratures and mortar methods. In *Computational Science for the 21st Century*, pages 119–128. J. Wiley et al. (Eds.), 1997.

[13] T.F. Chan, B.F. Smith, and J. Zou. Overlapping Schwarz methods on unstructured meshes using non-matching coarse grids. *Numer. Math.*, 73(2):149–167, 1996.

[14] Ph.G. Ciarlet. *The Finite Element Method for Elliptic Problems*. Classics in applied mathematics 40. Society for Industrial and Applied Mathematics, 2nd edition, 2002.

[15] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology*, volume 2. Springer, Berlin, 2000.

[16] S. Deparis, D. Forti, P. Gervasio, and A. Quarteroni. INTERNODES: an accurate interpolation-based method for coupling the Galerkin solutions of PDEs on subdomains featuring non-conforming interfaces. *Computers & Fluids*, 141:22–41, 2016.

[17] S. Deparis, D. Forti, and A. Quarteroni. A rescaled localized radial basis function interpolation on non-Cartesian and nonconforming grids. *SIAM J. Sci. Comput.*, 36(6):A2745–A2762, 2014.

[18] S. Deparis, D. Forti, and A. Quarteroni. *A fluid-structure interaction algorithm using radial basis function interpolation between non-conforming interfaces*, pages 439–450. Modeling and Simulation in Science, Engineering and Technology. Springer, 2016.

[19] C.A. Duarte, I. Babuška, and J.T. Oden. Generalized finite element methods for three-dimensional structural mechanics problems. *Comput. & Structures*, 77(2):215–232, 2000.

[20] M. Feistauer. On the finite element approximation of functions with noninteger derivatives. *Numer. Funct. Anal. Optim.*, 10(1-2):91–110, 1989.

[21] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[22] V. Gupta, C.A. Duarte, I. Babuška, and U. Banerjee. Stable GFEM (SGFEM): improved conditioning and accuracy of GFEM/XFEM for three-dimensional fracture mechanics. *Comput. Methods Appl. Mech. Engrg.*, 289:355–386, 2015.

[23] J.-L. Lions and E. Magenes. *Nonhomogeneous Boundary Value Problems and Applications*. Springer Verlag, Berlin, 1972.

[24] Y. Maday, F. Rapetti, and B.I. Wohlmuth. The influence of quadrature formulas in 2D and 3D mortar element methods. In *Recent developments in domain decomposition methods (Zürich, 2001)*, volume 23 of *Lect. Notes Comput. Sci. Eng.*, pages 203–221. Springer, 2002.

[25] J.M. Melenk and I. Babuška. The partition of unity finite element method: basic theory and applications. *Comput. Methods Appl. Mech. Engrg.*, 139(1-4):289–314, 1996.

[26] R.A. Nicolaides. Existence, uniqueness and approximation for generalized saddle point problems. *SIAM J. Numer. Anal.*, 19(2):349–357, 1982.

[27] A. Popp. *Mortar Methods for Computational Contact Mechanics and General Interface Problems*. PhD thesis, Technische Universität München, Münich, 2012.

[28] A. Popp, B. I. Wohlmuth, M. W. Gee, and W. A. Wall. Dual quadratic mortar finite element methods for 3d finite deformation contact. *SIAM Journal on Scientific Computing*, 34(4):B421–B446, 2012.

[29] M. A Puso. A 3d mortar method for solid mechanics. *International Journal for Numerical Methods in Engineering*, 59(3):315–336, 2004.

[30] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*. Springer Verlag,

Heidelberg, 1994.

[31] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations.* Oxford University Press, 1999.

[32] A. Toselli and O. Widlund. *Domain decomposition methods—algorithms and theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

[33] B. I. Wohlmuth. A mortar finite element method using dual spaces for the Lagrange multiplier. *SIAM J. Numer. Anal.*, 38:989–1012, 2000.

[34] B.I. Wohlmuth. *Discretization methods and iterative solvers based on domain decomposition*, volume 17 of *Lecture Notes in Computational Science and Engineering.* Springer-Verlag, Berlin, 2001.