Stabilized Spectral Element Approximation for the Navier–Stokes Equations

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Received February 7, 1997; accepted July 16, 1997

The conforming spectral element methods are applied to solve the linearized Navier–Stokes equations by the help of stabilization techniques like those applied for finite elements. The stability and convergence analysis is carried out and essential numerical results are presented demonstrating the high accuracy of the method as well as its robustness. © 1998 John Wiley & Sons, Inc. Numer Methods Partial Differential Eq **14:** 115–141, 1998

Keywords: Navier–Stokes equations; spectral elements; stabilization techniques

I. INTRODUCTION

In this article, we face the approximation of the Navier–Stokes equations for viscous incompressible flows in two-dimensional bounded domains. We propose the use of continuous spectral elements, defined on the Gauss–Lobatto Legendre quadrature formulas, stabilized by techniques similar to those introduced in the finite element context. In the last years the spectral element methods have been used massively for the approximation of the Navier–Stokes equations [1]–[2] and, more recently, even in the context of triangular elements [3].

It is well known that in the numerical approximation of the Navier–Stokes equations for incompressible flows, two possible types of instability can may arise: one is due to the presence of dominating convection terms, while the other is intrinsic to the mixed formulation (in velocity and pressure) of the problem. The first type of instability generates oscillations on the velocity field for high Reynolds number flows, the second type of instability produces spurious modes on the pressure when the finite dimensional spaces of pressure and velocity do not obey the so called inf-sup or Ladyzenskaya–Brezzi–Babuška (LBB) condition [4]–[5]. In particular, in the

Contract grant sponsors: M.U.R.S.T. through "Fondi 40%" and C.N.R. "Prog. Speciale."
 CCC 0749-159X/98/010115-27

spectral context, the LBB condition is satisfied if one uses polynomials of degree N for the velocity and polynomials of degree $N-2$ for the pressure, the so-called approach $(\mathbb{P}_N - \mathbb{P}_{N-2})$, or if one uses polynomials with the same degree for both the velocity and the pressure but with *a-posteriori* filtering of parasitic modes for the pressure or, again, other methods [6]. The LBB condition does not necessarily have to be fulfilled if the incompressibility condition is relaxed. A possible relaxation in the finite element context was proposed by Brezzi and Pitkaranta [7], and it has been reinterpreted by Hughes, Franca, and Balestra [8] in terms of streamline diffusion-type perturbation. They showed that the stabilizing quality of the streamline diffusion method was beneficial in a context beyond that of convective flows.

Hansbo and Szepessy [9] introduced a streamline diffusion finite element method for the time-dependent incompressible Navier–Stokes equations, which was based on a mixed velocity–pressure formulation using the same finite element discretization $(\mathbb{P}_1 - \mathbb{P}_1$ or $\mathbb{Q}_1 - \mathbb{Q}_1)$ of space–time for the velocity and the pressure spaces. Later Franca, Frey, and Hughes generalized the stabilization schemes for the nonstationary linearized Navier–Stokes equations [10]–[11] to high-order interpolations and they designed a stability parameter ‘‘τ’’ which is satisfactory both for low- and high-order interpolation. Tobiska and Verfürth [12] furnish an optimal error estimate for the streamline diffusion finite element method on the linearized and nonlinear Navier–Stokes equations. Canuto and Van Kemenade [13] proposed a bubble-stabilized spectral method for the incompressible Navier–Stokes equations; they made a comparison between the bubble-stabilized spectral method and the $(\mathbb{P}_N - \mathbb{P}_{N-2})$ method, and they designed and tested a stabilization parameter ‘‘τ’’ based on bubbles.

In this article, we extend the streamline diffusion stabilization to spectral element context, where the interpolation degree is generally higher than in the finite element one, and the Galerkin formulation is replaced by a generalized one that makes use of Gaussian integration on each element. A stabilization parameter ‘‘τ°, depending on both the element size of the elements and the interpolation polynomial degree, is furnished.

The temporal discretization is made by a semi-implicit finite difference scheme, so that at each time-level we approximate a linearized Navier–Stokes system. Stability and convergence results are proved for the stabilized approach on the linearized Navier–Stokes equations. The stability analysis gives both lower and upper bounds on the choice of the time-step Δt (see Theorem 6.3); while the spectral accuracy of high polynomial interpolation is not damaged by the use of the stabilizing terms (see Theorem 6.5). In order to prove stability and convergence, we need some basic estimates on the spectral element approximation. To this aim, we prove the inverse inequality for spectral elements on a quasi-uniform grid of parallelograms (Theorem 6.1), an interpolation result in L^2- and H^1-norms for conformal spectral elements (Theorem 6.2), a projection result in the one-dimensional case (Theorem 6.3) and we use an approximation result of Babuška and Suri [14] for the two-dimensional case.

At each time level, a linear system of large dimension has to be solved; the matrix of the system is sparse and its condition number strongly depends on the parameters of the spectral element discretization, say the mesh size H and the polynomial degree N on each element. In this article, the BiCGStab algorithm is used, with a preconditioner based on bilinear finite element discretization. Owing to the block-wise structure of the matrix, the preconditioner is inverted following the idea of ‘‘Element-by-Element’’ techniques [15]–[16], we can say that this preconditioner is optimal from the point of view of the efficiency because of its potential parallelization and its limited computational cost. In the last part of the article, several test cases are presented showing the high accuracy of the method, and a comparison with the results of Ghia, Ghia, and Shin [17] for finite differences discretization is reported. We refer to [18] for other significant test cases.

An outline of the article is as follows.

In Section II, the incompressible Navier–Stokes equations are given, jointly with their weak formulation. In Sections III and IV, the space and the time discretizations are presented, while in Sections V and VI the stabilized approach is given, and the stability and the convergence of the approximation is proved for the linearized Navier–Stokes equations. In Section VII some numerical results are given.

II. NAVIER–STOKES EQUATIONS

The Navier–Stokes equations for viscous homogeneous incompressible fluids read: find the vector field $\mathbf{u}$ and the scalar field p so that

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega\times(0,T) \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,T) \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega\times(0,T) \\ \mathbf{u} = \mathbf{u}_0 & \text{in } \Omega\times\{0\}, \end{cases} \tag{2.1}$$

where $T > 0, \Omega$ is an open spatial domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$; $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the velocity field, $p = p(\mathbf{x}, t)$ is the pressure scalar field, ν is the viscosity, $\mathbf{x} \in \Omega$ denotes the space variable, and $t \in (0, T)$ is the time variable. The function $\mathbf{u}_0$ is the initial data. As usual in the viscous incompressible Navier–Stokes equations, the density ρ of the fluid is normalized, and all the equations are considered in their adimensional form. The *Reynolds* number associated to the system (2.1) is:

$$Re = \frac{D\|\mathbf{u}_\infty\|}{\nu}, \tag{2.2}$$

where $\mathbf{u}_\infty$ is the drift velocity of the fluid and D is a reference length.

In order to write the weak formulation of the problem (2.1), we introduce the notations:

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : \gamma_{\partial\Omega} u = 0\}, \tag{2.3}$$

where $\gamma_{\partial\Omega}$ stands for the trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ [19] define

$$V = H_0^1(\Omega), \quad \mathbf{V} = V^2, \tag{2.4}$$

$$Q = \left\{ q \in L^2(\Omega) : \int_\Omega q d\Omega = 0 \right\}, \tag{2.5}$$

$$a : \mathbf{V}\times\mathbf{V} \to \mathbb{R} \quad a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nu\nabla\mathbf{u}\cdot\nabla\mathbf{v} d\Omega, \tag{2.6}$$

$$b : \mathbf{V}\times Q \to \mathbb{R} \quad b(\mathbf{v}, q) = -\int_\Omega q \mathrm{div}\mathbf{v} d\Omega, \tag{2.7}$$

$$c : \mathbf{V}\times\mathbf{V}\times\mathbf{V} \to \mathbb{R} \quad c(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} d\Omega, \tag{2.8}$$

and finally,

$$\tilde{c} : \mathbf{V}\times\mathbf{V}\times\mathbf{V} \to \mathbb{R} \quad \tilde{c}(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \frac{1}{2}c(\mathbf{w}, \mathbf{u}, \mathbf{v}) - \frac{1}{2}c(\mathbf{w}, \mathbf{v}, \mathbf{u}). \tag{2.9}$$

The variational formulation of the problem (2.1) reads:
find $\mathbf{u} : (0,T) \to \mathbf{u}(t) \in \mathbf{V}, p : (0,T) \to p(t) \in Q$ such that for almost every $t \in (0,T)$

$$\begin{cases} \frac{d}{dt}\int_\Omega \mathbf{u}(t)\cdot \mathbf{v} d\Omega + a(\mathbf{u}(t),\mathbf{v}) + \tilde{c}(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) + b(\mathbf{v},p(t)) = \\ \int_\Omega \mathbf{f}(t)\mathbf{v}d\Omega & \forall \mathbf{v}\in\mathbf{V} \\ b(\mathbf{u}(t),q) = 0 & \forall q \in Q. \end{cases} \tag{2.10}$$

It is well known that in $\mathbb{R}^2$ the weak solution of (2.10) exists and it is unique for finite time.

III. SPACE DISCRETIZATION

Let us denote by $\mathcal{T}_H$ a conformal, regular, and quasi-uniform (see, e.g., [20]) partition of Ω in N_e quadrilaterals T_k such that

$$\bar{\Omega} = \bigcup_{k=1}^{N_e} \bar{T}_k, \tag{3.1}$$

with

$$H = \max_{T_k\in\mathcal{T}_H} H_k, H_k = \mathrm{diam}(T_k), k = 1,\dots,N_e. \tag{3.2}$$

We denote the reference domain $(-1,1)^2$ by $\hat{T}$ and we suppose that, for all k, there exists a sufficiently smooth one-to-one mapping $\mathbf{F}_k : \hat{T} \to T_k$ with a sufficiently smooth inverse $\mathbf{F}_k^{-1} : T_k \to \hat{T}$ such that

$$\forall(\xi,\eta)\in\hat{T}, \mathbf{F}_k(\xi,\eta) = (F_{1_k}(\xi,\eta), F_{2_k}(\xi,\eta)) \in T_k. \tag{3.3}$$

If $\mathbf{F}_k$ is an affine map, then the domain T_k is a parallelogram and $\mathbf{F}_k$ has the form $\mathbf{F}_k(\hat{\mathbf{x}}) = B_k\hat{\mathbf{x}} + \mathbf{b}_k$. We denote by $\{\xi_i\}_{i=1}^{N+1}$ and $\{\omega_i\}_{i=1}^{N+1}$ the nodes and the weights of the Gauss–Lobatto Legendre quadrature formulas defined on $(-1,1)$ [21] and by $\{\xi_i,\xi_j\}_{i,j=1}^{N+1}$and $\omega_{ij} = \omega_i\omega_j$ the corresponding nodes and weights on the two-dimensional reference domain $\hat{T}$.

Let $\mathbb{Q}_N(\hat{T})$ be the set of algebraic polynomials, defined on $\hat{T}$, of degree less than or equal to N in each direction, and set

$$\mathbb{Q}_{\mathcal{H}}(\Omega) = \{v \in \mathcal{C}^0(\bar{\Omega}) : v|_{T_k} \in \mathbb{Q}_N(T_k), \forall T_k \in \mathcal{T}_H\}. \tag{3.4}$$

For $u_N, v_N \in \mathbb{Q}_N(\hat{T})$ we define the discrete inner product:

$$(u_N,v_N)_{N,\hat{T}} = \sum_{i,j=1}^{N+1} u_N(\xi_i,\xi_j)v_N(\xi_i,\xi_j)\omega_{ij}, \tag{3.5}$$

while for $u_N, v_N \in \mathbb{Q}_N(T_k)$ we set:

$$(u_N,v_N)_{N,T_k} = \sum_{i,j=1}^{N+1} u_N(x_i^k,y_j^k)v_N(x_i^k,y_j^k)\omega_{ij}|\det J_{F_k}(\xi_i,\xi_j)|, \tag{3.6}$$

where

$$(x_i^k,y_j^k) = \mathbf{F}_k(\xi_i,\xi_j), i,j = 1,\dots,N+1, k = 1,\dots,N_e, \tag{3.7}$$

and J_{F_k} is the jacobian of $\mathbf{F}_k$. Given $u_{\mathcal{H}}, v_{\mathcal{H}} \in \mathbb{Q}_{\mathcal{H}}(\Omega)$, we set

$$(u_{\mathcal{H}}, v_{\mathcal{H}})_{\mathcal{H}} = \sum_{k=1}^{N_e} (u_{N,k}, v_{N,k})_{N,T_k}, \tag{3.8}$$

where $u_{N,k} = u_{\mathcal{H}}|_{T_k}, v_{N,k} = v_{\mathcal{H}}|_{T_k}$.

From now on the index $\mathcal{H}$ will characterize the spectral element discretization we are considering; it stands for the couple $\mathcal{H} = (H, N)$, i.e., the mesh size and the number of degrees of freedom on each element T_k.

If S_{lk} is a side of $\partial T_k, l = 1, \ldots, 4$, and we denote by

$$g_{lk} : (-1, 1) \to S_{lk} \tag{3.9}$$

the one-to-one affine map, we can define

$$(u_{N,k}, v_{N,k})_{N,S_{lk}} = \sum_{i=1}^{N+1} u_N(g_{lk}(\xi_i)) v_N(g_{lk}(\xi_i)) \omega_i |g'_{lk}(\xi_i)|. \tag{3.10}$$

If Σ is any subset of ∂T_k such that $\bar{\Sigma} = \cup_l \bar{S}_{lk}$, we set:

$$(u_{\mathcal{H}}, v_{\mathcal{H}})_{\mathcal{H},\Sigma} = \sum_{l,k} (u_{N,k}, v_{N,k})_{N,S_{lk}}. \tag{3.11}$$

Finally, we define the finite dimensional space (spectral element subspaces):

$$V_{\mathcal{H}} = V \cap \mathbb{Q}_{\mathcal{H}}(\Omega), \mathbf{V}_{\mathcal{H}} = V_{\mathcal{H}}^2, Q_{\mathcal{H}} = Q \cap \mathbb{Q}_{\mathcal{H}}(\Omega). \tag{3.12}$$

The generalized Galerkin spectral element approximation of problem (2.10) reads: $\forall t \in (0, T)$ find $(\mathbf{u}_{\mathcal{H}}(t), p_{\mathcal{H}}(t)) \in \mathbf{V}_{\mathcal{H}} \times Q_{\mathcal{H}}$:

$$\begin{cases} \frac{d}{dt}(\mathbf{u}_{\mathcal{H}}(t), \mathbf{v}_{\mathcal{H}})_{\mathcal{H}} + a_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}(t), \mathbf{v}_{\mathcal{H}}) + \tilde{c}_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}(t), \mathbf{u}_{\mathcal{H}}(t), \mathbf{v}_{\mathcal{H}}) & \\ \quad + b_{\mathcal{H}}(\mathbf{v}_{\mathcal{H}}, p_{\mathcal{H}}(t)) = (\mathbf{f}(t), \mathbf{v}_{\mathcal{H}})_{\mathcal{H}} & \forall \mathbf{v}_{\mathcal{H}} \in \mathbf{V}_{\mathcal{H}} \\ b_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}(t), q_{\mathcal{H}}) = 0 & \forall q_{\mathcal{H}} \in Q_{\mathcal{H}} \\ \mathbf{u}_{\mathcal{H}}(0) = \mathbf{u}_{0_{\mathcal{H}}} & \text{in } \Omega, \end{cases} \tag{3.13}$$

where

$$a_{\mathcal{H}} : \mathbf{V}_{\mathcal{H}} \times \mathbf{V}_{\mathcal{H}} \to \mathbb{R} \quad a_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}, \mathbf{v}_{\mathcal{H}}) = \sum_{k=1}^{N_e} (\nu \nabla \mathbf{u}_{N,k}, \nabla \mathbf{v}_{N,k})_{N,T_k}$$

$$b_{\mathcal{H}} : \mathbf{V}_{\mathcal{H}} \times Q_{\mathcal{H}} \to \mathbb{R} \quad b_{\mathcal{H}}(\mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}) = -\sum_{k=1}^{N_e} (q_{N,k}, \nabla \cdot \mathbf{v}_{N,k})_{N,T_k}$$

$$c_{\mathcal{H}} : \mathbf{V}_{\mathcal{H}} \times \mathbf{V}_{\mathcal{H}} \times \mathbf{V}_{\mathcal{H}} \to \mathbb{R} \quad c_{\mathcal{H}}(\mathbf{w}_{\mathcal{H}}, \mathbf{u}_{\mathcal{H}}, \mathbf{v}_{\mathcal{H}}) = \sum_{k=1}^{N_e} ((\mathbf{w}_{N,k} \cdot \nabla) \mathbf{u}_{N,k}, \mathbf{v}_{N,k})_{N,T_k}, \tag{3.14}$$

$$\tilde{c}_{\mathcal{H}}(\mathbf{w}_{\mathcal{H}}, \mathbf{u}_{\mathcal{H}}, \mathbf{v}_{\mathcal{H}}) = \frac{1}{2} c_{\mathcal{H}}(\mathbf{w}_{\mathcal{H}}, \mathbf{u}_{\mathcal{H}}, \mathbf{v}_{\mathcal{H}}) - \frac{1}{2} c_{\mathcal{H}}(\mathbf{w}_{\mathcal{H}}, \mathbf{v}_{\mathcal{H}}, \mathbf{u}_{\mathcal{H}}), \tag{3.15}$$

and $\mathbf{u}_{0_{\mathcal{H}}}$ is a suitable approximation of $\mathbf{u}_0$.

IV. TIME DISCRETIZATION

We advance in time the problem (3.13) by a suitable finite difference scheme. Given $\Delta t \in (0, T)$, we set $t^0 = 0$ and $t^n = t^0 + n \cdot \Delta t$ with $n = 1, \ldots, M$ and $M = \left[\frac{T}{\Delta t}\right]$.

Given $\mathbf{u}^n$, for $n \geq 0$, we look for the solution $(\mathbf{u}^{n+1}, p^{n+1})$ of the system:

$$\begin{cases} \frac{1}{\Delta t}(\mathbf{u}^{n+1} - \mathbf{u}^n) - \nu\Delta\mathbf{u}^{n+1} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1} & \text{in } \Omega \\ \nabla \cdot \mathbf{u}^{n+1} = 0 & \text{in } \Omega \\ \mathbf{u}^{n+1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Setting now $\mathbf{u}_{\mathcal{H}}^n = \mathbf{u}_{\mathcal{H}}(t^n), p_{\mathcal{H}}^n = p_{\mathcal{H}}(t^n)$, and $\mathcal{F}_{\mathcal{H}}^n(\mathbf{v}_{\mathcal{H}}) = (\mathbf{f}(t^n), \mathbf{v}_{\mathcal{H}})_{\mathcal{H}}$, for each n, the fully discrete formulation of (3.13) reads: for $n = 0, \ldots, M-1$ find $(\mathbf{u}^{n+1}, p^{n+1}) \in \mathbf{V}_{\mathcal{H}} \times Q_{\mathcal{H}}$:

$$\begin{cases} \frac{1}{\Delta t}(\mathbf{u}_{\mathcal{H}}^{n+1} - \mathbf{u}_{\mathcal{H}}^n, \mathbf{v}_{\mathcal{H}})_{\mathcal{H}} + a_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}^{n+1}, \mathbf{v}_{\mathcal{H}}) + \tilde{c}_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}^n, \mathbf{u}_{\mathcal{H}}^{n+1}, \mathbf{v}_{\mathcal{H}}) & \\ \quad + b_{\mathcal{H}}(\mathbf{v}_{\mathcal{H}}, p_{\mathcal{H}}^{n+1}) = (\mathbf{f}^{n+1}, \mathbf{v}_{\mathcal{H}})_{\mathcal{H}} & \forall \mathbf{v}_{\mathcal{H}} \in \mathbf{V}_{\mathcal{H}} \\ b_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}^{n+1}, q_{\mathcal{H}}) = 0 & \forall q_{\mathcal{H}} \in Q_{\mathcal{H}} \\ \mathbf{u}_{\mathcal{H}}^0 = \mathbf{u}_{0_{\mathcal{H}}} & \text{in } \Omega. \end{cases} \tag{4.2}$$

V. STABILIZATION METHODS

It is well known that the general Galerkin approximation to the Navier–Stokes equations for incompressible flows can show two different types of instability. The first one is due to the advective–diffusive nature of the problem that, for high Reynolds number, implies the presence of spurious oscillations on the velocity field. Moreover, the mixed formulation of the problem can imply the presence of pressure spurious modes, i.e., functions $\hat{p}_{\mathcal{H}} \in Q_{\mathcal{H}}$ such that

$$(\hat{p}_{\mathcal{H}}, \nabla \cdot \mathbf{v}_{\mathcal{H}}) = 0 \qquad \forall \mathbf{v}_{\mathcal{H}} \in \mathbf{V}_{\mathcal{H}}. \tag{5.1}$$

It follows that, if $(\mathbf{u}_{\mathcal{H}}, p_{\mathcal{H}}) \in \mathbf{V}_{\mathcal{H}} \times Q_{\mathcal{H}}$ is a solution of (3.13), also

$$(\mathbf{u}_{\mathcal{H}}, p_{\mathcal{H}} + \alpha\hat{p}_{\mathcal{H}}) \qquad \forall \alpha \in \mathbb{R}, \ \forall \hat{p}_{\mathcal{H}} \text{ satisfying } (5.1), \tag{5.2}$$

should be a solution of (3.13). This instability can be avoided by requiring that the compatibility condition (or inf-sup Ladyzenskaya–Brezzi–Babuška condition) be satisfied, so that the uniqueness of the solution is guaranteed.

From a numerical viewpoint, the LBB condition requires that the space $\mathbf{V}_h$ is sufficiently rich compared with the space $Q_{\mathcal{H}}$. In the spectral element context, several approaches satisfying the LBB condition have been proposed for the Stokes problem. These approaches require the use of different discretizations for the velocity and the pressure, and one or more staggered grids.

One possibility is given by the use of polynomials of degree N to approximate the velocity and polynomials of degree $N-2$ to approximate the pressure (briefly $\mathbb{P}_N - \mathbb{P}_{N-2}$). In order to implement this method, two strategies can be used: the first one consists of using $(N+1)$ Gauss–Lobatto Legendre nodes for the velocity (in each spatial direction) and the internal $(N-1)$ Gauss–Lobatto Legendre nodes for the pressure (in each spatial direction), the second one consists of using $(N+1)$ Gauss–Lobatto Legendre nodes for the velocity and $(N-1)$ Gauss–Legendre nodes for the pressure.

The first method has the drawback that the compressibility condition does not use a quadrature formula, the second one that it requires the use of two staggered grids (see [6] for a detailed description of these approaches).

A valid alternative to these approaches is to use a stabilized method, which essentially consists of adding residual dependent terms to the standard Galerkin formulation, so that the problems connected to either the convective terms or the LBB conditions can be overcome. In particular, the same computational grid can be used for both the velocity and the pressure. In this work, stabilization techniques for the spectral element discretization are proposed following the schemes introduced by Franca and Hughes [11], [22], [23] for the finite element method.
Such an approach was formerly used for spectral approximations: for advection diffusion equations ([24], and also for Navier–Stokes equations [13]).

Let us denote

$$L_k(\mathbf{w}, \mathbf{v}_{N,k}, q_{N,k}) = -\nu\Delta\mathbf{v}_{N,k} + (\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k} + \nabla q_{N,k} \qquad \forall k = 1,\ldots,N_e. \tag{5.3}$$

By adapting the idea in [10] to our problem (4.2), we propose the following stabilized problem: for $n = 0,\ldots,M-1$ find $(\mathbf{u}_{\mathcal{H}}^{n+1}, p_{\mathcal{H}}^{n+1}) \in \mathbf{V}_{\mathcal{H}} \times Q_{\mathcal{H}}$:

$$\begin{cases} \left(\frac{\mathbf{u}_{\mathcal{H}}^{n+1}-\mathbf{u}_{\mathcal{H}}^{n}}{\Delta t}, \mathbf{v}_{\mathcal{H}}\right)_{\mathcal{H}} + a_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}^{n+1}, \mathbf{v}_{\mathcal{H}}) + b_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}^{n+1}, q_{\mathcal{H}}) \\ \quad + \tilde{c}_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}^{n}, \mathbf{u}_{\mathcal{H}}^{n+1}, \mathbf{v}_{\mathcal{H}}) + b_{\mathcal{H}}(\mathbf{v}_{\mathcal{H}}, p_{\mathcal{H}}^{n+1}) + \sum_{T_k\in\mathcal{T}_H} (\nabla\cdot\mathbf{u}_{N,k}^{n+1}, \gamma_k(\mathbf{x})\nabla\cdot\mathbf{v}_{N,k})_{N,T_k} \\ \quad + \sum_{T_k\in\mathcal{T}_H} \left(\frac{\mathbf{u}_{N,k}^{n+1}}{\Delta t} + L_k(\mathbf{u}_{N,k}^{n}, \mathbf{u}_{N,k}^{n+1}, p_{N,k}^{n+1}), \tau_k(\mathbf{x}) L_k(\mathbf{u}_{N,k}^{n}, \mathbf{v}_{N,k}, q_{N,k})\right)_{N,T_k} \\ = (\mathbf{f}^{n+1}, \mathbf{v}_{\mathcal{H}})_{\mathcal{H}} + \sum_{T_k\in\mathcal{T}_{\mathcal{H}}} \left(\mathbf{f}^{n+1} + \frac{\mathbf{u}_{N,k}^{n}}{\Delta t}, \tau_k(\mathbf{x}) L_k(\mathbf{u}_{N,k}^{n}, \mathbf{v}_{N,k}, q_{N,k})\right)_{N,T_k} \\ \quad \forall \mathbf{v}_{\mathcal{H}} \in \mathbf{V}_{\mathcal{H}}, q_{\mathcal{H}} \in Q_{\mathcal{H}} \\ \mathbf{u}_{\mathcal{H}}^{0} = \mathbf{u}_{0_{\mathcal{H}}}. \end{cases} \tag{5.4}$$

We observe that if in (5.4) we put $q_{\mathcal{H}} = 0$, then we obtain the momentum equation plus the stabilization, i.e.:

$$\begin{aligned} &\left(\frac{\mathbf{u}_{\mathcal{H}}^{n+1}-\mathbf{u}_{\mathcal{H}}^{n}}{\Delta t}, \mathbf{v}_{\mathcal{H}}\right)_{\mathcal{H}} + a_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}^{n+1}, \mathbf{v}_{\mathcal{H}}) \\ &\quad + \tilde{c}_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}^{n}, \mathbf{u}_{\mathcal{H}}^{n+1}, \mathbf{v}_{\mathcal{H}}) + b_{\mathcal{H}}(\mathbf{v}_{\mathcal{H}}, p_{\mathcal{H}}^{n+1}) + \sum_{T_k\in\mathcal{T}_H} (\nabla\cdot\mathbf{u}_{N,k}^{n+1}, \gamma_k(\mathbf{x})\nabla\cdot\mathbf{v}_{N,k})_{N,T_k} \\ &\quad + \sum_{T_k\in\mathcal{T}_H} \left(\frac{\mathbf{u}_{N,k}^{n+1}}{\Delta t} + L_k(\mathbf{u}_{N,k}^{n}, \mathbf{u}_{N,k}^{n+1}, p_{N,k}^{n+1}), \tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k} + (\mathbf{u}_{N,k}^{n}\cdot\nabla)\mathbf{v}_{N,k})\right)_{N,T_k} \\ &= (\mathbf{f}^{n+1}, \mathbf{v}_{\mathcal{H}})_{\mathcal{H}} + \sum_{T_k\in\mathcal{T}_H} \left(\mathbf{f}^{n+1} + \frac{\mathbf{u}_{N,k}^{n}}{\Delta t}, \tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k} + (\mathbf{u}_{N,k}^{n}\cdot\nabla)\mathbf{v}_{N,k})\right)_{N,T_k} \\ &\quad \forall \mathbf{v}_{\mathcal{H}} \in \mathbf{V}_{\mathcal{H}}, \end{aligned} \tag{5.5}$$

while if we put $\mathbf{v}_{\mathcal{H}} = \mathbf{0}$, then we obtain the continuity equation plus the stabilization, i.e.:

$$\begin{aligned} &b_{\mathcal{H}}(\mathbf{u}_{\mathcal{H}}^{n+1}, q_h) + \sum_{T_k\in\mathcal{T}_H} \left(\frac{\mathbf{u}_{N,k}^{n+1}}{\Delta t} + L_k(\mathbf{u}_{N,k}^{n}, \mathbf{u}_{N,k}^{n+1}, p_{N,k}^{n+1}), \tau_k(\mathbf{x})\nabla q_{N,k}\right)_{N,T_k} \\ &\qquad = \sum_{T_k\in\mathcal{T}_H} \left(\mathbf{f}^{n+1} + \frac{\mathbf{u}_{N,k}^{n}}{\Delta t}, \tau_k(\mathbf{x})\nabla q_{N,k}\right)_{N,T_k} \forall q_{\mathcal{H}} \in Q_{\mathcal{H}}, \end{aligned} \tag{5.6}$$

The stabilization parameters $\tau_k(\mathbf{x})$ and $\gamma_k(\mathbf{x})$ are defined as follows at each time-step t^n:

$$\tau_k(\mathbf{x}) = \frac{H_k}{2|\mathbf{u}^n_{\mathcal{H}}(\mathbf{x})|_p N^2}\xi(Re_k(\mathbf{x})), \gamma_k(\mathbf{x}) = \frac{\lambda|\mathbf{u}^n_{\mathcal{H}}(\mathbf{x})|_p H_k}{N^2}\xi(Re_k(\mathbf{x})), \tag{5.7}$$

where

$$Re_k(\mathbf{x}) = \frac{m|\mathbf{u}^n_{\mathcal{H}}(\mathbf{x})|_p H_k}{2\nu N^2}, \tag{5.8}$$

$$\xi(Re_k(\mathbf{x})) = \begin{cases} Re_k(\mathbf{x}) & \text{if } 0 \leq Re_k(\mathbf{x}) < 1 \\ 1 & \text{if } 1 \leq Re_k(\mathbf{x}), \end{cases} \tag{5.9}$$

$$|\mathbf{u}^n_{\mathcal{H}}(\mathbf{x})|_p = \begin{cases} (|(\mathbf{u}^n_{\mathcal{H}})_1(\mathbf{x})|^p + |(\mathbf{u}^n_{\mathcal{H}})_2(\mathbf{x})|^p)^{1/p} & \text{if } 1 \leq p < \infty \\ \max\limits_{i=1,2} |(\mathbf{u}^n_{\mathcal{H}})_i(\mathbf{x})| & \text{if } p = \infty, \end{cases} \tag{5.10}$$

and finally (see Lemma 6.2)

$$0 < m \leq \frac{1}{6\tilde{C}^2}. \tag{5.11}$$

The constant $\tilde{C}$ is the constant of the inverse inequality for the spectral element discretization (see Theorem 6.1).
Since $\xi(Re_k(\mathbf{x}))/Re_k(\mathbf{x}) \leq 1$, we have the following bound on τ_k:

$$\tau_k(\mathbf{x}) \leq \frac{mH_k^2}{4\nu N^4} \qquad \forall \mathbf{x} \in \Omega. \tag{5.12}$$

VI. PROOF OF STABILITY AND CONVERGENCE

We prove now a stability result about the discretization of the linearized Navier–Stokes system:

$$\begin{cases} \alpha\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{w}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

with $\alpha = 1/\Delta t$, in view of the fact that at each time-setp of the finite difference scheme (4.2) we have to solve such a problem with $\mathbf{u} := \mathbf{u}^{n+1}$ and $\mathbf{w} := \mathbf{u}^n$.

We set $X_{\mathcal{H}} = \mathbf{V}_{\mathcal{H}} \times Q_{\mathcal{H}}$ and let $[\mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}]$ denote an element in $X_{\mathcal{H}}$. We define the following norm in $X_{\mathcal{H}}$:

$$|||[\mathbf{u}_{\mathcal{H}}, p_{\mathcal{H}}]|||_\Omega = \Bigg[\alpha\|\mathbf{u}_{\mathcal{H}}\|^2_{L^2(\Omega)} + \nu\|\nabla\mathbf{u}_{\mathcal{H}}\|^2_{L^2(\Omega)} + \sum_{T_k\in\mathcal{T}_H} \|\gamma_k^{1/2}(\mathbf{x})\nabla\cdot\mathbf{u}_{N,k}\|^2_{L^2(T_k)} + \sum_{T_k\in\mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x})\nabla p_{N,k}\|^2_{L^2(T_k)}\Bigg]^{1/2}. \tag{6.2}$$

The space $X_{\mathcal{H}}$ is a Hilbert space endowed with the norm (6.2). Given $\mathbf{w} \in \mathbf{V}_{\mathcal{H}}$, we define the following bilinear form on $X_{\mathcal{H}}$:

$$\begin{aligned}
&\mathcal{B}_{\mathcal{H}\mathbf{w}}(\mathbf{u}_\mathcal{H}, p_\mathcal{H}; \mathbf{v}_\mathcal{H}, q_\mathcal{H}) = \alpha(\mathbf{u}_\mathcal{H}, \mathbf{v}_\mathcal{H})_\mathcal{H} + (\nu\nabla\mathbf{u}_\mathcal{H}, \nabla\mathbf{v}_\mathcal{H})_\mathcal{H} \\
&\quad + \frac{1}{2}[((\mathbf{w}\cdot\nabla)\mathbf{u}_\mathcal{H}, \mathbf{v}_\mathcal{H})_\mathcal{H} - ((\mathbf{w}\cdot\nabla)\mathbf{v}_\mathcal{H}, \mathbf{u}_\mathcal{H})_\mathcal{H}] - (p_\mathcal{H}, \nabla\cdot\mathbf{v}_\mathcal{H})_\mathcal{H} \\
&\quad + (\nabla\cdot\mathbf{u}_\mathcal{H}, q_\mathcal{H})_\mathcal{H} + \sum_{T_k\in\mathcal{T}_H} (\nabla\cdot\mathbf{u}_{N,k}, \gamma_k(\mathbf{x})\nabla\cdot\mathbf{v}_{N,k})_{N,T_k} \\
&\quad + \sum_{T_k\in\mathcal{T}_H} (\alpha\mathbf{u}_{N,k} + L_k(\mathbf{w}, \mathbf{u}_{N,k}, p_{N,k}), \tau_k(\mathbf{x})L_k(\mathbf{w}, \mathbf{v}_{N,k}, q_{N,k}))_{N,T_k} \\
&\quad \forall\mathbf{v}_\mathcal{H} \in \mathbf{V}_\mathcal{H}, \forall q_\mathcal{H} \in Q_\mathcal{H},
\end{aligned} \tag{6.3}$$

and the linear functional

$$\mathcal{F}_{\mathcal{H}\mathbf{w}}(\mathbf{v}_\mathcal{H}, q_\mathcal{H}) = (\mathbf{f}, \mathbf{v}_\mathcal{H})_\mathcal{H} + \sum_{T_k\in\mathcal{T}_H} (\mathbf{f}, \tau_k(\mathbf{x})L_k(\mathbf{x}, \mathbf{v}_{N,k}, q_{N,k}))_{N,T_k}\ \forall\mathbf{v}_\mathcal{H} \in \mathbf{V}_\mathcal{H},\ \forall q_\mathcal{H} \in Q_\mathcal{H}.$$

Given $\mathbf{w} \in \mathbf{V}_\mathcal{H}$, the stabilization on the linearized model problem (6.1) can be rewritten as:

$$\mathcal{B}_{\mathcal{H}\mathbf{w}}(\mathbf{u}_\mathcal{H}, p_\mathcal{H}; \mathbf{v}_\mathcal{H}, q_\mathcal{H}) = \mathcal{F}_{\mathcal{H}\mathbf{w}}(\mathbf{v}_\mathcal{H}, q_\mathcal{H}) \qquad \forall\mathbf{v}_\mathcal{H} \in \mathbf{V}_\mathcal{H},\ \forall q_\mathcal{H} \in Q_\mathcal{H}. \tag{6.4}$$

Lemma 6.1. *If the decomposition $\mathcal{T}_H$ satisfies* (3.1)–(3.2) *then*

$$\|u_\mathcal{H}\|_{L^2(\Omega)} \le \|u_\mathcal{H}\|_\mathcal{H} \le 3\|u_\mathcal{H}\|_{L^2(\Omega)} \qquad \forall u_\mathcal{H} \in \mathbb{Q}_\mathcal{H}(\Omega), \tag{6.5}$$

with $\|u_\mathcal{H}\|_\mathcal{H} = (u_\mathcal{H}, u_\mathcal{H})_{\mathcal{H},\Omega}$.

Proof. The proof of this lemma follows from the equivalence between the L^2 norm and the discrete norm on a reference element $\hat{T}$, $\|u_N\|_{N,\hat{T}} = \sqrt{(u_N, u_N)_{N,\hat{T}}}$ for any function $u_N \in \mathbb{Q}_N(\hat{T})$. Indeed (see [6]) $\|u_N\|_{L^2(\hat{T})} \le \|u_N\|_{N,\hat{T}} \le 3\|u_N\|_{L^2(\hat{T})}$. ∎

Remark. We consider now a regular, quasi-uniform, and affine equivalent family $\mathcal{T}_H$ of quadrilaterals $T_k \subset \bar{\Omega}$, it means that

$$\exists\sigma > 1 : \forall T_k \in \mathcal{T}_H\ \frac{H_k}{\rho_k} \le \sigma, \text{ with } \rho_k = \sup\{\text{diam}(B) | B \text{ is a ball contained in } T_k\},$$

and it holds (see Theorem 15.2 in [25])

$$\|J_{F_k}\| \le \frac{H_k}{\hat{\rho}} \qquad \|J_{F_k}^{-1}\| \le \frac{\hat{H}}{\rho_k} \tag{6.6}$$

with $\hat{H} = \sqrt{2}\hat{\rho} = 2\sqrt{2}$ and $|\det J_{F_k}| \le CH_k^2$.

The following scaling results are a consequence of Theorem 15.1 of [25]:

$$|\hat{v}|_{H^m(\hat{T})} \le CH_k^{m-1}|v|_{H^m(T_k)}, \quad \forall v \in H^m(T_k) \tag{6.7}$$

and

$$|v|_{H^m(T_k)} \le CH_k^{1-m}|\hat{v}|_{H^m(\hat{T})}, \quad \forall\hat{v} \in H^m(\hat{T}). \tag{6.8}$$

We analyze now the so-called inverse inequality for spectral elements.

Theorem 6.1 (Inverse inequality). *Let $\mathcal{T}_H$ be a regular, quasi-uniform, and affine equivalent family of quadrilaterals T_k in $\bar{\Omega}$. Then there exists a positive constant $\tilde{C}$ independent of N and H such that*

$$\|\nabla v_\mathcal{H}\|_{L^2(\Omega)} \le \tilde{C}N^2H^{-1}\|v_\mathcal{H}\|_{L^2(\Omega)} \qquad \forall v_\mathcal{H} \in \mathbb{Q}_\mathcal{H}. \tag{6.9}$$

Proof. It is enough to prove that

$$\int_{T_k} |\nabla v_N|^2 \le \tilde{C}^2 N^4 H^{-2} \int_{T_k} |v_N|^2 \qquad \forall T_k \in \mathcal{T}_H,\ \forall v_N \in \mathbb{Q}_N(T_k),\ \text{and } v_N = v_{\mathcal{H}}|_{T_k}. \quad (6.10)$$

Following the proof of proposition 6.3.2 in [26] (the inverse inequality for piecewise polynomials) we consider $\hat{T} = (-1,1)^2$ and the affine one-to-one map $\mathbf{F}_k$ introduced in Section III. Then, $\forall v_N \in \mathbb{Q}_N(T_k)$, we define $\hat{v}_N \in \mathbb{Q}_N(\hat{T})$ such that $\hat{v}_N = v_N \circ \mathbf{F}_k$.

By the inverse inequality for algebraic polynomials [27], there exists C_1 independent of N such that

$$\int_{\hat{T}} |\nabla \hat{v}_N|^2 \le C_1 N^4 \int_{\hat{T}} |\hat{v}_N|^2, \quad \forall \hat{v}_N \in \mathbb{Q}_N(\hat{T}), \quad (6.11)$$

and by (6.7)–(6.8), we have

$$|v_N|^2_{H^1(T_k)} \le C^2 |\hat{v}_N|^2_{H^1(\hat{T})} \le C^2 \cdot C_1^2 N^4 |\hat{v}_N|^2_{L^2(\hat{T})} \le \tilde{C}^2 H_k^{-2} N^4 |v_N|_{L^2(T_k)}. \quad (6.12)$$

The thesis follows by summation on all elements $\{T_k\}$ and by (3.2). ∎

Remark. From Theorem 6.1 we easily obtain a second inverse inequality:

$$\sum_{T_k \in \mathcal{T}_H} \|\Delta v_{N,k}\|^2_{L^2(T_k)} \le \tilde{C}^2 N^4 H^{-2} \|\nabla v_{\mathcal{H}}\|^2_{L^2(\Omega)} \qquad \forall v_{\mathcal{H}} \in \mathbb{Q}_{\mathcal{H}}. \quad (6.13)$$

Lemma 6.2 (Coercivity). *Given* $\mathbf{w} \in \mathbf{V}_{\mathcal{H}}$, *and* $\bar{\tau} = \max_{\substack{1 \le k \le N_e \\ x \in \Omega}} \tau_k(\mathbf{x})$, *if*

$$m \le \frac{1}{6\tilde{C}^2} \quad \textit{and} \quad \frac{2\sqrt{2}\|\mathbf{w}\|^2_{L^\infty(\Omega)}}{120\nu + \sqrt{2}\bar{\tau}\|\mathbf{w}\|^2_{L^\infty(\Omega)}} < \alpha < \frac{2}{\bar{\tau}} \quad (6.14)$$

then a positive constant $\alpha^* > 0$ *exists such that*

$$\mathcal{B}_{\mathcal{H}\mathbf{w}}(\mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}; \mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}) \ge \alpha^* |||[\mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}]|||^2_{\Omega}\ \forall [\mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}] \in X_{\mathcal{H}}. \quad (6.15)$$

Proof. By definition (6.3), we have

$$\begin{aligned}
&\mathcal{B}_{\mathcal{H}\mathbf{w}}(\mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}; \mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}) \\
&\quad = \alpha\|\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}} + \nu\|\nabla\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}} + \sum_{T_k \in \mathcal{T}_H} \|\gamma_k^{1/2} \nabla \cdot \mathbf{v}_{N,k}\|^2_{N,T_k} \\
&\qquad + \sum_{T_k \in \mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k} + (\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k} + \nabla q_{N,k})\|^2_{N,T_k} \\
&\qquad + \sum_{T_k \in \mathcal{T}_H} (\alpha\mathbf{v}_{N,k}, \tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k} + (\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k} + \nabla q_{N,k}))_{N,T_k} \\
&\qquad\quad \text{(by Young inequality)} \\
&\quad \ge \frac{\alpha}{2}\|\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}} + \nu\|\nabla\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}} + \sum_{T_k \in \mathcal{T}_H} \|\gamma_k^{1/2} \nabla \cdot \mathbf{v}_{N,k}\|^2_{N,T_k} \\
&\qquad + \left(1 - \frac{\alpha}{2}\bar{\tau}\right) \sum_{T_k \in \mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k} + (\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k} + \nabla q_{N,k})\|^2_{N,T_k}
\end{aligned}$$

$$
\begin{aligned}
&\text{(by Young inequality and if } \alpha \leq 2/\bar{\tau}) \\
\geq\ & \frac{\alpha}{2}\|\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}} + \nu\|\nabla\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}} + \sum_{T_k\in\mathcal{T}_H} \|\gamma_k^{1/2}\nabla\cdot\mathbf{v}_{N,k}\|^2_{N,T_k} \\
&+ \left(1-\frac{\alpha}{2}\bar{\tau}\right)\Bigg[-21\sum^{T_k\in\mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x})\nu\Delta\mathbf{v}_{N,k}\|^2_{N,T_k} \\
&\qquad - \frac{1}{5}\sum_{T_k\in\mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k}\|^2_{N,T_k} \\
&\qquad + \frac{1}{132}\sum_{T_k\in\mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x})\nabla q_{N,k}\|^2_{N,T_k}\Bigg].
\end{aligned}
$$

Using now the inverse inequality (6.13) and the fact that $\tau_k(\mathbf{x}) \leq \frac{mH^2}{4\nu N^4}$, we have

$$
\sum_{T_k\in\mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x})\nu\Delta\mathbf{v}_{N,k}\|^2_{N,T_k} \leq \frac{mH^2}{4\nu N^4}\nu^2\tilde{C}^2N^4H^{-2}\|\nabla\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}} \leq \frac{1}{24}\nu\|\nabla\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}} \tag{6.16}
$$

and

$$
\begin{aligned}
\sum_{T_k\in\mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k}\|^2_{N,T_k} &\leq \sqrt{2}\|\mathbf{w}\|^2_{L^\infty(\Omega)}\frac{mH^2}{4\nu N^4}\tilde{C}^2N^4H^{-2}\|\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}} \\
&\leq \frac{\sqrt{2}}{24\nu}\|\mathbf{w}\|^2_{L^\infty(\Omega)}\|\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}}.
\end{aligned} \tag{6.17}
$$

It follows that

$$
\begin{aligned}
\mathcal{B}_{\mathcal{H}\mathbf{w}}(\mathbf{v}_{\mathcal{H}},q_{\mathcal{H}};\mathbf{v}_{\mathcal{H}},q_{\mathcal{H}}) \geq\ & \left[\frac{\alpha}{2} - \left(1-\frac{\alpha}{2}\bar{\tau}\right)\frac{\sqrt{2}}{120\nu}\|\mathbf{w}\|^2_{L^\infty(\Omega)}\right]\|\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}} \\
&+ \left[\frac{1}{8}+\frac{7\alpha\bar{\tau}}{16}\right]\nu\|\nabla\mathbf{v}_{\mathcal{H}}\|^2_{\mathcal{H}} + \sum_{T_k\in\mathcal{T}_H}\|\gamma_k^{1/2}\nabla\cdot\mathbf{v}_{N,k}\|^2_{N,T_k} \\
&+ \frac{1}{132}\left(1-\frac{\alpha}{2}\bar{\tau}\right)\sum_{T_k\in\mathcal{T}_H}\|\tau_k^{1/2}(\mathbf{x})\nabla q_{N,k}\|^2_{N,T_k}.
\end{aligned} \tag{6.18}
$$

We set now: $C_1 = \frac{\alpha}{2} - \left(1-\frac{\alpha}{2}\bar{\tau}\right)\frac{\sqrt{2}}{120\nu}\|\mathbf{w}\|^2_{L^\infty(\Omega)}, C_2 = \frac{1}{8}+\frac{7\alpha\bar{\tau}}{16}$, and $C_3 = \frac{1}{132}\left(1-\frac{\alpha\bar{\tau}}{2}\right)$; under the assumptions (6.14) on α, the constants C_1, C_2, and C_3 are nonnegative constants and the thesis follows by Lemma 6.1 and by setting $\alpha^* = \min\{C_1, C_2, C_3\}$. ∎

Remark. We observe that the assumption on α given in (6.14) yields the following ones on Δt:

$$
\frac{\bar{\tau}}{2} < \frac{\Delta t}{\alpha_1} < \frac{\bar{\tau}}{2} + \frac{60\nu}{\sqrt{2}\|\mathbf{w}\|^2_{L^\infty(\Omega)}}. \tag{6.19}
$$

The lower bound on Δt reflects a similar one given in [12].

Remark. If one uses spectral elements in space–time with piecewise constant elements in time, then the term

$$
\sum_{T_k\in\mathcal{T}_H}(\alpha\mathbf{u}_{N,k},\tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k}+(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k}+\nabla q_{N,k}))_{N,T_k} \tag{6.20}
$$

should be omitted into the stabilizing term. In such case, we obtain a less restrictive bound on α, i.e.,

$$\alpha > \frac{\sqrt{2}}{120} \frac{\|\mathbf{w}\|^2_{L^\infty(\Omega)}}{\nu} \tag{6.21}$$

and consequently, Δt must only satisfy the upper bound:

$$\frac{\Delta t}{\alpha_1} < \frac{120}{\sqrt{2}} \frac{\nu}{\|\mathbf{w}\|^2_{L^\infty(\Omega)}} \tag{6.22}$$

in space–time with piecewise constant elements in time.

Lemma 6.3 (Stability). *If $(\mathbf{u}_\mathcal{H}, p_\mathcal{H})$ is the solution of (6.4), then there exists a positive constant β such that*

$$|||[\mathbf{u}_\mathcal{H}, p_\mathcal{H}]|||_\Omega \le \frac{\beta}{\alpha^*} \|\mathbf{f}\|_\mathcal{H}. \tag{6.23}$$

Proof. By definition of $\mathcal{F}_{\mathcal{H}\mathbf{w}}$, we have

$$\begin{aligned}
&\mathcal{F}_{\mathcal{H}\mathbf{w}}(\mathbf{v}_\mathcal{H}, q_\mathcal{H}) \\
&\quad = (\mathbf{f}, \mathbf{v}_\mathcal{H})_\mathcal{H} \sum_{T_k \in \mathcal{T}_H} (\mathbf{f}, \tau_k(\mathbf{x}) L_k(\mathbf{w}, \mathbf{v}_{N,k}, q_{N,k}))_{N,T_k} \\
&\quad \le \|\mathbf{f}\|_\mathcal{H} \left[\|\mathbf{v}_\mathcal{H}\|_\mathcal{H} + \sum_{T_k \in \mathcal{T}_H} \|\tau_k(\mathbf{x})(-\nu \Delta \mathbf{v}_{N,k} + (\mathbf{w} \cdot \nabla)\mathbf{v}_{N,k} + \nabla q_{N,k})\|_{N,T_k} \right].
\end{aligned} \tag{6.24}$$

Since we choose $\tau_k(\mathbf{x})$ less than 1, we have

$$\begin{aligned}
&\left[\|\mathbf{v}_\mathcal{H}\|_\mathcal{H} + \sum_{T_k \in \mathcal{T}_H} \|\tau_k(\mathbf{x})(-\nu \Delta \mathbf{v}_{N,k} + (\mathbf{w} \cdot \nabla)\mathbf{v}_{N,k} + \nabla q_{N,k})\|_{N,T_k} \right]^2 \\
&\quad \le 2\|\mathbf{v}_\mathcal{H}\|^2_\mathcal{H} + 6 \sum_{T_k \in \mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x}) \nu \Delta \mathbf{v}_{N,k}\|^2_{N,T_k} + 6 \sum_{T_k \in \mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x})(\mathbf{w} \cdot \nabla)\mathbf{v}_{N,k}\|^2_{N,T_k} \\
&\qquad + 6 \sum_{T_k \in \mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x}) \nabla q_{N,k}\|^2_{N,T_k} \quad \text{(by inverse inequality and (6.14))} \\
&\quad \le 6 \left[\left(1 + \frac{1}{24} \frac{\|\mathbf{w}\|^2_{L^\infty(\Omega)}}{\nu}\right) \|\mathbf{v}_\mathcal{H}\|^2_\mathcal{H} + \frac{\nu}{24} \|\nabla \mathbf{v}_\mathcal{H}\|^2_\mathcal{H} + \sum_{T_k \in \mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x}) \nabla q_{N,k}\|^2_{N,T_k} \right].
\end{aligned} \tag{6.25}$$

We have that

$$\mathcal{F}_{\mathcal{H}\mathbf{w}}(\mathbf{v}_\mathcal{H}, q_\mathcal{H}) \le \beta \|\mathbf{f}\|_\mathcal{H} |||[\mathbf{v}_\mathcal{H}, q_\mathcal{H}]|||_\Omega \qquad \forall [\mathbf{v}_\mathcal{H}, q_\mathcal{H}] \in X_\mathcal{H},$$

by taking $\beta = \max\left\{ \sqrt{\frac{1}{\alpha}\left(1 + \frac{1}{24}\frac{\|\mathbf{w}\|^2_{L^\infty(\Omega)}}{\nu}\right)}, \sqrt{\frac{1}{24\nu}} \right\}$ and using Lemma 6.1.

The thesis follows by Lemma 6.2. ∎

Now we need to introduce some approximation estimates for spectral elements. First of all we give an interpolation result in two dimensions, then a projection result for the one-dimensional case and, finally, we recall Babuška and Suri results for the approximation in two dimensions [14].

For every $T_k \in \mathcal{T}_H$ let $I_N^k : \mathcal{C}^0(T_k) \to \mathbb{Q}_N(T_k)$ the local Lagrange interpolation operator on the Gauss–Lobatto Legendre nodes in T_k, and $I_{\mathcal{H}} : \mathcal{C}^0(\Omega) \to \mathbb{Q}_{\mathcal{H}}(\Omega)$ the global interpolation operator such that $(I_{\mathcal{H}} u)|_{T_k} = I_N^k(u|_{T_k})$, for every $T_k \in \mathcal{T}_H$.
We have the following interpolation result for spectral elements.

Theorem 6.2. *For all $u \in H^s(\Omega)$ with $s \geq 2$, there exists a constant $C > 0$ independent of H and N such that*

$$\|u - I_N u\|_{H^m(T_k)} \leq C H_k^{\min(N+1,s)-m} N^{m-s} \|u\|_{H^s(T_k)} \qquad m = 0,1 \qquad \forall T_k \in \mathcal{T}_H \quad (6.26)$$

and

$$\|u - I_{\mathcal{H}} u\|_{H^m(\Omega)} \leq C H^{\min(N+1,s)-m} N^{m-s} \|u\|_{H^s(\Omega)} \qquad m = 0,1. \quad (6.27)$$

Proof. The proof follows the guidelines in [28] and it is extended to the two-dimensional case.

Let $\hat{T} = (-1,1)^2$, the following error estimate is known in the spectral methods context (see [19], [21]): $\forall \hat{u} \in H^s(\hat{T})$

$$\|\hat{u} - I_N \hat{u}\|_{H^m(\hat{T})} \leq C N^{m-s} \|\hat{u}\|_{H^s(\hat{T})} \qquad m = 0,1, s \geq 2. \quad (6.28)$$

We take an arbitrary $v_{N-1} \in \mathbb{Q}_{N-1}(\hat{T})$, we have

$$\begin{aligned} \|\hat{u} - I_N \hat{u}\|_{H^m(\hat{T})} &= \|(\hat{u} - v_{N-1}) - (I_N \hat{u} - v_{N-1})\|_{H^m(\hat{T})} \\ &= \|(\hat{u} - v_{N-1}) - I_N(\hat{u} - v_{N-1})\|_{H^m(\hat{T})} \\ &\leq C(s) N^{m-s} \inf_{v_{N-1} \in \mathbb{Q}_{N-1}(\hat{T})} \|\hat{u} - v_{N-1}\|_{H^s(\hat{T})}. \end{aligned}$$

From Deny–Lions lemma (see [26]) it is easy to prove ([14]) that

$$\inf_{p_N \in \mathbb{Q}_N(T_k)} \|u + p_N\|_{H^s(T_k)} \leq C H_k^{\min(N+1,s)} |u|_{H^s(T_k)} \qquad \forall u \in H^s(T_k), \quad (6.29)$$

and the ®rst estimate of the thesis follows by a scaling argument. Moreover, in view of the fact that $I_{\mathcal{H}} u \in \mathbb{Q}_{\mathcal{H}}(\Omega)$ and by summation on T_k, we have the estimate (6.27). ∎

We give now a projection result for spectral elements in $\mathbb{R}$.

Theorem 6.3. *Let I be an open subset in $\mathbb{R}$ and $\mathcal{T}_H$ a quasi-uniform decomposition of I in N_e disjoint subintervals I_k, for $k = 1, \ldots, N_e$ and let it be $H = \max_k meas(I_k)$. Given $u \in H^s(I)$, if $u_{\mathcal{H}}$ is the spectral element approximation of u, then there exists a positive constant C such that*

$$\|u - u_N^k\|_{H^m(I_k)} \leq C H^{\min(N+1,s)-m} N^{m-s} \|u\|_{H^s(I_k)}, \quad 0 \leq m \leq s, s \geq 1, \quad (6.30)$$

with $u_N^k = u_{\mathcal{H}}|_{I_k}$, and

$$\|u - u_{\mathcal{H}}\|_{H^m(I)} \leq C H^{\min(N+1,s)-m} N^{m-s} \|u\|_{H^s(I)} \qquad m = 0,1 \qquad s \geq 2. \quad (6.31)$$

Proof. Let us denote by a_k and b_k the left and right extrema, respectively, of the interval I_k, and let $\bar{u}_k \in \mathbb{P}_1(I_k)$ be such that $\bar{u}_k(a_k) = u(a_k)$ and $\bar{u}_k(b_k) = u(b_k)$; finally, let $\mathring{u} \in H^s(I) \cap H_0^1(I)$ be such that $\mathring{u}_k = \mathring{u}|_{I_k} = u|_{I_k} - \bar{u}_k$, for $k = 1, \ldots, N_e$. We set

$$V_{\mathcal{H}}^0 = \{v \in \mathcal{C}^0(I) : v|_{I_k} \in \mathbb{P}_N(I_k), v = 0 \text{ on } \partial I_k, k = 1, \ldots, N_e\}. \quad (6.32)$$

Then we define the orthogonal projector $\Pi_s^k : H^s(I_k) \cap H_0^1(I_k) \to \mathbb{P}_N(I_k)$ with respect to the inner product in $H^s(I_k)$, such that, given $\mathring{v}_k \in H^s(I_k) \cap H_0^1(I_k)$:

$$(\Pi_s^k \mathring{v}_k, w_N)_{H^s(I_k)} = (\mathring{v}_k, w_N)_{H^s(I_k)} \qquad \forall w_N \in \mathbb{P}_N(I_k). \tag{6.33}$$

From [29] we have that there exists a constant $C > 0$ independent of N such that

$$\|\mathring{u}_k - \Pi_s^k \mathring{u}_k\|_{H^m(I_k)} \leq C N^{m-s} \|\mathring{u}_k\|_{H^s(I_k)} \qquad 0 \leq m \leq s, s \geq 0, \tag{6.34}$$

and, for any $v_{N-1} \in \mathbb{P}_{N-1}(I_k)$ we have

$$\begin{aligned}
\|\mathring{u}_k - \Pi_s^k \mathring{u}_k\|_{H^m(I_k)} &= \|(\mathring{u}_k - v_{N-1}) - \Pi_s^k(\mathring{u}_k - v_{N-1})\|_{H^m(I_k)} \\
&\leq C N^{m-s} \|\mathring{u}_k - v_{N-1}\|_{H^s(I_k)} \quad \text{[by a scaling argument and (6.29)]} \\
&\leq C(s) H^{\min(N+1,s)-m} N^{m-s} |\mathring{u}_k|_{H^s(I_k)}. \qquad (6.35)
\end{aligned}$$

In order to prove the second estimate we define the following operator: $\Pi_s : H^s(I) \cap H_0^1(I) \to V_{\mathcal{H}}^0$ such that

$$\Pi_s \mathring{u} \in V_{\mathcal{H}}^0, \text{ and } (\Pi_s \mathring{u})|_{I_k} = \Pi_s^k \mathring{u}_k. \tag{6.36}$$

We have

$$\begin{aligned}
\|\mathring{u} - \Pi_s \mathring{u}\|_{H^1(I)} &= \left(\sum_{I_k \in \mathcal{T}_H} \|\mathring{u}_k - \Pi_s^k \mathring{u}\|^2_{H^m(I_k)} \right)^{1/2} \\
&\leq C H^{\min(N+1,s)-m} N^{m-s} \|\mathring{u}\|_{H^s(I)}. \qquad (6.37)
\end{aligned}$$

■

We recall the two following approximation results of Babuška and Suri, whose proof is in [14].

Lemma 6.4. *Let $\mathcal{T}_H$ be a quasi-uniform mesh and $T_k \in \mathcal{T}_H$ with vertices A_i; let $u \in H^s(T_k)$. There exist a positive constant C independent of u, N, and H and a sequence $z_{N,k} \in \mathbb{Q}_N(T_k)$ for $N = 1, 2, \ldots,$ such that, for any $0 \leq m \leq s$,*

$$\|u - z_{N,k}\|_{H^m(T_k)} \leq C N^{m-s} H_k^{\min(N+1,s)-m} \|u\|_{H^s(T_k)} \qquad s \geq 0. \tag{6.38}$$

If $s > 3/2$, then we can assume that $z_{N,k}(A_i) = u(A_i)$.

Theorem 6.4. *Let $u \in H^s(\Omega) \cap H_0^1(\Omega)$, with $s > 3/2$, then for any $N \geq 1$ and $H > 0$ there exists $z_{\mathcal{H}} \in V_{\mathcal{H}}$ such that*

$$\|u - z_{\mathcal{H}}\|_{H^k(\Omega)} \leq C H^{\min(N+1,s)-k} N^{k-s} \|u\|_{H^s(\Omega)} k = 0, 1, \tag{6.39}$$

where C is independent of u, N, H, and $\mathcal{T}_H$.

Remark. The function $z_{\mathcal{H}}$ is obtained starting by the local polynomials $z_{N,k}$ and by matching continuously at the interfaces between two adjacent elements (see Ref. 14, proof of Theorem 4.6). The proof of the following lemma follows from Theorem 6.2 and it is similar to the proof of (4.3.44) in [26] for one-dimension quadrature formulas.

Lemma 6.5. *If $u \in H^s(T_k)$ with $s \geq 2$, then $\exists C > 0$ independent of N and H_k such that*

$$\begin{aligned}
&|(u, v_N)_{L^2(T_k)} - (u, v_N)_{N,T_k}| \\
&\qquad \leq C(s) N^{-s} H^{\min(N+1,s)} \|u\|_{H^s(T_k)} \|v_N\|_{L^2(T_k)} \qquad \forall v_N \in \mathbb{Q}_N(T_k). \qquad (6.40)
\end{aligned}$$

Remark. If $u \in H^s(T_k)$ and $z_{N-1,k} \in \mathbb{Q}_{N-1}(T_k)$ is a polynomial satisfying (6.38), then

$$\|\nabla z_{N-1,k}\|_{H^{s-1}(T_k)} \le C\|u\|_{H^s(T_k)}. \tag{6.41}$$

Theorem 6.5 (Convergence). *There exists a unique solution* $[\mathbf{u}_{\mathcal{H}}, p_{\mathcal{H}}]$ *of* (5.4). *Moreover, if* $\mathbf{u} \in [H_0^1(\Omega) \cap H^{s+1}(\Omega)]^2$ *with* $s \ge 1, p \in L_0^2(\Omega) \cap H^l(\Omega)$ *with* $l \ge 1, \mathbf{w} \in W^{\max(s,l,r),\infty}(\Omega)$, $\mathbf{f} \in H^r(\Omega)$, *then there exists a positive constant* C *such that*

$$\begin{aligned}|||[\mathbf{u} - \mathbf{u}_{\mathcal{H}}, p - p_{\mathcal{H}}]|||_\Omega \le C(&H^{\min(N+1,s)-1}N^{1-s}\|\mathbf{u}\|_{H^s(\Omega)} + H^{\min(N+1,l)}N^{-l}\|p\|_{H^l(\Omega)} \\ &+ H^{\min(N+1,r)}N^{-r}\|\mathbf{f}\|_{H^r(\Omega)}).\end{aligned} \tag{6.42}$$

In particular,

$$\begin{aligned}C = \max\Bigg\{&\sqrt{v} + \|\mathbf{w}\|_{W^{s-1,\infty}(\Omega)} + \frac{H}{N}\left(\sqrt{\alpha} + \frac{\|\mathbf{w}\|_{W^{s,\infty}(\Omega)}}{\nu}\right) + \frac{1}{N}\|\mathbf{w}\|_{W^{s-2,\infty}(\Omega)} \\ &+ \frac{H}{N^2}\left(\sqrt{\frac{\lambda|\mathbf{w}|_p}{2\nu}} + \frac{1}{\nu}\|\mathbf{w}\|^2_{W^{s-2,\infty}(\Omega)}\right), \frac{1}{N\nu}(1 + \|\mathbf{w}\|_{W^{l-1,\infty}(\Omega)}), \\ &\|\mathbf{w}\|_{W^{r,\infty}(\Omega)}\left(1 + \frac{H}{\nu N^2}\right)\Bigg\}.\end{aligned} \tag{6.43}$$

Proof. Lemmas 6.2 and 6.3 ensure the coercivity of $\mathcal{B}_{\mathcal{H}\mathbf{w}}$ and the continuity of $\mathcal{F}_{\mathcal{H}\mathbf{w}}$; the Strang Lemma [26] ensures the existence and the uniqueness of the solution of (5.4) and it is used here to give the error convergence estimate:

$$\begin{aligned}|||[\mathbf{u} - \mathbf{u}_{\mathcal{H}}, p - p_{\mathcal{H}}]|||_\Omega \le \inf_{\substack{\mathbf{z}_{\mathcal{H}} \in \mathbf{V}_{\mathcal{H}} \\ r_{\mathcal{H}} \in Q_{\mathcal{H}}}} \Bigg[&\left(1 + \frac{\beta}{\alpha^*}\right)|||[\mathbf{u} - \mathbf{z}_{\mathcal{H}}, p - r_{\mathcal{H}}]|||_\Omega \\ &+ \sup_{\substack{\mathbf{v}_{\mathcal{H}} \in \mathbf{V}_{\mathcal{H}} \\ q_{\mathcal{H}} \in Q_{\mathcal{H}}}} \frac{|\mathcal{B}_{\mathbf{w}}(\mathbf{z}_{\mathcal{H}}, r_{\mathcal{H}}; \mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}) - \mathcal{B}_{\mathcal{H}\mathbf{w}}(\mathbf{z}_{\mathcal{H}}, r_{\mathcal{H}}; \mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}})|}{|||[\mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}]|||_\Omega}\Bigg] \\ &+ \sup_{\substack{\mathbf{v}_{\mathcal{H}} \in \mathbf{V}_{\mathcal{H}} \\ q_{\mathcal{H}} \in Q_{\mathcal{H}}}} \frac{|\mathcal{F}_{\mathbf{w}}(\mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}) - \mathcal{F}_{\mathcal{H}\mathbf{w}}(\mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}})|}{|||[\mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}]|||_\Omega},\end{aligned} \tag{6.44}$$

where $\mathcal{B}_{\mathbf{w}}$ and $\mathcal{F}_{\mathbf{w}}$ are the counterparts, in the Galerkin context, of $\mathcal{B}_{\mathcal{H}\mathbf{w}}$ and $\mathcal{F}_{\mathcal{H}\mathbf{w}}$, respectively. We begin by controlling the term $|||[\mathbf{u} - \mathbf{z}_{\mathcal{H}}, p - r_{\mathcal{H}}]|||_\Omega$ by using (6.38) and definitions (5.7) for $\tau_k(\mathbf{x})$ and $\gamma_k(\mathbf{x})$, respectively. By definition (6.3) and by taking an arbitrary element $[\mathbf{z}_{\mathcal{H}}, r_{\mathcal{H}}] \in \mathbf{V}_{\mathcal{H}} \times Q_{\mathcal{H}}$, we have

$$
\begin{aligned}
&|||[\mathbf{u}-\mathbf{z}_{\mathcal{H}}, p-r_{\mathcal{H}}]|||_{\Omega} \\
&\quad = \Bigg[\alpha\|\mathbf{u}-\mathbf{z}_{\mathcal{H}}\|^2_{L^2(\Omega)} + \nu|\mathbf{u}-\mathbf{z}_{\mathcal{H}}|^2_{H^1(\Omega)} \\
&\qquad + \sum_{T_k\in\mathcal{T}_H} \|\gamma_k^{1/2}\nabla(\mathbf{u}-\mathbf{z}_{N-1,k})\|^2_{L^2(T_k)} \\
&\qquad + \sum_{T_k\in\mathcal{T}_H} \|\tau_k^{1/2}(\mathbf{x})\nabla(p-r_{N-1,k})\|^2_{L^2(T_k)}\Bigg]^{1/2} \qquad \text{(by Theorem 6.4)} \\
&\quad \le \Bigg[\alpha(CH^{\min(N+1,s)}N^{-s}\|\mathbf{u}\|_{H^s(\Omega)})^2 + \nu(CH^{\min(N+1,s)-1}N^{1-s}\|\mathbf{u}\|_{H^s(\Omega)})^2 \\
&\qquad + \frac{\lambda|\mathbf{w}|_p H^2}{2\nu N^4}(CH^{\min(N+1,s)-1}N^{1-s}\|\mathbf{u}\|_{H^s(\Omega)})^2 \\
&\qquad + \frac{mH^2}{4\nu N^4}(CH^{\min(N+1,l)-1}N^{1-l}\|p\|_{H^l(\Omega)})^2\Bigg]^{1/2} \\
&\quad \le CH^{\min(N+1,s)-1}N^{1-s}\|\mathbf{u}\|_{H^s(\Omega)}\left(\sqrt{\nu}+\sqrt{\alpha}\frac{H}{N}+\sqrt{\frac{\lambda|\mathbf{w}|_p}{2\nu}}\frac{H}{N^2}\right) \\
&\qquad + \frac{C}{N\nu}H^{\min(N+1,l)}N^{-l}\|p\|_{H^l(\Omega)}.
\end{aligned} \tag{6.45}
$$

We analyze now the error due to the generalized Galerkin approach. By the exactness of the Gauss–Lobatto quadrature formulas on polynomials of degree $(2N-1)$, we choose $[\mathbf{z}_{\mathcal{H}}, r_{\mathcal{H}}] \in X_{\mathcal{H}}$ such that $\mathbf{z}_{\mathcal{H}}|_{T_k} = \mathbf{z}_{N-1,k} \in (\mathbb{Q}_{N-1}(T_k))^2, r_{\mathcal{H}}|_{T_k} = r_{N-1,k} \in \mathbb{Q}_{N-1}(T_k)$, so that all the terms that do not involve the function $\mathbf{w}$ and that arise from $\mathcal{B}_{\mathbf{w}}$ annul those arising from $\mathcal{B}_{\mathcal{H}\mathbf{w}}$. It holds:

$$
\begin{aligned}
&|\mathcal{B}_{\mathbf{w}}(\mathbf{z}_{\mathcal{H}}, r_{\mathcal{H}}; \mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}}) - \mathcal{B}_{\mathcal{H}\mathbf{w}}(\mathbf{z}_{\mathcal{H}}, r_{\mathcal{H}}; \mathbf{v}_{\mathcal{H}}, q_{\mathcal{H}})| \\
&\quad \le |\tilde{c}(\mathbf{w}, \mathbf{z}_{\mathcal{H}}, \mathbf{v}_{\mathcal{H}}) - \tilde{c}_{\mathcal{H}}(\mathbf{w}, \mathbf{z}_{\mathcal{H}}, \mathbf{v}_{\mathcal{H}})| \\
&\qquad + \Bigg|\sum_{T_k\in\mathcal{T}_H}[((\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k}, \tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k}+\nabla q_{N,k}))_{L^2(T_k)} \\
&\qquad\qquad - ((\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k}, \tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k}+\nabla q_{N,k}))_{N,T_k}]\Bigg| \\
&\qquad + \Bigg|\sum_{T_k\in\mathcal{T}_H}[(-\nu\Delta\mathbf{z}_{N-1,k}+\nabla r_{N-1,k}, \tau_k(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k})_{L^2(T_k)} \\
&\qquad\qquad - (-\nu(\Delta\mathbf{z}_{N-1,k}+\nabla r_{N-1,k}, \tau_k(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k})_{N,T_k}]\Bigg| \\
&\qquad + \Bigg|\sum_{T_k\in\mathcal{T}_H}[((\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k}, \tau_k(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k})_{L^2(T_k)} \\
&\qquad\qquad - ((\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k}, \tau_k(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k})_{N,T_k}]\Bigg|,
\end{aligned} \tag{6.46}
$$

where

$$
\begin{aligned}
&|\tilde{c}(\mathbf{w},\mathbf{z}_{\mathcal{H}},\mathbf{v}_{\mathcal{H}})-\tilde{c}_{\mathcal{H}}(\mathbf{w},\mathbf{z}_{\mathcal{H}},\mathbf{v}_{\mathcal{H}})| \\
&= \frac{1}{2}|c(\mathbf{w},\mathbf{z}_{\mathcal{H}},\mathbf{v}_{\mathcal{H}})-c(\mathbf{w},\mathbf{v}_{\mathcal{H}},\mathbf{z}_{\mathcal{H}})-c_{\mathcal{H}}(\mathbf{w},\mathbf{z}_{\mathcal{H}},\mathbf{v}_{\mathcal{H}})+c_{\mathcal{H}}(\mathbf{w},\mathbf{v}_{\mathcal{H}},\mathbf{z}_{\mathcal{H}})| \\
&\le \frac{1}{2}\left|\sum_{T_k\in\mathcal{T}_H}[((\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k},\mathbf{v}_{N,k})_{L^2(T_k)}-((\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k},\mathbf{v}_{N,k})_{N,T_k}]\right| \\
&\quad +\frac{1}{2}\left|\sum_{T_k\in\mathcal{T}_H}[((\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k},\mathbf{z}_{N-1,k})_{L^2(T_k)}-((\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k},\mathbf{z}_{N-1,k})_{N,T_k}]\right|. \qquad (6.47)
\end{aligned}
$$

We analyze each term of (6.47) on an arbitrary element $T_k \in \mathcal{T}_H$:

$$
\begin{aligned}
&\bullet|((\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k},\mathbf{v}_{N,k})_{L^2(T_k)}-((\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k},\mathbf{v}_{N,k})_{N,T_k}| \quad \text{(by Lemma 6.5)} \\
&\le CH_k^{\min(N+1,s)-1}N^{1-s}\|(\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k}\|_{H^{s-1}(T_k)}\|\mathbf{v}_{N,k}\|_{L^2(T_k)} \quad \text{(by (6.41))} \\
&\le CH^{\min(N+1,s)-1}N^{1-s}\|\mathbf{w}\|_{W^{s-1,\infty}(T_k)}\|\mathbf{u}\|_{H^s(T_k)}\|\mathbf{v}_{N,k}\|_{L^2(T_k)},
\end{aligned}
$$

$$
\begin{aligned}
&\bullet|((\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k},\mathbf{z}_{N-1,k})_{L^2(T_k)}-((\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k},\mathbf{z}_{N-1,k})_{N,T_k}| \\
&= |(\nabla\mathbf{v}_{N,k},\mathbf{w}\otimes\mathbf{z}_{N-1,k})_{L^2(T_k)}-(\nabla\mathbf{v}_{N,k},\mathbf{w}\otimes\mathbf{z}_{N-1,k})_{N,T_k}| \quad \text{(by Lemma 6.5)} \\
&\le CH_k^{\min(N+1,s)}N^{-s}\|\mathbf{w}\otimes\mathbf{z}_{N-1,k}\|_{H^s(T_k)}\|\nabla\mathbf{v}_{N,k}\|_{L^2(T_k)} \quad \text{(by (6.41))} \\
&\le CH^{\min(N+1,s)}N^{-s}\|\mathbf{w}\|_{W^{s,\infty}(T_k)}\|\mathbf{u}\|_{H^s(T_k)}\|\nabla\mathbf{v}_{N,k}\|_{L^2(T_k)},
\end{aligned}
$$

and the terms of (6.46):

$$
\begin{aligned}
&\bullet|((\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k},\tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k}+\nabla q_{N,k}))_{L^2(T_k)} \\
&\quad -((\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k},\tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k}+\nabla q_{N,k}))_{N,T_k}| \quad \text{(by Lemma 6.5 and (6.41))} \\
&\le CH_k^{\min(N+1,s)-1}N^{1-s}\|\mathbf{w}\|_{W^{s-1,\infty}(T_k)}\|\mathbf{u}\|_{H^s(T_k)}\|\tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k}+\nabla q_{N,k})\|_{L^2(T_k)} \\
&\quad \text{(by Theorem 6.1 and definition of } \tau_k(x)) \\
&\le CH^{\min(N+1,s)-1}N^{1-s}\|\mathbf{w}\|_{W^{s-1,\infty}(T_k)}\|\mathbf{u}\|_{H^s(T_k)}[\|\mathbf{v}_{N,k}\|_{L^2(T_k)}+\|\tau_k(\mathbf{x})\nabla q_{N,k}\|_{L^2(T_k)}].
\end{aligned}
$$

$$
\begin{aligned}
&\bullet|(-\nu\Delta\mathbf{z}_{N-1,k}+\nabla r_{N-1,k},\tau_k(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k})_{L^2(T_k)} \\
&\quad -(-\nu\Delta\mathbf{z}_{N-1,k}+\nabla r_{N-1,k},\tau_k(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k})_{N,T_k}| \\
&= |(\tau_k(\mathbf{x})\mathbf{w}\otimes(-\nu\Delta\mathbf{z}_{N-1,k}+\nabla r_{N-1,k}),\nabla\mathbf{v}_{N,k})_{L^2(T_k)} \\
&\quad -(\tau_k(\mathbf{x})\mathbf{w}\otimes(-\nu\Delta\mathbf{z}_{N-1,k}+\nabla r_{N-1,k}),\nabla\mathbf{v}_{N,k})_{N,T_k}| \quad \text{(by Lemma 6.5)} \\
&\le CH_k^{\min(N+1,s)-2}N^{2-s}\|\mathbf{w}\|_{W^{s-2,\infty}(T_k)}\|\tau_k(\mathbf{x})\nu\Delta\mathbf{z}_{N-1,k}\|_{H^{s-2}(T_k)}\|\nabla\mathbf{v}_{N,k}\|_{L^2(T_k)} \\
&\quad + CH_k^{\min(N+1,l)-1}N^{1-l}\|\mathbf{w}\|_{W^{l-1,\infty}(T_k)}\|\tau_k(\mathbf{x})\nabla r_{N-1,k})\|_{H^{l-1}(T_k)}\|\nabla\mathbf{v}_{N,k}\|_{L^2(T_k)} \\
&\quad \text{(by (6.41) and Theorem 6.1)} \\
&\le CH^{\min(N+1,s)-2}N^{2-s}\|\mathbf{w}\|_{W^{s-2,\infty}(T_k)}\frac{mH^2}{4N^4}\|\mathbf{u}\|_{H^s(\Omega)}\tilde{C}N^2H^{-1}\|\mathbf{v}_{N,k}\|_{L^2(T_k)}
\end{aligned}
$$

$$
\begin{aligned}
&+ CH^{\min(N+1,l)-1}N^{1-l}\|\mathbf{w}\|_{W^{l-1,\infty}(T_k)}\frac{mH^2}{4\nu N^4}\|p\|_{H^l(T_k)}\tilde{C}N^2H^{-1}\|\mathbf{v}_{N,k}\|_{L^2(T_k)}\\
&\quad (\text{by } m < 1/(6\tilde{C}^2) \text{ and } (6.41))\\
\le\ & \Bigg[CH^{\min(N+1,s)-1}N^{-s}\|\mathbf{w}\|_{W^{s-2,\infty}(T_k)}\|\mathbf{u}\|_{H^s(T_k)}\\
&\quad + \frac{C}{N\nu}H^{\min(N+1,l)}N^{-l}\|\mathbf{w}\|_{W^{l-1,\infty}(T_k)}\|p\|_{H^lT_k)}\Bigg]\|\mathbf{v}_{N,k}\|_{L^2(T_k)}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\bullet|(\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k},\tau_k(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k})_{L^2(T_k)} - (\mathbf{w}\cdot\nabla)\mathbf{z}_{N-1,k},\tau_k(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k})_{N,T_k}|\\
&\le\ CH^{\min(N+1,s)-1}N^{1-s}\|\mathbf{w}\|^2_{W^{s-1,\infty}(T_k)}\|\nabla\mathbf{z}_{N-1,k}\|_{H^{s-1}(T_k)}\|\tau_k(\mathbf{x})\nabla\mathbf{v}_N\|_{L^2(T_k)}\\
&\quad (\text{by Theorem 6.1 and } (6.41))\\
&\le\ \frac{H}{N^2}\frac{C}{\nu}H^{\min(N+1,s)-1}N^{1-s}\|\mathbf{w}\|^2_{W^{s-1,\infty}(T_k)}\|\mathbf{u}\|_{H^s(T_k)}\|\mathbf{v}_{N,k}\|_{L^2(T_k)}.
\end{aligned}
$$

Then, by summation on $T_k \in \mathcal{T}_H$ we obtain:

$$
\begin{aligned}
&\sup_{\substack{\mathbf{v}_{\mathcal{H}}\in\mathbf{V}_{\mathcal{H}}\\ q_{\mathcal{H}}\in Q_{\mathcal{H}}}}\frac{|\mathcal{B}_{\mathbf{w}}(\mathbf{z}_{\mathcal{H}},r_{\mathcal{H}};\mathbf{v}_{\mathcal{H}},q_{\mathcal{H}}) - \mathcal{B}_{\mathcal{H}\mathbf{w}}(\mathbf{z}_{\mathcal{H}},r_{\mathcal{H}};\mathbf{v}_{\mathcal{H}},q_{\mathcal{H}})|}{|||[\mathbf{v}_{\mathcal{H}},q_{\mathcal{H}}]|||_\Omega}\\
&\le\ CH^{\min(N+1,s)-1}N^{1-s}\|\mathbf{u}\|_{H^s(\Omega)}\cdot\Bigg[\|\mathbf{w}\|_{W^{s-1,\infty}(\Omega)} + \frac{H}{N\nu}\|\mathbf{w}\|_{W^{s,\infty}(\Omega)}\\
&\qquad + \frac{1}{N}\|\mathbf{w}\|_{W^{s-2,\infty}(\Omega)} + \frac{H}{\nu N^2}\|\mathbf{w}\|^2_{W^{s-2,\infty}(\Omega)}\Bigg]\\
&\quad + \frac{C}{N\nu}H^{\min(N+1,l)}N^{-l}\|\mathbf{w}\|_{W^{l-1,\infty}(\Omega)}\|p\|_{H^l(\Omega)}. \qquad (6.48)
\end{aligned}
$$

We have to control the last term in (6.44):

$$
\begin{aligned}
&|\mathcal{F}_{\mathbf{w}}(\mathbf{v}_{\mathcal{H}},q_{\mathcal{H}}) - \mathcal{F}_{\mathcal{H}\mathbf{w}}(\mathbf{v}_{\mathcal{H}},q_{\mathcal{H}})| \le |(\mathbf{f},\mathbf{v}_{\mathcal{H}})_{L^2(\Omega)} - (\mathbf{f},\mathbf{v}_{\mathcal{H}})_{\mathcal{H}})|\\
&\quad + \Bigg|\sum_{T_k\in\mathcal{T}_H}[(\mathbf{f},\tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k} + (\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k} + \nabla q_{N,k}))_{L^2(T_k)}\\
&\qquad - (\mathbf{f},\tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k} + (\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k} + \nabla q_{N,k}))_{N,T_k}]\Bigg|; \qquad (6.49)
\end{aligned}
$$

where, by Lemma 6.5,

$$
|(\mathbf{f},\mathbf{v}_{\mathcal{H}})_{L^2(\Omega)} - (\mathbf{f},\mathbf{v}_{\mathcal{H}})_{\mathcal{H}}| \le CH^{\min(N+1,r)}N^{-r}\|\mathbf{f}\|_{H^r(\Omega)}\|\mathbf{v}_{\mathcal{H}}\|_{L^2(\Omega)}
$$

and

$$
\begin{aligned}
&|(\mathbf{f},\tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k} + (\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k} + \nabla q_{N,k}))_{L^2(T_k)}\\
&\quad - (\mathbf{f},\tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k} + (\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k} + \nabla q_{N,k}))_{N,T_k}|\\
&\le\ |(\mathbf{f},\tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k} + \nabla q_{N,k})_{L^2(T_k)} - (\mathbf{f},\tau_k(\mathbf{x})(-\nu\Delta\mathbf{v}_{N,k} + \nabla q_{N,k})_{N,T_k}|\\
&\quad + |(\mathbf{f},\tau_k(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k})_{L^2(T_k)} - (\mathbf{f},\tau_k(\mathbf{x})(\mathbf{w}\cdot\nabla)\mathbf{v}_{N,k})_{N,T_k}| \quad (\text{by Lemma 6.5})
\end{aligned}
$$

$$\leq \; CH^{\min(N+1,r)} N^{-r} \|\mathbf{f}\|_{H^r(T_k)} \cdot [\nu \|\tau_k(\mathbf{x}) \Delta \mathbf{v}_{N,k}\|_{L^2(T_k)} + \|\tau_k(\mathbf{x}) \nabla q_{N,k}\|_{L^2(T_k)}]$$
$$+ \, |(\mathbf{f} \otimes \mathbf{w}, \tau_k(\mathbf{x}) \nabla \mathbf{v}_{N,k})_{L^2(T_k)} - (\mathbf{f} \otimes \mathbf{w}, \tau_k(\mathbf{x}) \nabla \mathbf{v}_{N,k})_{N,T_k}| \quad \text{(by Theorem 6.1)}$$
$$\leq \; CH^{\min(N+1,r)} N^{-r} \|\mathbf{f}\|_{H^r(T_k)} [\|\mathbf{v}_{N,k}\|_{L^2(T_k)} + \|\tau_k(\mathbf{x}) \nabla q_{N,k}\|_{L^2(T_k)}]$$
$$+ \, \frac{H}{N^2} \frac{C}{\nu} H^{\min(N+1,r)} N^{-r} \|\mathbf{f}\|_{H^r(T_k)} \|\mathbf{w}\|_{W^{r,\infty}(T_k)} \|\mathbf{v}_{N,k}\|_{L^2(T_k)}.$$

Then

$$\sup_{\substack{\mathbf{v}_\mathcal{H} \in \mathbf{V}_\mathcal{H} \\ q_\mathcal{H} \in Q_\mathcal{H}}} \frac{|\mathcal{F}_\mathbf{w}(\mathbf{v}_\mathcal{H}, q_\mathcal{H}) - \mathcal{F}_{\mathcal{H}\mathbf{w}}(\mathbf{v}_\mathcal{H}, q_\mathcal{H})|}{|||[\mathbf{v}_\mathcal{H}, q_\mathcal{H}]|||_\Omega}$$
$$\leq \; CH^{\min(N+1,r)} N^{-r} \|\mathbf{f}\|_{H^r(T_k)} \|\mathbf{w}\|_{W^{r,\infty}(T_k)} \left(1 + \frac{H}{\nu N^2}\right). \qquad (6.50)$$

The thesis follows by applying now $\inf_{\substack{\mathbf{z}_\mathcal{H} \in \mathbf{V}_\mathcal{H} \\ r_\mathcal{H} \in Q_\mathcal{H}}}$ on the sum between (6.45) and (6.48) and then by summing with (6.50). ∎

VII. NUMERICAL RESULTS

At each time level we have to solve a linear system of large dimension but with sparse structure. If N is the polynomial interpolation degree in each dimension and N_e is the global number of spectral elements, then we have a system of dimension $3N_e \cdot N^2$, with a density of about 10% of nonzero coefficients.

Numerical results show that the condition number of the matrix $\mathcal{A}$ arising from the stabilized approximation method is

$$K(\mathcal{A}) = \mathcal{O}(H^{-4} N^6), \qquad (7.1)$$

where H is the maximum diameter on the elements of the decomposition.

So, it is mandatory to solve the system by a preconditioned iterative method. We choose to use the Bi-CGStab of Van-der Vorst (see [30]), preconditioned by a local *finite element preconditioner* on the style of ''EBE'' (see [15] and [16]).

It is well known ([31]–[33]) that finite element preconditioners are optimal for the spectral approximation of boundary value problems. However, instead of a global preconditioner, which requires many storage locations to be inverted, we considered an element-by-element bilinear preconditioner, based on the same stabilized approximation used for the primary problem, with a no-friction boundary condition on the internal boundaries, i.e., $\forall T_k \in \mathcal{T}_H$:

$$\tilde{T}\mathbf{n} := -p\mathbf{n} + \nu(\mathbf{n} \cdot \nabla)\mathbf{u} = \mathbf{0} \quad \text{on } \partial T_k \setminus (\partial\Omega \cap \partial T_k), \qquad (7.2)$$

($\hat{T}$ is the stress tensor) while on $\partial T_k \cap \partial\Omega$ the original boundary condition given on the problem is imposed. No boundary condition for the pressure has been imposed explicitly.

Remark. Even if this preconditioner is not optimal, numerical results point out substantial reduction of the iterations number with respect to an unpreconditioned system or even to a diagonally preconditioned system. All the results presented in the following section are obtained using the local finite element preconditioner.

In order to stop the iterations we control that the euclidean norm of the residual, normalized over the Euclidean norm of the right-hand side, is less than a given tolerance ϵ. If it is not specified we

TABLE I. Spectral accuracy of the stabilized scheme for the Kim and Moin analytical solution to the Navier–Stokes equations on 16 spectral elements.

N	$\|\mathbf{u} - \mathbf{u}_{\mathcal{H}}\|_{H^1(\Omega)}$	$\|p - p_{\mathcal{H}}\|_{L^2(\Omega)}$
4	0.4276e-2	0.1883e-1
5	0.3589e-3	0.3674e-2
6	0.2586e-4	0.3279e-3
7	0.1472e-5	0.8756e-4

choose $\epsilon = 10^{-10}$. At each time-step we take the maximum number of iterations of the BiCGStab equal to 400.

Now we present some numerical results attesting to the high accuracy of the stabilized approximation. First, we consider the analytical solution of Kim and Moin (see [34]) for the time-dependent Navier-Stokes equations and we show the accuracy of the schemes we have used in space and in time. We have considered the problem (2.1) on the computational domain $\Omega = (0, 1)^2$, with exact solution

$$u(x, y) = -\cos(\alpha\pi x)\sin(\alpha\pi y)e^{(-2\alpha^2\pi^2 t\nu)}$$

$$v(x, y) = \sin(\alpha\pi x)\cos(\alpha\pi y)e^{(-2\alpha^2\pi^2 t\nu)}$$

$$p(x, y) = -\frac{1}{4}[\cos(2\alpha\pi x) + \cos(2\alpha\pi y)]e^{(-4\alpha^2\pi^2 t\nu)}. \tag{7.3}$$

and we have chosen $\alpha = 2$. We have considered a partition of Ω in 4×4 squared elements, viscosity $\nu = 10^{-2}$, the Euler semi-implicit scheme (4.1) for the time discretization. The stabilization parameters are chosen as follows: $p = 2, m = 5 \cdot 10^{-3}, \lambda = 1$. In Table I we show the space approximation errors, here $(0, T) = (0, 0.01)$ and $\Delta t = 0.0001$, while Table II shows the first-order accuracy in Δt for the semi-implicit Euler scheme (4.1). Again, we considered a partition of Ω in 4×4 squared elements, with polynomial degree $N = 5$, the other parameters are chosen as for the results of Table I.

A. Driven Cavity Flow

This test case shows the motion of a flow inside a plane square domain $\Omega = (0, 1)^2$ with tangential velocity prescribed on the top boundary $\mathbf{u}_\infty = (1, 0)^T$. The parameter D in the definition of Re is the measure of the side of Ω. A no-slip boundary condition is imposed on the vertical sides as well as on the bottom horizontal side.

The problem has been solved by the stabilized spectral elements (SSE), and by the Euler semi-implicit time advancing scheme in order to linearize the nonlinear terms.

The iterative pseudo-temporal procedure is stopped when the Euclidean norm of the difference between two successive numerical solutions is less than 10^{-6}. The numerical solutions shown in Figs. 1–4 are obtained with a time-step Δt given in Table III, as the number of time-steps needed to obtain the stationary solutions. The stabilization parameters have been chosen as follows: $m = 8 \cdot 10^{-3}, \lambda = 1$, and $p = 2$.

TABLE II. $\|\mathbf{u} - \mathbf{u}_{\mathcal{H}}\|_{H^1(\Omega)}$ error of the semi-implicit Euler scheme at $T = 1$.

Δt	0.01	0.025	0.05	0.1	0.25
$\|\mathbf{u} - \mathbf{u}_{\mathcal{H}}\|_{H^1(\Omega)}$	0.2018e-2	0.4951e-2	0.9707e-2	0.2077e-1	0.4329e-1

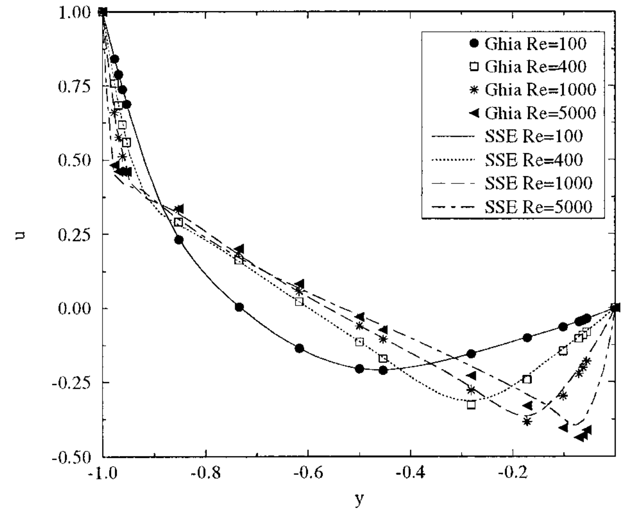

FIG. 1. Profiles of u velocities along vertical lines through geometric center of the cavity.

First of all we present the comparison with the results of Ghia et al. [17] about the profiles of u and v velocities along vertical and horizontal (respectively) lines through geometric center of the cavity (see Figs. 1 and 2). The solid lines represent the numerical solution obtained by SSE at different Reynolds numbers, while the symbols represent the numerical data found in Tables I and II in [17].

In Table IV we compare the space discretization used in [17] and by SSE. N and M stand for the polynomial degree in each spectral element (in each direction) and the number of elements in each direction, respectively. Unless otherwise specified, the decomposition of Ω is uniform.

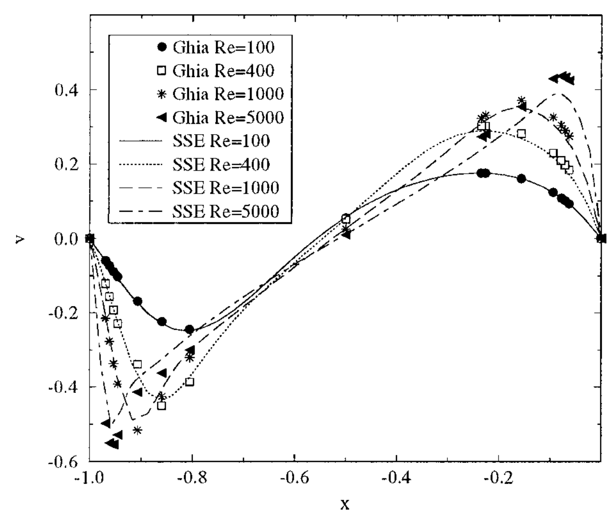

FIG. 2. Profiles of v velocities along horizontal lines through geometric center of the cavity.

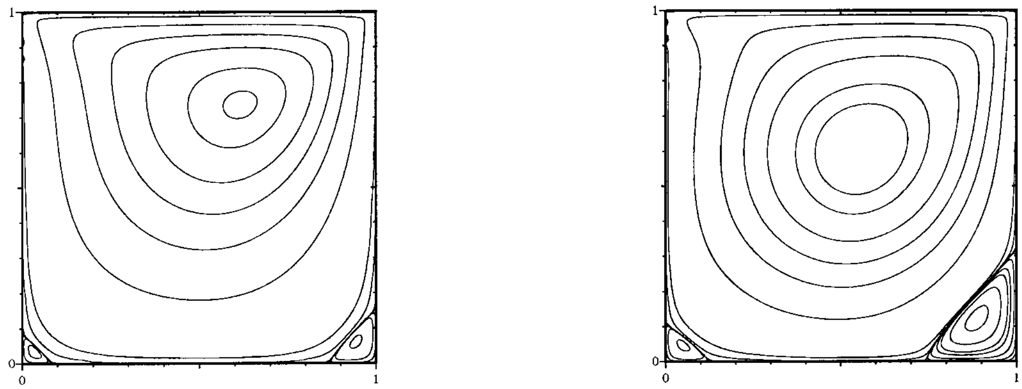

FIG. 3. Streamlines for $Re = 100$ (left) and $Re = 400$ (right).

In Figs. 3 and 4, we present the streamline contours for the cavity flow configurations, for $Re = 100, 400, 1000, 5000$.

B. Uniform Flow Past a Circular Cylinder

This is a problem unsteady in nature and it represents the motion of a flow past a circular obstacle. The computational domain $\Omega = (-4.5, 15.5) \times (-4.59, 4.5)$ is considered and the circular obstacle, with diameter $D = 1$, is centered at (0.0, 0.0). The unsymmetry of the geometry has been adopted in order to generate the periodic motion of the fluid [35].

The computational domain has been discretized in 64 spectral elements with polynomial degree $N = 6$ in each direction. The global number of nodes is 1700. The boundary conditions are assigned as they can be observed in Fig. 5, where t_1 and t_2 represent the two components of the normal component of the stress tensor: $\tilde{T}\mathbf{n} := -p\mathbf{n} + \nu(\mathbf{n} \cdot \nabla)\mathbf{u}$.

The semi-implicit Euler scheme has been used with $\Delta t = 0.1$. At the beginning of the motion, two symmetric eddies past the cylinder are generated and, approximately, at $t = 22.0$ the unsymmetry on the geometry generates the beginning of the vortices. The motion is transitory

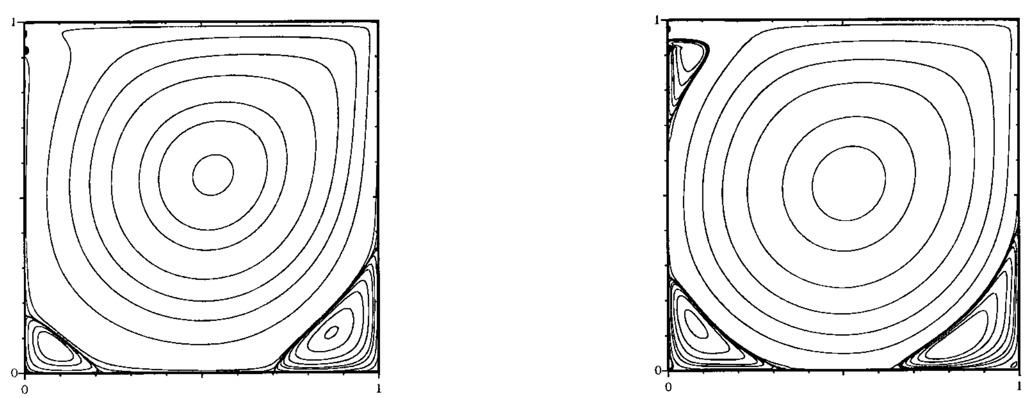

FIG. 4. Streamlines for $Re = 1000$ (left) and $Re = 5000$ (right).

TABLE III. The time-step and the number of time-steps needed to obtain the stationary solution of the driven cavity test case.

Re	Δt	# time-steps
100	0.1	186
400	0.1	425
1000	0.1	593
5000	0.05	1323

TABLE IV. The number of nodes of the space discretization used in the ''Driven Cavity'' test case. M denotes the number of elements in each space direction.

Re	100	400	1000	5000
Ghia Discretization	129×129 (16641)	129×129 (16641)	129×129 (16641)	129×129 (16641)
SSE Discretization	$N = 8$ $M = 6$ (2401)	$N = 8$ $M = 6$ (2401)	$N = 6$ $M = 10$ (3721)	$N = 6$ $M = 12$ (5329)

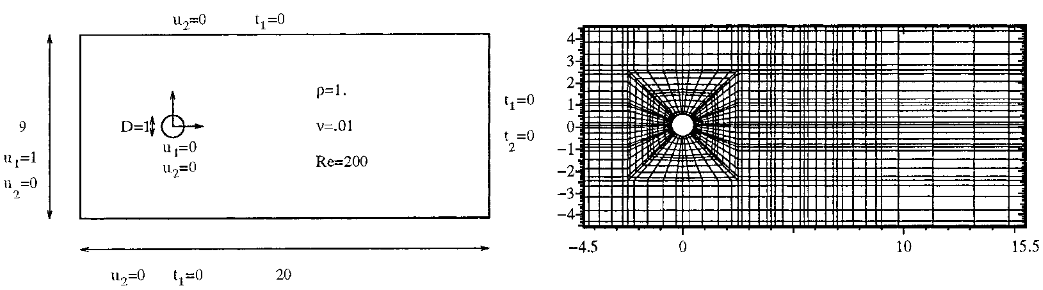

FIG. 5. The geometry (left) and the partition of the computational domain (right) for the test case *uniform flow past a circular cylinder*.

until $t = 110.0$, when it becomes periodic with a period $T = 5.6$. This period corresponds to a Strouhal number $St = D/(\|\mathbf{u}_\infty\|T) = 0.178$, comparable with those found in [35].

We show the data and the discretization of the computational domain in Fig. 5 and the stationary streamlines inside a period of the motion in Figs. 6–8.

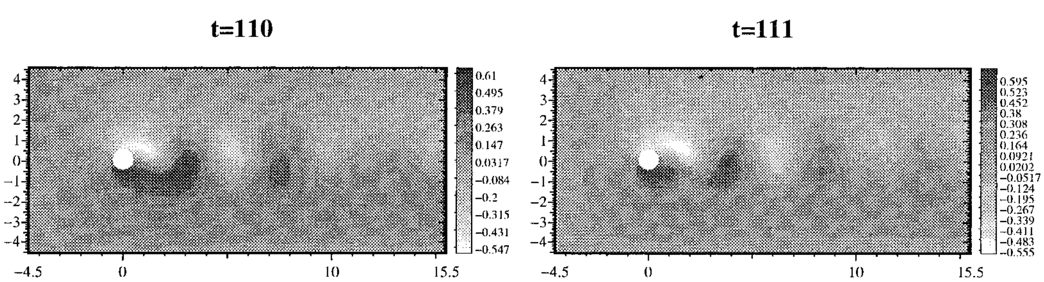

FIG. 6. The stationary streamlines inside a period of the motion for $Re = 100$.

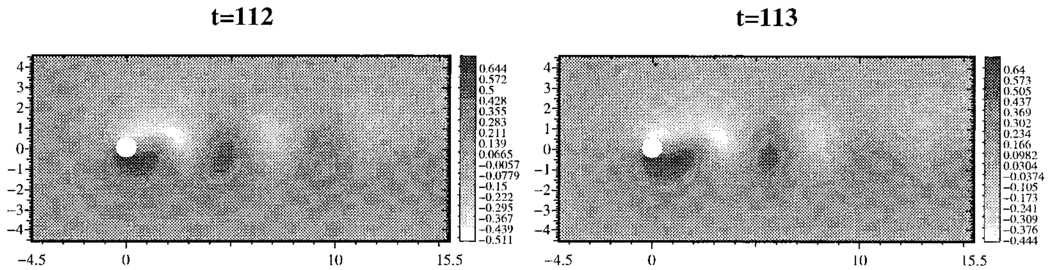

FIG. 7. The stationary streamlines inside a period of the motion for $Re = 100$.

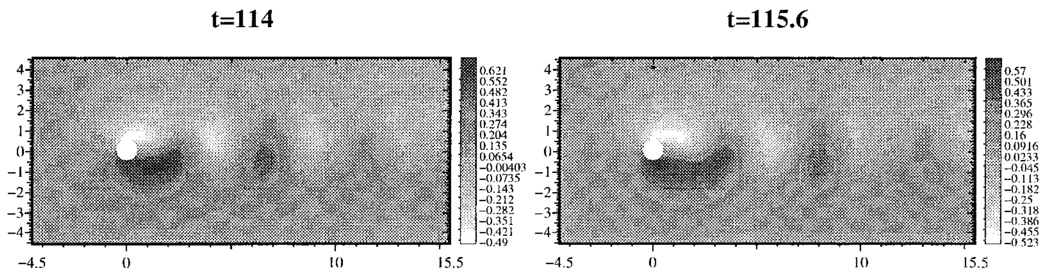

FIG. 8. The stationary streamlines inside a period of the motion for $Re = 100$.

In Figs. 9–11 the vorticity for $Re = 200$ is shown in a zoom of the computational domain, inside a period of the motion. Here we have used $\Delta t = 0.05$ and we have obtained a period $T = 6.0$ with $St = 1.67$.

VIII. CONCLUSIONS

In this article, we have considered the approximation of the incompressible linearized Navier–Stokes equations on bidimensional domains by a stabilized spectral element method. We used a

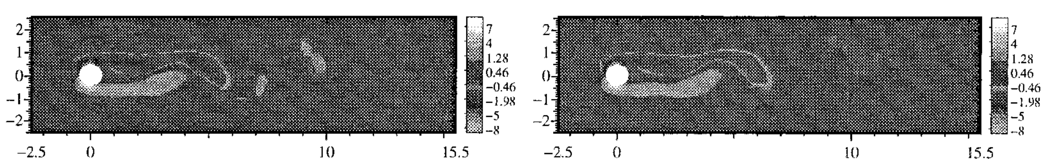

FIG. 9. The vorticity inside a period of the motion ($t = 110$ and $t = 111$).

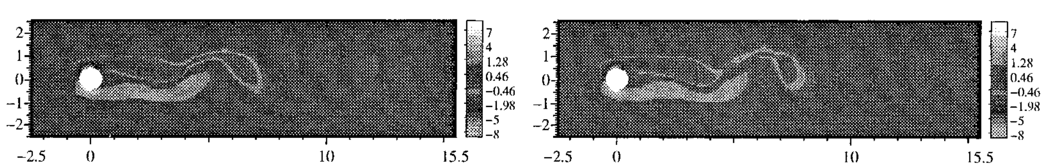

FIG. 10. The vorticity inside a period of the motion ($t = 112$ and $t = 113$).

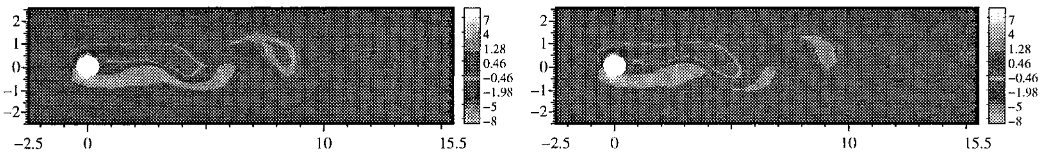

FIG. 11. The vorticity inside a period of the motion ($t = 114$ and $t = 115$).

spectral element discretization in space, plus SUPG-like stabilization techniques. An automatic design of the stabilization parameters is given in Section V. The time variable has been discretized by a finite difference scheme. Results of stability and convergence are proved for the approximation we have used. Finally, we reported numerical tests demonstrating the spectral accuracy of the approximation.

We thank Prof. A. Quarteroni for fruitful discussions during the preparation of this report.

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