Mathematical Models and Methods in Applied Sciences
Vol. 14, No. 5 (2004) 701–733
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NUMERICAL APPROXIMATION OF CONTROLLABILITY OF TRAJECTORIES FOR EULER–BERNOULLI THERMOELASTIC PLATES

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Received 13 May 2003
Revised 26 September 2003
Communicated by N. Bellomo

Euler–Bernoulli thermoelastic plate model with a control function in the thermal equation is considered. This paper is devoted to the analysis and construction of the minimization procedure related to the controllability of its trajectories by applying both penalty and duality arguments. Numerical approximation of the optimality system is carried out through the use of spectral element methods in space and finite difference schemes in time. Numerical results obtained on several test cases are shown.

Keywords: Null controllability; thermoelastic plate; optimization and variational techniques; spectral element methods.

AMS Subject Classification: 35B37, 74F05, 93B05, 65K10, 65M70

1. Introduction

Let Ω be a bounded, open, connected subset of $\mathbb{R}^2$, with a Lipschitz boundary and ω any open subset of Ω. Let $T > 0$ and set $Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$. We consider a model which describes the small vibrations of a homogeneous, elastically and thermally isotropic Euler–Bernoulli plate, under the influence of a control function $f \in L^2(\omega \times (0,T))$. In an absence of other exterior forces, and with hinged mechanical and Dirichlet thermal boundary conditions, the system we are going to study is the following:

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta\theta = 0 & \text{in } Q \\ \theta_t - \Delta\theta - \Delta u_t = \chi_\omega f & \text{in } Q \\ u = 0,\ \Delta u = 0,\ \theta = 0 & \text{on } \Sigma \\ u(0) = u_0\,,\ u_t(0) = u_1\,,\ \theta(0) = \theta_0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

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Here, u is the *vertical deflection* of the plate and θ is the *variation of temperature* of the plate with respect to its reference temperature. The subscript "t" denotes time derivative, χ_ω is the characteristic function of ω, and u_0, u_1, θ_0 are initial data in a suitable space.

1.1. *The control problem*

We introduce the Hilbert space

$$H := \big(H^2(\Omega) \cap H^1_0(\Omega)\big) \times L^2(\Omega) \times L^2(\Omega)$$

equipped with the inner product

$$\langle \mathbf{z}, \mathbf{s} \rangle_H := \int_\Omega (\Delta z_1 \, \Delta s_1 + z_2 \, s_2 + z_3 \, s_3) \, dx \,,$$

where $\mathbf{z} := [z_1,\ z_2,\ z_3]^\top$, $\mathbf{s} := [s_1,\ s_2,\ s_3]^\top$ and $M^\top$ is the transpose of the related matrix M. The induced norm is denoted by $\| \cdot \|_H$. Putting $v := u_t$ and

$$\mathbf{z}(t) := [u(t),\ v(t),\ \theta(t)]^\top, \quad \mathbf{z}^0 := [u_0,\ u_1,\ \theta_0]^\top,$$

problem (1.1) can be rewritten as an abstract linear evolution equation in H of the form

$$\begin{cases} \mathbf{z}_t = A\,\mathbf{z} + B\,f & \text{in } Q \\ \mathbf{z}(0) = \mathbf{z}^0 & \text{in } \Omega \\ z_1 = 0,\ \Delta z_1 = 0,\ z_3 = 0 & \text{on } \Sigma \end{cases} \tag{1.2}$$

where we set the operator $A : D(A) \to H$ by

$$A = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & 0 & -\Delta \\ 0 & \Delta & \Delta \end{bmatrix} \tag{1.3}$$

with domain

$$D(A) = \big\{ \mathbf{z} \in H :\ \Delta u, v, \theta \in H^2(\Omega) \cap H^1_0(\Omega) \big\} \,.$$

The initial data are $\mathbf{z}^0 \in H$, the control operator $B : L^2(\omega) \to H$ is defined as

$$B\,f = \big[0, 0, \chi_\omega\, f\big]^\top \,. \tag{1.4}$$

In our paper we present some results about the *controllability of trajectories* at any time $T > 0$ for thermoelastic model (1.1). This problem can be formulated in the following way. Given $\hat{\mathbf{z}}^0 \in H$ and $\hat{f} \in L^2(\omega \times (0,T))$, let $\hat{\mathbf{z}} = \mathbf{z}(T; \hat{\mathbf{z}}^0, \hat{f})$ be the related solution for PDE system (1.2). Given any different initial data $\tilde{\mathbf{z}}^0 \in H$, we look for a control function $\tilde{f} \in L^2(\omega \times (0,T))$ such that

$$\mathbf{z}(T; \tilde{\mathbf{z}}^0, \tilde{f}) = \hat{\mathbf{z}}.$$

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We can solve this problem as follows. We consider system (1.2) with initial data $\mathbf{z}^0 := \hat{\mathbf{z}}^0 - \tilde{\mathbf{z}}^0$ and $f := \hat{f} - \tilde{f}$. We try to find a control function $f \in L^2(\omega \times (0,T))$ such that the solution

$$\mathbf{z}(T; \mathbf{z}^0, f) = 0\,,$$

or equivalently, such that the state $\mathbf{z}^0$ can be transferred to 0 at the time T.

Definition 1.1. The PDE system (1.2) is said to be *null controllable*, if for any $T > 0$ and arbitrary initial data $\mathbf{z}^0 \in H$, there exists a control function $f \in L^2(\omega \times (0,T))$ such that the corresponding solution $\mathbf{z}(t; \mathbf{z}^0, f)$ to (1.2) satisfies

$$\mathbf{z}(T; \mathbf{z}^0, f) = 0\,. \tag{1.5}$$

Thus, the above problem of the controllability of trajectoires corresponds to prove the null controllability for system (1.2).

Our aim is: for each $T > 0$, we look for a control f which steers the solution $\mathbf{z}$ of (1.2) to zero, and such that its $L^2(\omega \times (0,T))$-measurement is minimal with respect to all such steering controls. This is to say to look for the solution of the minimum problem

$$\min_{f \in V} J(f), \qquad J(f) := \frac{1}{2}\|f\|^2_{L^2(\omega \times (0,T))}\,, \tag{1.6}$$

where

$$V := \left\{ f \in L^2(\omega \times (0,T)) : \text{ the solution } \mathbf{z} \text{ of (1.2) satisfies (1.5)} \right\}\,.$$

It is easy to verify that V is a convex, closed, non-empty subset of $L^2(\omega \times (0,T))$ and then the minimum problem (1.6) has a unique solution. Nevertheless, V is not a vector space and we cannot construct the optimality system related to the minimum problem (1.6). We choose to replace the cost functional $J(f)$ with the *penalized* functional

$$J_k(f) := \frac{1}{2}\int_0^{\mathrm{T}} \|f(t)\|^2_{L^2(\omega)} dt + \frac{k}{2}\|\mathbf{z}(T; \mathbf{z}^0, f) - \mathbf{z}_T\|^2_H\,, \tag{1.7}$$

with $k > 0$ "large", and to look for the solution of

$$\min_{f \in L^2(\omega \times (0,T))} J_k(f)\,. \tag{1.8}$$

In this way the constraint is penalized inside the functional and we look for the minimum on the vector space $L^2(\omega \times (0,T))$. This choice enables us to reformulate the minimum problem (1.8) in terms of an optimality system.

The question of the null controllability for thermoelastic plates has already received attention in the literature (see Sec. 1.2). In Ref. 8 the approximation, based on FEM, is introduced for thermoelastic systems with boundary control and point observation. No numerical results are known about the approximation of control problems for thermoelastic system.

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Our paper contains the numerical implementation of the theoretical result presented in Ref. 5, in particular we focus our attention on both the construction of the cost functional $J_k(f)$ (as defined in (1.7)) and numerical solution of the minimum problem (1.8) in order to choose the control f in two-dimensional geometries.

Being the state equation (1.2) linear and the cost functional quadratic, instead of solving directly a minimum problem on the control f (say $\min_f J_k(f)$), we solve the equivalent problem

$$\min_{\boldsymbol{\zeta}^T} J_k^*(\boldsymbol{\zeta}^T)\,, \tag{1.9}$$

obtained by applying Convex Duality Theory.[11,29,30] The dual problem (1.9), that can be viewed as an identification problem for the final data ($\boldsymbol{\zeta}^T \in H$) of a backward in time adjoint system,[23] is better suited to numerical calculations than the original one.

To solve the minimum problem (1.9) we extend the ideas used by Glowinski and Lions[16] in the context of linear diffusion equations to the thermoelastic system. We rewrite the minimum problem in a variational form and we solve it by the Conjugate Gradient (CG) method. At each CG-iteration, both a primal system (1.2) and an adjoint system to (1.2) have to be solved.

Spectral Element Methods[15] are used to approximate the solution in space variables, while classical finite difference schemes like Newmark and Crank-Nicolson[27] are used for time advancing. The fourth-order term in the first equation of system (1.2) is faced by a mixed approach, for which a new unknown $w := -\Delta u$ is introduced in the system.

In this paper we show several numerical results obtained through the approximations described above. We refer to a future work[14] for both the study of the discretized problem and the convergence of the approximate solution to the solution of (1.2).

The paper is organized as follows. In Sec. 2 we sketch the modelling procedure of a thin homogeneous thermoelastic plate subject to thermal deformations. The resulting model is derived in the framework of the well-established theory of heat flow due to Fourier and according to the standard approximation for the Kirchhoff plate.

Section 3 contains the formulation of the dual problem and the construction of the optimality systems. Finally, Secs. 4 and 5 are devoted to the numerical approximation of the control problem and numerical results, respectively.

1.2. *Literature*

Questions related to controllability of thermoelastic plates have attracted considerable attention in recent years.

In $(1.1)_1$ the plate component does not contain a rotational inertia term which otherwise gives hyperbolic characteristics. Then, underlying dynamics of thermoelastic model, system (1.1) is governed by analytic semigroups (cf. Refs. 1, 9 and

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21). Owing to smoothing effect associated to analyticity, the exact controllability for thermoelastic plates has been proved for large spaces of controls. Avalos[2] shows the exact controllability at any time $T > 0$ for thermoelastic plates, with and without rotational inertia,

$$\begin{cases} u_{tt} - \gamma\,\Delta u_{tt} + \Delta^2 u + \alpha\,\Delta\theta = f_1 & \text{in } Q \\ \theta_t - \Delta\theta + \sigma\,\theta - \alpha\,\Delta u_t = f_2 & \text{in } Q \\ u = 0,\ \dfrac{\partial u}{\partial\nu} = 0,\ \theta = 0 & \text{on } \Sigma \\ u(0) = u_0\,,\ u_t(0) = u_1\,,\ \theta(0) = \theta_0 & \text{on } \Omega \end{cases} \tag{1.10}$$

in the absence of control forces ($f_1 \equiv 0$), and by means a control $f_2 \in L^2(0,T;H^{-1}(\Omega))$ in the whole Ω. In the control space $L^2(0,T;H^{-1}(\Omega))$ this result is optimal.

De Teresa and Zuazua[10] study the thermoelastic plate system (1.10) in the presence of a control function $f_1 \in L^1(0,T;H^{-1}(\Omega))$, with $\operatorname{supp} f_1(\cdot,t) \subset \omega \subset \Omega$, in the absence of heat sources ($f_2 \equiv 0$), and with $\sigma \equiv 0$ and $\gamma \neq 0$. Clamped boundary conditions are imposed on u. By using both a decoupling result (see Ref. 17) for three-dimensional thermoelasticity, and a variational approach to controllability (see Ref. 12), and some observability inequalities for the system of thermoelastic plate, a result of *exact-approximate controllability* is obtained. In other words, by the geometric control conditions introduced in Ref. 4, they find sufficient conditions on control time T and control region ω such that for every initial and final data $(u_0, u_1, \theta_0), (v_0, v_1, \vartheta_0)$, belonging to the space of states where system (1.10) evolves, and for every $\varepsilon > 0$, there exists a control function f_1 such that the solution of (1.10) satisfies

$$u(T) = v_0\,, \quad u_t(T) = v_1\,, \quad \|\theta(T) - \vartheta_0\|_{L^2(\Omega)} \le \varepsilon\,.$$

Lasiecka and Triggiani[20] consider the controllability problem for the thermoelastic plate equation, without rotational inertia term, with hinged mechanical and Dirichlet thermal boundary conditions, under the influence of either mechanical or thermal control on the whole domain, namely

$$\begin{cases} u_{tt} + \mathcal{A}^2\,u - \mathcal{A}\,\theta = g_1 & \text{in } Q \\ \theta_t + \mathcal{A}\,\theta + \mathcal{A}\,u_t = g_2 & \text{in } Q \\ u = 0,\ \mathcal{A}\,u = 0,\ \theta = 0 & \text{on } \Sigma \\ u(0) = u_0\,,\ u_t(0) = u_1\,,\ \theta(0) = \theta_0 & \text{on } \Omega\,, \end{cases}$$

where $\mathcal{A}$ is a strictly positive, self-adjoint partial differential operator with compact resolvent, and either $(g_1, g_2) = (0, h)$ or $(g_1, g_2) = (k, 0)$, with $h, k \in L^2(Q)$ and $h, k \not\equiv 0$. With respect to result of Ref. 2, in this paper the set of controls is taken in the narrower space $L^2(Q)$ and the null controllability is proved for any $T > 0$. This result has been complemented by providing optimal blow-up estimates of norms of fast controls in Refs. 3 and 31.

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The case where $g_1 \equiv 0$ and the control function g_2 is such that $\operatorname{supp} g_2(\cdot, t) \subset \omega \subset \Omega$ has been tackled by Benabdallah and Naso in Ref. 5. By applying an iterative method and thanks to the observability estimates on the eigenfunctions of the Laplacian operator (see Ref. 22), the null controllability for system (1.1) is proved at any time $T > 0$ by $L^2(\omega \times (0,T))$-thermal control. In this proof both the analyticity property of semigroup associated to the thermoelastic system (see Ref. 21) and the commutative property of the operators, which comes from the hinged boundary conditions, are crucial.

2. Preliminary Results

2.1. *The plate model*

We consider a plate of uniform thickness d. When the plate is in equilibrium, we assume it occupies a fixed bounded domain $\mathcal{D} \subset \mathbb{R}^3$ placed in a reference frame $\mathbf{x} := (x_1, x_2, x_3)$. The plate has a middle surface midway between its faces in a region $\Omega \subset \mathbb{R}^2$ of the plane $x_3 = 0$. We suppose that the plate is hinged along its Lipschitz boundary $\partial\Omega$.

The material composing the plate is homogeneous and (elastically and thermally) *isotropic*, so that its stress–strain law is given by

$$\mathbf{T}(\mathbf{x}, t) = \mathbb{L}_0 \left[\boldsymbol{\varepsilon}(\mathbf{x}, t) - \alpha_0\, \theta(\mathbf{x}, t)\, \mathbf{I} \right] , \tag{2.1}$$

where the *elastic strain* $\boldsymbol{\varepsilon}$, the *stress* $\mathbf{T}$ are second-order tensors, $\mathbf{I}$ is the second-order identity tensor, and $\mathbb{L}_0$ is a fourth-order tensor. The last term in (2.1) represents the *thermal strain* and the positive constant α_0 is called the *coefficient of thermal expansion*. Moreover, $\theta := \Theta - \Theta_0$ denotes the *temperature variation* with respect to the reference value Θ_0. According to this constitutive equation, in Ref. 19 a mathematical model for a Kirchhoff thermoelastic plate is derived.

Let $\mathbf{q}\colon \Omega \times \mathbb{R} \to \mathbb{R}^3$ be the *mean heat flux vector* in the plate. Fourier law of the heat conduction for a thermally isotropic body in the approximation theory is written as

$$\mathbf{q}(\mathbf{x}, t) = -k_0\, \nabla\theta(\mathbf{x}, t)\,, \tag{2.2}$$

where the positive constant k_0 denotes the *coefficient of thermal conductivity.*

The usual energy balance equation is replaced by

$$\rho_0\, h(\mathbf{x}, t) = -\nabla \cdot \mathbf{q}(\mathbf{x}, t) + \rho_0\, r(\mathbf{x}, t)\,, \tag{2.3}$$

where h is the *thermal power*, which denotes the rate of heat absorption per unit of volume, $\rho_0 > 0$ is the *mass density in the reference initial configuration*, and r is the *external heat supply per unit of mass*. Neglecting any hereditary contribution to mechanical dissipation, h is described by the following linearized constitutive equation (see Ref. 13):

$$\rho_0\, h(\mathbf{x}, t) = \Theta_0\, \mathbf{B}\colon \boldsymbol{\varepsilon}_t(\mathbf{x}, t) + \rho_0\, c_v\, \theta_t(\mathbf{x}, t)\,, \tag{2.4}$$

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where $\mathbf{B}$ is a symmetric second-order tensor, $c_v > 0$ is the *specific heat* of the body and ":" represents the tensorial scalar product.

The motion equation for the bending component u, via a variational formulation, can be obtained by application of the Principle of Virtual Work. Finally, substituting (2.2) and (2.4) into (2.3), and paralleling the procedures of Ref. 19 in the framework of thermoelastic materials, we obtain system (1.1).

2.2. *Existence results*

We define the positive self-adjoint operator

$$\mathcal{A}\, h := -\Delta h \quad \text{with} \quad \mathcal{A} : D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)\,,$$

so that

$$\mathcal{A}^2\, h = \Delta^2 h \quad \text{with} \quad D(\mathcal{A}^2) = \left\{ h \in H^4(\Omega) :\ h|_{\partial\Omega} = 0,\ \Delta h|_{\partial\Omega} = 0 \right\}\,.$$

Theorem 2.1. *Let $T > 0$ be arbitrary and $f \in L^2(\omega \times (0,T))$. For any initial data $(u_0, u_1, \theta_0) \in H$, there exists a unique solution to problem* (1.1), *such that*

$$(u, u_t, \theta) \in L^2(0,T; H^3(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega))\,.$$

Proof. The proof follows the same guidelines given in Refs. 25 and 26, where different regularity assumptions are taken on both f and initial data. Here we report only the part of the proof to obtain *a priori* estimate on the energy.

Multiplying $(1.1)_1$ for u_t and $(1.1)_2$ for θ, integrating in Ω, we found

$$\frac{d}{dt}\mathcal{E}(t) = -\|(-\Delta)^{1/2}\theta(t)\|^2 + \langle f(t), \theta(t)\rangle \le -\left(1 - \frac{\varepsilon}{2}\right)\|(-\Delta)^{1/2}\theta(t)\|^2 + \frac{1}{2\varepsilon}\|f(t)\|^2\,,$$

where

$$\mathcal{E}(t) := \frac{1}{2}\left[\|(-\Delta)u(t)\|^2 + \|u_t(t)\|^2 + \|\theta(t)\|^2\right]\,.$$

We consider

$$\begin{aligned}\frac{d}{dt}\langle(-\Delta)u, u_t\rangle &= \langle(-\Delta)u, u_{tt}\rangle + \|(-\Delta)^{1/2}u_t\|^2\\ &= -\|(-\Delta)^{3/2}u\|^2 + \|(-\Delta)^{1/2}u_t\|^2 + \langle(-\Delta)^{1/2}\theta, (-\Delta)^{3/2}u\rangle\\ &\le -\frac{1}{2}\|(-\Delta)^{3/2}u\|^2 + \|(-\Delta)^{1/2}u_t\|^2 + \frac{1}{2}\|(-\Delta)^{1/2}\theta\|^2 \qquad (2.5)\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\langle\theta, u_t\rangle &= \langle\theta_t, u_t\rangle + \langle\theta, u_{tt}\rangle\\ &= -\|(-\Delta)^{1/2}u_t\|^2 + \|(-\Delta)^{1/2}\theta\|^2 - \langle(-\Delta)\theta, u_t\rangle - \langle\theta, (-\Delta)^2u\rangle + \langle f, u_t\rangle\\ &\le -\frac{1}{2}\|(-\Delta)^{1/2}u_t\|^2 + \frac{\varepsilon}{2}\|(-\Delta)^{3/2}u\|^2 + \left(\frac{3}{2} + \frac{1}{2\varepsilon}\right)\|(-\Delta)^{1/2}\theta\|^2 + \langle f, u_t\rangle\end{aligned}$$

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$$\leq -\frac{1}{4}\|(-\Delta)^{1/2}u_t\|^2 + \frac{\varepsilon}{2}\|(-\Delta)^{3/2}u\|^2 + \left(\frac{3}{2} + \frac{1}{2\varepsilon}\right)\|(-\Delta)^{1/2}\theta\|^2$$

$$+ \frac{1}{2\delta}\|f\|^2\,, \tag{2.6}$$

with $\delta > 0$. Let us introduce the functional

$$\mathcal{L}(t) := N\,\mathcal{E}(t) + \langle u_t, (-\Delta)u\rangle + M\langle\theta, u_t\rangle$$

with $N, M > 0$. We have

$$\frac{d}{dt}\mathcal{L} \leq N\left[-\frac{1}{2}\|(-\Delta)^{1/2}\theta\|^2 + \frac{1}{2}\|f\|^2\right] - \frac{1}{2}\|(-\Delta)^{3/2}u\|^2 + \|(-\Delta)^{1/2}u_t\|^2$$

$$+ \frac{1}{2}\|(-\Delta)^{1/2}\theta\|^2 - \frac{M}{4}\|(-\Delta)^{1/2}u_t\|^2 + \frac{\bar{\varepsilon}}{2}\|(-\Delta)^{3/2}u\|^2$$

$$+ M\left(\frac{3}{2} + \frac{M}{2\bar{\varepsilon}}\right)\|(-\Delta)^{1/2}\theta\|^2 + \frac{M}{2\delta}\|f\|^2$$

$$\leq -\left(\frac{1}{2} - \frac{\bar{\varepsilon}}{2}\right)\|(-\Delta)^{3/2}u\|^2 - \left(\frac{M}{4} - 1\right)\|(-\Delta)^{1/2}u_t\|^2$$

$$- \left[\frac{N}{2} - \frac{1}{2} - \frac{1}{2}\left(\frac{M^2}{\bar{\varepsilon}} + 3\right)\right]\|(-\Delta)^{1/2}\theta\|^2 + \left(\frac{N}{2} + \frac{M}{2\delta}\right)\|f\|^2\,.$$

We set

$$\mathcal{E}_1(t) := \frac{1}{2}\left[\|(-\Delta)^{3/2}u(t)\|^2 + \|(-\Delta)^{1/2}u_t(t)\|^2 + \|(-\Delta)^{1/2}\theta(t)\|^2\right]\,,$$

and choose

$$\bar{\varepsilon} < 1, \quad M > 4, \quad N > 4 + \frac{M^2}{\bar{\varepsilon}}\,.$$

Then, there exists $c_0 > 0$ such that

$$\frac{d}{dt}\mathcal{L} \leq -c_0\,\mathcal{E}_1 + \left(\frac{N}{2} + \frac{M}{2\delta}\right)\|f\|^2\,.$$

If we choose N such that $N > \max(1, \sqrt{M^2+1})$ there exist two positive constants c_1 and c_2 such that

$$c_1\,\mathcal{E}(t) \leq \mathcal{L}(t) \leq c_2\,\mathcal{E}(t)\,.$$

Thus, we obtain

$$\mathcal{E}(t) + C\int_0^t \mathcal{E}_1(\tau)\,d\tau \leq \mathcal{E}(0) + C\int_0^t \|f(\tau)\|^2\,d\tau$$

and our conclusion follows. □

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3. Formulation of the Controllability Problem

Let us introduce the operator $L_t : L^2(\omega \times (0,T)) \to H$ defined by

$$L_t\, f := \int_0^t e^{A(t-s)}\, B\, f(s)\, ds\,, \qquad \forall\; t \in [0,T]\,. \tag{3.1}$$

Suppose that the null controllability property, as defined in Definition 1.1, holds true for system (1.2).

Remark 3.1. In terms of the previous notation, the null controllability property is equivalent to the statement that

$$\operatorname{Im} e^{A\,T} \subset \operatorname{Im} L_T\,.$$

This containment is in turn equivalent to establishing the *observability inequality* (see for instance Ref. 32):

$$\exists\; C_T > 0 : \quad \|e^{A^*T}\boldsymbol{\zeta}^T\|_H^2 \le C_T \|L_T^*\,\boldsymbol{\zeta}^T\|^2_{L^2(\omega\times(0,T))}, \quad \forall\; \boldsymbol{\zeta}^T \in H\,, \tag{3.2}$$

where

$$\begin{aligned} L_T^* :\; H &\longrightarrow L^2(\omega \times (0,T)) \\ \boldsymbol{\zeta}^T &\longmapsto B^*\; e^{A^*(T-\cdot)}\; \boldsymbol{\zeta}^T\,, \end{aligned} \tag{3.3}$$

and A^* and B^* are the adjoint operators of A and B, respectively.

Remark 3.2. It is proved that the observability constant $C_T = \mathcal{O}(T^{-5})$ is optimal (see Refs. 3 and 31).

3.1. *The dual problem and the optimality system*

The minimum problem (1.8) is solved by the duality theorem of Fenchel–Rockafeller.[11,29,30]

The idea is to construct the "*optimality system*" equivalent to the minimum problem (1.8), whose solution immediately gives the optimal control f_k.

To this aim we introduce the functionals $F : L^2(\omega \times (0,T)) \to \mathbb{R}$ and $G_k : H \to \mathbb{R}$, such that

$$F(f) := \frac{1}{2}\,\|f\|^2_{L^2(\omega\times(0,T))}\,, \tag{3.4}$$

$$G_k(L_T\,f) := \frac{k}{2}\|\mathbf{z}(T;\mathbf{z}^0,f) - \mathbf{z}_T\|_H^2 = \frac{k}{2}\|L_T\,f + e^{A\,T}\,\mathbf{z}^0 - \mathbf{z}_T\|_H^2\,. \tag{3.5}$$

We set

$$\begin{aligned} J_k(f_k) &:= \inf_{f\in L^2(\omega\times(0,T))} J_k(f) = \inf_{f\in L^2(\omega\times(0,T))} [F(f) + G_k(L_T\,f)] \\ J_k^*(\boldsymbol{\zeta}_k^T) &:= \inf_{\boldsymbol{\zeta}^T\in H} J_k^*(\boldsymbol{\zeta}^T) = \inf_{\boldsymbol{\zeta}^T\in H}\left[F^*(L_T^*\,\boldsymbol{\zeta}^T) + G_k^*(-\boldsymbol{\zeta}^T)\right]\,, \end{aligned} \tag{3.6}$$

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where F^* and G_k^* are the conjugate functions of F and G_k, respectively, and L_T^* is defined in (3.3). We denote by $\boldsymbol{\zeta}(t) = e^{(T-t)A^*}\,\boldsymbol{\zeta}^T$ the solution of the following adjoint system with respect to (1.1):

$$\begin{cases} \boldsymbol{\zeta}_t = -A^*\,\boldsymbol{\zeta} & \text{in } Q \\ \boldsymbol{\zeta}(T) = \boldsymbol{\zeta}^T & \text{in } \Omega \\ \zeta_1 = 0,\ \Delta\zeta_1 = 0,\ \zeta_3 = 0 & \text{on } \Sigma, \end{cases} \tag{3.7}$$

where

$$A^* = \begin{bmatrix} 0 & -I & 0 \\ \Delta^2 & 0 & \Delta \\ 0 & -\Delta & \Delta \end{bmatrix}$$

and $\boldsymbol{\zeta}^T := [\zeta_1^T,\ \zeta_2^T,\ \zeta_3^T]^\top \in H$, $\boldsymbol{\zeta} := [\zeta_1,\ \zeta_2,\ \zeta_3]^\top \in D(A^*)$. By simple calculation $D(A^*) = D(A)$ holds.

Moreover, by putting

$$\boldsymbol{\xi}(x,t) = \boldsymbol{\zeta}(x, T-t)\,, \tag{3.8}$$

system (3.7) is equivalent to find $\boldsymbol{\xi} := [\xi_1,\ \xi_2,\ \xi_3]^\top \in D(A)$:

$$\begin{cases} \boldsymbol{\xi}_t = A^*\,\boldsymbol{\xi} & \text{in } Q \\ \boldsymbol{\xi}(0) = [\zeta_1^T,\ -\zeta_2^T,\ \zeta_3^T]^\top & \text{in } \Omega \\ \xi_1 = 0,\ \Delta\xi_1 = 0,\ \xi_3 = 0 & \text{on } \Sigma, \end{cases} \tag{3.9}$$

or, again, to find $[\varphi,\ \varphi_t,\ \psi]^\top \in D(A)$:

$$\begin{cases} \varphi_{tt} + \Delta^2\varphi + \Delta\psi = 0 & \text{in } Q \\ \psi_t - \Delta\psi - \Delta\varphi_t = 0 & \text{in } Q \\ \varphi = 0\,,\ \Delta\varphi = 0\,,\ \psi = 0 & \text{on } \Sigma \\ \varphi(0) = \zeta_1^T\,,\ \varphi_t(0) = -\zeta_2^T\,,\ \psi(0) = \zeta_3^T & \text{in } \Omega\,. \end{cases} \tag{3.10}$$

By considering the homogeneous problem (3.10), with initial data $[\varphi_0,\ \varphi_1,\ \psi_0]^\top$ in H, inequality (3.2) is equivalent to (see Refs. 2, 5 and 20)

$$\exists\ C_T > 0: \quad \|[\varphi(T),\ \varphi_t(T),\ \psi(T)]^\top\|_H^2 \le C_T\|\psi\|^2_{L^2(\omega\times(0,T))} \tag{3.11}$$

for any solution $[\varphi(t),\ \varphi_t(t),\ \psi(t)]^\top$ of problem (3.10).

By application of the duality theorem of Fenchel–Rockafeller, we obtain the so-called conditions of optimality

$$J_k(f_k) = -J_k^*(\boldsymbol{\zeta}_k^T), \quad L_T^*\boldsymbol{\zeta}_k^T = F'(f_k), \quad -\boldsymbol{\zeta}_k^T = G_k'(L_T\, f_k)\,. \tag{3.12}$$

Let $\tilde{\mathbf{h}} := \hat{\mathbf{h}} + \mathbf{h}^0$ and $\hat{\mathbf{h}}$, $\mathbf{h}^0$ be in H. For any $\mathbf{h} \in H$, the conjugate function G_k^* of G_k is

$$\begin{aligned} G_k^*(\mathbf{h}) &= \sup_{\hat{\mathbf{h}}\in H}\left[\langle \mathbf{h}, \hat{\mathbf{h}}\rangle_H - \frac{k}{2}\|\hat{\mathbf{h}} + \mathbf{h}^0\|_H^2\right] \\ &= -\langle \mathbf{h}, \mathbf{h}^0\rangle_H + \sup_{\tilde{\mathbf{h}}\in H}\underbrace{\left[\langle \mathbf{h}, \tilde{\mathbf{h}}\rangle_H - \frac{k}{2}\|\tilde{\mathbf{h}}\|_H^2\right]}_{:=\mathcal{H}(\tilde{\mathbf{h}})} = -\langle \mathbf{h}, \mathbf{h}^0\rangle_H + \frac{1}{2k}\,\|\mathbf{h}\|_H^2 \end{aligned}$$

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since $\mathcal{H}'(\tilde{\mathbf{h}}) = 0$ when $\widetilde{\mathbf{h}} = \dfrac{1}{k}\,\mathbf{h}$. Considering $\hat{\mathbf{h}} = L_T\, f_k$ and $\mathbf{h}^0 = e^{A\,T}\,\mathbf{z}^0 - \mathbf{z}_T$, we obtain

$$G_k^*(\mathbf{h}) = -\langle \mathbf{h}, e^{A\,T}\,\mathbf{z}^0 - \mathbf{z}_T\rangle_H + \frac{1}{2k}\,\|\mathbf{h}\|_H^2\,. \tag{3.13}$$

By integrating by parts, the functional $J_k^*(\boldsymbol{\zeta}^T)$ becomes

$$\begin{aligned} J_k^*(\boldsymbol{\zeta}^T) :&= F^*(L_T^*\,\boldsymbol{\zeta}^T) + G_k^*(-\boldsymbol{\zeta}^T) \\ &= \frac{1}{2}\int_0^T \|B^*\,\boldsymbol{\zeta}\|_{L^2(\omega)}^2\,dt + \langle \boldsymbol{\zeta}(0), \mathbf{z}^0\rangle_H - \langle \boldsymbol{\zeta}^T, \mathbf{z}_T\rangle_H + \frac{1}{2k}\,\|\boldsymbol{\zeta}^T\|_H^2\,. \end{aligned}$$

Recalling definition (1.4) of the operator B, it holds

$$J_k^*(\boldsymbol{\zeta}^T) = \frac{1}{2}\int_0^T \|\zeta_3\|_{L^2(\omega)}^2\,dt + \langle \boldsymbol{\zeta}(0), \mathbf{z}^0\rangle_H - \langle \boldsymbol{\zeta}^T, \mathbf{z}_T\rangle_H + \frac{1}{2k}\,\|\boldsymbol{\zeta}^T\|_H^2\,.$$

Remark 3.3. Since we are interested to study the null controllability problem, the term $\mathbf{z}_T$ is equal to zero and the dual functional becomes

$$J_k^*(\boldsymbol{\zeta}^T) = \frac{1}{2}\int_0^T \|\zeta_3\|_{L^2(\omega)}^2\,dt + \langle \boldsymbol{\zeta}(0), \mathbf{z}^0\rangle_H + \frac{1}{2k}\,\|\boldsymbol{\zeta}^T\|_H^2\,. \tag{3.14}$$

From observability inequality (3.11), rewritten as

$$\|\boldsymbol{\zeta}(0)\|_H^2 \le C_T \int_0^T \|\zeta_3\|_{L^2(\omega)}^2\,dt\,, \tag{3.15}$$

and Young inequality, we find that for any $\delta > 0$

$$J_k^*(\boldsymbol{\zeta}^T) \ge \left(\frac{1}{2C_T} - \delta\right)\|\boldsymbol{\zeta}(0)\|_H^2 - C_\delta\|\mathbf{z}^0\|_H^2 + \frac{1}{2k}\,\|\boldsymbol{\zeta}^T\|_H^2, \quad \forall\ k > 0\,, \tag{3.16}$$

so that the functional $J_k^*(\boldsymbol{\zeta}^T)$ is coercive. Moreover, it is convex and continuous, then the minimum problem $\min_{\boldsymbol{\zeta}^T} J_k^*(\boldsymbol{\zeta}^T)$ admits a unique solution, denoted by $\boldsymbol{\zeta}_k^T$. By both conditions of optimality (3.12) and definitions (3.4)–(3.5), we get

$$\begin{gathered} L_T^*\boldsymbol{\zeta}_k^T = F'(f_k) = f_k \\ -\boldsymbol{\zeta}_k^T = G_k'(L_T\,f_k) = k\left(L_T\,f_k + e^{A\,T}\,\mathbf{z}^0 - \mathbf{z}_T\right) = k[\mathbf{z}(T;\mathbf{z}^0, f_k) - \mathbf{z}_T]\,, \end{gathered}$$

and recalling that $L_T^*\boldsymbol{\zeta}_k^T = \zeta_{k3}$ and $\mathbf{z}_T = \mathbf{0}$, we have

$$f_k = \zeta_{k3} \quad \text{and} \quad \boldsymbol{\zeta}_k^T = -k\,\mathbf{z}(T;\mathbf{z}^0, f_k)\,. \tag{3.17}$$

By setting $\mathbf{z}_k(T) := \mathbf{z}(T;\mathbf{z}^0, f_k)$, from (3.17) we obtain

$$\|\mathbf{z}_k(T)\|_H = \frac{1}{k}\|\boldsymbol{\zeta}_k^T\|_H\,. \tag{3.18}$$

By considering (3.7) with $\boldsymbol{\zeta}^T \equiv \mathbf{0}$, we find that $\boldsymbol{\zeta}(t) \equiv \mathbf{0}$, for any $t \in [0, T]$. This implies that (3.14) evaluated for $\boldsymbol{\zeta}^T \equiv \mathbf{0}$ becomes $J_k^*(\mathbf{0}) = 0$, and recalling (3.6) we obtain

$$J_k^*(\boldsymbol{\zeta}_k^T) \le J_k^*(\mathbf{0}) = 0\,, \quad \text{for any } k > 0\,,$$

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and, from the following inequalities

$$\begin{aligned}\int_0^{\mathrm{T}}\int_\omega |\zeta_{3k}(x,t)|^2\,dx\,dt &\le -2\langle \boldsymbol{\zeta}_k(0),\mathbf{z}^0\rangle_H - \frac{1}{k}\,\|\boldsymbol{\zeta}_k^T\|_H^2\\ &\le 2\|\boldsymbol{\zeta}_k(0)\|_H\,\|\mathbf{z}^0\|_H\\ &\le 2\left[C_T\int_0^{\mathrm{T}}\int_\omega |\zeta_{3k}(x,t)|^2\,dx\,dt\right]^{1/2}\|\mathbf{z}^0\|_H\end{aligned}$$

we have

$$\|\zeta_{3k}\|_{L^2(\omega\times(0,T))} = \left[\int_0^{\mathrm{T}}\int_\omega |\zeta_{3k}(x,t)|^2\,dx\,dt\right]^{1/2} \le 2\,C_T^{1/2}\,\|\mathbf{z}^0\|_H\,. \tag{3.19}$$

By (3.19), ζ_{3k} is bounded in $L^2(\omega\times(0,T))$, and there exists a subsequence (ζ_{3k_n}) such that

$$\zeta_{3k_n} \rightharpoonup f \quad \text{in } L^2(\omega\times(0,T))\,.$$

Then, f satisfies the same previous estimate (3.19) and it is chosen as the control function. We observe that $\|\mathbf{z}_{k_n}(t)\|_H$ is bounded in $(0,T)$. In fact

$$\begin{aligned}\|\mathbf{z}_{k_n}(t)\|_H^2 &\le \|e^{At}\,\mathbf{z}^0\|_H^2 + \left\|\int_0^t e^{A(t-\tau)}\,B\,\zeta_{3k_n}\,d\tau\right\|_H^2\\ &\le (1+4C_T)\,\|\mathbf{z}^0\|_H^2\,.\end{aligned}$$

Thus, we can extract a subsequence $\mathbf{z}_{k_{n_m}}$ such that

$$\mathbf{z}_{k_{n_m}} \rightharpoonup \mathbf{z} \quad \text{in } H, \quad \text{a.e. } t\in(0,T)\,. \tag{3.20}$$

Recalling that

$$\|\mathbf{z}(T)\|_H \le \liminf_{k\to\infty}\|\mathbf{z}_k(T)\|_H = 0$$

we find $\mathbf{z}(T)=\mathbf{0}$.

Finally, given k, the *optimality system* for problem (1.8) follows from (3.17) and it reads

$$\begin{cases}\mathbf{z}_t = A\,\mathbf{z} + B\,\zeta_3 & \text{in } \Omega\times(0,T)\\ \mathbf{z}(0)=\mathbf{z}^0 & \text{in } \Omega\\ z_1=0,\ \Delta z_1=0,\ z_3=0 & \text{on } \Sigma\end{cases} \tag{3.21}$$

$$\begin{cases}\boldsymbol{\zeta}_t = -A^*\,\boldsymbol{\zeta} & \text{in } \Omega\times(0,T)\\ \boldsymbol{\zeta}(T) = -k\,\mathbf{z}(T;\mathbf{z}^0,\zeta_3) & \text{in } \Omega,\\ \zeta_1=0,\ \Delta\zeta_1=0,\ \zeta_3=0 & \text{on } \Sigma\,.\end{cases} \tag{3.22}$$

The optimal control f_k is given by $\chi_\omega\zeta_3$ where ζ_3 is the third component of the vector function $\boldsymbol{\zeta}$ obtained by solving (3.21)–(3.22).

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The next step is to solve the optimality system. We introduce the operator $\Lambda : H \to H$ such that

$$\Lambda \boldsymbol{\zeta}^T := L_T L_T^* \boldsymbol{\zeta}^T = \mathbf{z}(T; \mathbf{z}^0, \zeta_3) - e^{AT}\mathbf{z}^0 , \tag{3.23}$$

and the identity operator I, so that the solution of the optimality system (3.21)–(3.22) satisfies the functional equation

$$(k^{-1}I + \Lambda)\boldsymbol{\zeta}^T = -e^{AT}\mathbf{z}^0 . \tag{3.24}$$

Problem (3.24) admits the following variational formulation:

$$\text{find } \boldsymbol{\zeta}^T \in H : \qquad \langle (k^{-1}I + \Lambda)\boldsymbol{\zeta}^T, \boldsymbol{\eta}^T\rangle_H = \langle -e^{AT}\mathbf{z}^0, \boldsymbol{\eta}^T\rangle_H \qquad \forall\, \boldsymbol{\eta}^T \in H . \tag{3.25}$$

By defining the bilinear form $a_k : H \times H \to \mathbb{R}$:

$$a_k(\boldsymbol{\zeta}^T, \boldsymbol{\eta}^T) := \langle \Lambda\boldsymbol{\zeta}^T, \boldsymbol{\eta}^T\rangle_H + k^{-1}\langle \boldsymbol{\zeta}^T, \boldsymbol{\eta}^T\rangle_H \tag{3.26}$$

and the linear functional $L : H \to \mathbb{R}$:

$$L(\boldsymbol{\eta}^T) := \langle -e^{AT}\mathbf{z}^0, \boldsymbol{\eta}^T\rangle_H , \tag{3.27}$$

problem (3.25) reads:

$$\text{find } \boldsymbol{\zeta}^T \in H : \qquad a_k(\boldsymbol{\zeta}^T, \boldsymbol{\eta}^T) = L(\boldsymbol{\eta}^T) \qquad \forall\, \boldsymbol{\eta}^T \in H . \tag{3.28}$$

We can apply the Lax–Milgram lemma to problem (3.28) (the space H is a Hilbert space, the bilinear form a_k is continuous and coercive, the linear functional L is continuous, the operator Λ is self-adjoint and nonnegative definite, cf. for instance Ref. 32) and conclude that it has a unique solution $\boldsymbol{\zeta}^T$. Moreover, there exists a positive constant C such that:

$$\|\boldsymbol{\zeta}^T\|_H \le C\|e^{AT}\mathbf{z}^0\|_H . \tag{3.29}$$

Remark 3.4. By $\langle \Lambda\boldsymbol{\zeta}^T, \boldsymbol{\zeta}^T\rangle_H = \|\zeta_3\|_{L^2(\omega\times(0,T))}$ and by both (3.13) and (3.14), it holds

$$J_k^*(\boldsymbol{\zeta}^T) = \frac{1}{2}\langle \Lambda\boldsymbol{\zeta}^T, \boldsymbol{\zeta}^T\rangle_H + \langle \boldsymbol{\zeta}^T, e^{AT}\mathbf{z}^0\rangle_H + \frac{1}{2k}\|\boldsymbol{\zeta}^T\|_H . \tag{3.30}$$

We recognize that minimizing (3.30) is equivalent to solve the variational equation (3.28).

Now we can summarize the steps to find the solution and the control for problem (1.2) in the following way:

(1) solve problem (3.28) to find $\boldsymbol{\zeta}^T$ (this is equivalent to solve the optimality system (3.21)–(3.22))
(2) solve the adjoint system (3.7) to find $\boldsymbol{\zeta}$
(3) set $f = \zeta_3$
(4) solve the primal system (1.2) to find $\mathbf{z}$.

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Remark 3.5. From the computational point of view the most heavy step is the first. Problem (3.28) is solved by the Conjugate Gradient method[27], an iterative method that, under suitable assumptions on the bilinear form a_k, converges to the solution in a finite number of iterations and it requires to solve a primal and an adjoint system at each iteration.

Remark 3.6. In view of (3.8), for given initial data and external forces, both problem (3.7) and (1.2) are of the same type. Their numerical approximation constitutes the computational kernel of the whole minimization procedure.

The following section is devoted to the numerical approximation of problem (3.26)–(3.28) and to the numerical approximation of a problem like (1.1) with a given right-hand side f.

4. Numerical Approximation

In this section we briefly describe the numerical methods used to discretize problems (3.25), (3.7) and (1.2).

Given an initial guess $(\boldsymbol{\zeta}^T)^{(0)} \in H$, the CG algorithm iteratively constructs a sequence $(\boldsymbol{\zeta}^T)^{(n)} \in H$, that converges, for $n = 1, 2, \ldots$, to the solution $\boldsymbol{\zeta}^T$ of (3.26)–(3.28). The most expensive step of a CG-iteration is the evaluation of a function $\mathbf{q} \in H$ such that

$$\langle \mathbf{q}, \boldsymbol{\eta}^T \rangle_H = a_k(\mathbf{p}, \boldsymbol{\eta}^T) \qquad \forall\, \boldsymbol{\eta}^T \in H\,, \tag{4.1}$$

given $\mathbf{p} \in H$.

This is equivalent to:

- given $\mathbf{p} \in H$, compute the solution $\boldsymbol{\sigma} = [\sigma_1,\ \sigma_2,\ \sigma_3]^\top$ of

$$\begin{cases} \boldsymbol{\sigma}_t = -A^* \boldsymbol{\sigma} & \text{in } \Omega \times (0,T) \\ \boldsymbol{\sigma}(T) = \mathbf{p} & \text{in } \Omega\,, \\ \sigma_1 = 0,\ \Delta\sigma_1 = 0,\ \sigma_3 = 0 & \text{on } \Sigma \end{cases} \tag{4.2}$$

- extract the third component σ_3 of $\boldsymbol{\sigma}$
- compute the solution $\mathbf{s}$ of

$$\begin{cases} \mathbf{s}_t = A\,\mathbf{s} + B\,\sigma_3 & \text{in } \Omega \times (0,T) \\ \mathbf{s}(0) = \mathbf{s}^0 & \text{in } \Omega \\ s_1 = 0,\ \Delta s_1 = 0,\ s_3 = 0 & \text{on } \Sigma \end{cases} \tag{4.3}$$

- set $\mathbf{q} = \mathbf{s}(T) + \dfrac{1}{k}\mathbf{p}$.

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Then, the CG algorithm to solve (3.28) reads:

$$
\begin{aligned}
&\text{find } \mathbf{q}: \quad \langle \mathbf{q}, \boldsymbol{\eta}^T\rangle_H = a_k((\boldsymbol{\zeta}^T)^{(0)}, \boldsymbol{\eta}^T) \qquad \forall\, \boldsymbol{\eta}^T \in H\\
&\text{set } \mathbf{r}^{(0)} = e^{-AT}\mathbf{z}^0 - \mathbf{q}\\
&\text{set } \mathbf{p}^{(0)} = \mathbf{r}^{(0)}\\
&\text{for } n \ge 0\\
&\qquad \text{find } \mathbf{q}^{(n)}: \quad \langle \mathbf{q}^{(n)}, \boldsymbol{\eta}^T\rangle_H = a_k(\mathbf{p}^{(n)}, \boldsymbol{\eta}^T) \qquad \forall\, \boldsymbol{\eta}^T \in H\\
&\qquad \alpha_n = \langle \mathbf{r}^{(n)}, \mathbf{r}^{(n)}\rangle_H / \langle \mathbf{q}^{(n)}, \mathbf{p}^{(n)}\rangle_H\\
&\qquad (\boldsymbol{\zeta}^T)^{(n+1)} = (\boldsymbol{\zeta}^T)^{(n)} + \alpha_n \mathbf{p}^{(n)}\\
&\qquad \mathbf{r}^{(n+1)} = \mathbf{r}^{(n)} - \alpha_n \mathbf{q}^{(n)}\\
&\qquad \beta_n = \langle \mathbf{r}^{(n+1)}, \mathbf{r}^{(n+1)}\rangle_H / \langle \mathbf{r}^{(n)}, \mathbf{r}^{(n)}\rangle_H\\
&\qquad \mathbf{p}^{(n+1)} = \mathbf{r}^{(n+1)} + \beta_n \mathbf{p}^{(n)}
\end{aligned}
\tag{4.4}
$$

Our aim now is to solve numerically both systems (4.2) and (4.3). We note that system (4.2) is like (3.7) while system (4.3) is like (1.2), and in view of Remark 3.6 they can be viewed as particular cases of problem (1.1), so that we focus our attention on the approximation of thermoelastic system (1.1) with f given.

Remark 4.1. In the next sections we describe the approximation used, but we refer to a work in progess[14] for a detailed analysis of the convergence of the numerical solution to the continuous one.

4.1. *Approximation of the thermoelastic system*

We introduce the following weak formulation of problem (1.1) for $f \in L^2(\omega \times (0,T))$ and $[u_0,\ u_1,\ \theta_0]^\top \in H$ given.

For any $t \in (0,T]$ find $\mathbf{z}(t) = [u(t),\ u_t(t),\ \theta(t)]^\top$ such that

$$
\begin{cases}
\dfrac{d^2}{dt^2}(u(t),\varphi) + (\Delta u(t), \Delta\varphi) - (\nabla\theta(t), \nabla\varphi) = 0 & \forall\, \varphi \in H^2(\Omega)\cap H_0^1(\Omega)\\
\dfrac{d}{dt}(\theta(t),\psi) + (\nabla\theta(t), \nabla\psi) + (\nabla u_t(t), \nabla\psi) = (\chi_\omega f, \psi) & \forall\, \psi \in H_0^1(\Omega)\\
u(0) = u_0,\ v(0) = u_1,\ \theta(0) = \theta_0 & \text{in } \Omega,
\end{cases}
\tag{4.5}
$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product. By Theorem 2.1, problem (4.5) is well-posed.

The approximation of the term $\Delta^2 u$ in (1.1) by variational numerical methods, such as finite elements or spectral elements, should require C^1-continuity across the interfaces between the elements, thus the use of Hermite's elements, which are cumbersome to implement.

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A classical alternative consists of using a mixed formulation for problem (4.5) in which we introduce a new unknown $w = -\Delta u$. Under the assumption of Theorem 2.1, problem (4.5) reads: for any $t \in (0, T]$ find $u(t),\ w(t),\ v(t) = u_t(t),\ \theta(t)$ such that

$$\begin{cases} (\nabla u(t), \nabla \varphi_1) - (w(t), \varphi_1) = 0 & \forall\ \varphi_1 \in H_0^1(\Omega) \\ \dfrac{d^2}{dt^2}(u(t), \varphi_2) + (\nabla w(t), \nabla \varphi_2) - (\nabla \theta(t), \nabla \varphi_2) = 0 & \forall\ \varphi_2 \in H_0^1(\Omega) \\ \dfrac{d}{dt}(\theta(t), \varphi_3) + (\nabla \theta(t), \nabla \varphi_3) + (\nabla v(t), \nabla \varphi_3) = (\chi_\omega f, \varphi_3) & \forall\ \varphi_3 \in H_0^1(\Omega) \\ u(0) = u_0,\ v(0) = u_1,\ \theta(0) = \theta_0 & \text{in } \Omega \\ u = 0,\ w = 0,\ \theta = 0 & \text{on } \Sigma. \end{cases} \tag{4.6}$$

Problem (4.6) has a unique solution. As a matter of fact, integrating Eq. $(4.6)_1$ by parts in the space variable, we get

$$(w(t), \varphi_1) + (\Delta u(t), \varphi_1) = 0 \qquad \forall\ \varphi_1 \in H_0^1(\Omega)\ \ \forall\ t \in (0, T]\,. \tag{4.7}$$

Hence $w(t) = -\Delta u(t)\ \forall\ t \in (0, T]$. This result joined to equations $(4.6)_{2-5}$ leads to system (4.5), which has a unique solution satisfying $u \in L^2(0, T; H^3(\Omega) \cap H_0^1(\Omega))$. Then $w(t) = -\Delta u(t)$ is also unique and belongs to $L^2(0, T; H_0^1(\Omega))$.

A first step to the approximation of problem (4.6) entails the discretization of the space variable only. This leads to a system of ordinary differential equations whose solution $[u_{\mathcal{H}}(t),\ w_{\mathcal{H}}(t),\ v_{\mathcal{H}}(t),\ \theta_{\mathcal{H}}(t)]^\top$ is an approximation of the exact solution for each $t \in (0, T]$.

The generalized Galerkin approach is followed to reformulate problem (4.6) in finite-dimensional spaces. This method is obtained from a Galerkin method in which each integral is replaced by suitable quadrature formulas. Spectral Element Methods are employed to choose finite-dimensional spaces, quadrature formulas and derivatives discretization.

4.2. *Spectral element approximation of the space variable*

In order to discretize space derivatives we consider the Spectral Element Methods.[24,15] They are among the most commonly used methods for the approximation of partial differential equations which join the high accuracy of Spectral Methods[6,7] with the great versatility of Finite Element Methods. Historically, spectral methods have been associated with Fourier expansion and they have been applied to approximate periodic functions. However, nowadays they are used indifferently for periodic as well as general boundary-value problems. For the latter, algebraic polynomial expansions (especially Chebyshev's and Legendre's) are used in lieu of Fourier trigonometric polynomials. We introduce a conformal, regular and

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quasi-uniform (see, e.g., Ref. 18) partition $\mathcal{T}_h$ of Ω in N_e quadrilaterals T_k such that

$$\bar{\Omega} = \bigcup_{k=1}^{N_e} \bar{T}_k \,, \tag{4.8}$$

with

$$h = \max_{T_k \in \mathcal{T}_h} h_k, \qquad h_k = \mathrm{diam}(T_k), \quad k = 1, \ldots, N_e \,. \tag{4.9}$$

Let $\mathbb{Q}_N(T_k)$ be the set of algebraic polynomials, defined on T_k, of degree less than or equal to N in each direction, and set

$$\mathbb{Q}_{\mathcal{H}}(\Omega) = \{v \in \mathcal{C}^0(\bar{\Omega}) : \ v|_{T_k} \in \mathbb{Q}_N(T_k), \ \forall \ T_k \in \mathcal{T}_h\} \,. \tag{4.10}$$

Given $u_{\mathcal{H}}, \ v_{\mathcal{H}} \in \mathbb{Q}_{\mathcal{H}}(\Omega)$, we set

$$(u_{\mathcal{H}}, v_{\mathcal{H}})_{\mathcal{H},\Omega} = \sum_{k=1}^{N_e} (u_{N,k}, v_{N,k})_{N,T_k} \,, \tag{4.11}$$

where $u_{N,k} = u_{\mathcal{H}}|_{T_k}$, $v_{N,k} = v_{\mathcal{H}}|_{T_k}$ and where $(\cdot,\cdot)_{N,T_k}$ denotes the discrete inner product in $L^2(T_k)$, based on the Gauss–Lobatto Legendre (GLL) quadrature formulas.[27]

From now on, the index $\mathcal{H}$ characterizes the spectral element discretization we are considering; it stands for the couple $\mathcal{H} = (h, N)$, i.e. the mesh size and the polynomial degree on each element T_k, while $N_{\mathcal{H}}$ denotes the total number of grid points in Ω.

We note that when $N = 1$, the Spectral Element Methods coincides with Finite Element Method $\mathbb{Q}_1$ with lumped mass matrix.

We define the finite-dimensional spectral element space:

$$V_{\mathcal{H}} = H_0^1(\Omega) \cap \mathbb{Q}_{\mathcal{H}}(\Omega) \tag{4.12}$$

and we look for the finite-dimensional solution $[u_{\mathcal{H}}, \ w_{\mathcal{H}}, \ v_{\mathcal{H}}, \ \theta_{\mathcal{H}}]^\top \in [V_{\mathcal{H}}]^4$ approximating the solution of (4.6).

Remark 4.2. For an extensive description of Spectral Methods we refer to Canuto *et al.*[7] and Bernardi and Maday.[6] Here it is worth noting the interpolation error estimate for spectral elements, which is proved in Ref. 15. For every $T_k \in \mathcal{T}_h$ let $I_N^k : \mathcal{C}^0(T_k) \to \mathbb{Q}_N(T_k)$ the local Lagrange interpolation operator on the GLL nodes in T_k, and let $I_{\mathcal{H}} : \mathcal{C}^0(\Omega) \to \mathbb{Q}_{\mathcal{H}}(\Omega)$ the global interpolation operator such that $(I_{\mathcal{H}}u)|_{T_k} = I_N^k(u|_{T_k})$, for every $T_k \in \mathcal{T}_h$. the following estimate holds:
there exists a constant $C > 0$ such that

$$\|u - I_{\mathcal{H}}u\|_{H^m(\Omega)} \le C h^{\min(N+1,s)-m} N^{m-s} \|u\|_{H^s(\Omega)} \qquad m = 0, 1, \tag{4.13}$$

provided $u \in H^s(\Omega)$ with $s \ge 2$.

The above estimate characterizes the high properties of approximation of spectral element methods.

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Let $\{\varphi_{\mathcal{H}i}\}_{i=1}^{N_\mathcal{H}}$ denote the Lagrange basis of $V_\mathcal{H}$ with respect to the GLL quadrature nodes in Ω, then $u_\mathcal{H},\ w_\mathcal{H},\ v_\mathcal{H},\ \theta_\mathcal{H}$ can be written as expansion of the Lagrange basis as

$$\begin{aligned} u_\mathcal{H} &= \sum_{i=1}^{N_\mathcal{H}} u_i(t)\varphi_{\mathcal{H}i},\ w_\mathcal{H} = \sum_{i=1}^{N_\mathcal{H}} w_i(t)\varphi_{\mathcal{H}i}\,, \\ v_\mathcal{H} &= \sum_{i=1}^{N_\mathcal{H}} v_i(t)\varphi_{\mathcal{H}i},\ \theta_\mathcal{H} = \sum_{i=1}^{N_\mathcal{H}} \theta_i(t)\varphi_{\mathcal{H}i}\,. \end{aligned} \tag{4.14}$$

Moreover, we set $\mathbf{u}(t) = [u_i(t)]_{i=1}^{N_\mathcal{H}}$, $\mathbf{w}(t) = [w_i(t)]_{i=1}^{N_\mathcal{H}}$, $\mathbf{v}(t) = [v_i(t)]_{i=1}^{N_\mathcal{H}}$, $\boldsymbol{\theta}(t) = [\theta_i(t)]_{i=1}^{N_\mathcal{H}}$. Lastly we define the *mass matrix* M

$$\mathsf{M}_{ij} := (\varphi_{\mathcal{H}i}, \varphi_{\mathcal{H}j})_{\mathcal{H},\Omega}, \qquad i,j = 1,\ldots,N_\mathcal{H} \tag{4.15}$$

and the *stiffness matrix* A

$$\mathsf{A}_{ij} := (\nabla\varphi_{\mathcal{H}j}, \nabla\varphi_{\mathcal{H}i})_{\mathcal{H},\Omega}, \qquad i,j = 1,\ldots,N_\mathcal{H}. \tag{4.16}$$

The semi-discretization of system (4.6) by Spectral Element Methods reads: given $u_{\mathcal{H}0},\ u_{\mathcal{H}1},\ \theta_{\mathcal{H}0}$, suitable approximations of $u_0,\ u_1,\ \theta_0$, respectively, in $V_\mathcal{H}$ and given $f \in L^2(\omega \times (0,T))$, for any $t \in (0,T]$, find $\mathbf{u}(t),\ \mathbf{w}(t),\ \mathbf{v}(t),\ \boldsymbol{\theta}(t) \in \mathbb{R}^{N_\mathcal{H}}$ such that

$$\begin{cases} \mathsf{A}\mathbf{u}(t) - \mathsf{M}\mathbf{w}(t) = \mathbf{0} \\ \mathsf{M}\dfrac{d^2}{dt^2}\mathbf{u}(t) + \mathsf{A}\mathbf{w}(t) - \mathsf{A}\boldsymbol{\theta}(t) = \mathbf{0} \\ \mathsf{M}\dfrac{d}{dt}\boldsymbol{\theta}(t) + \mathsf{A}\boldsymbol{\theta}(t) + \mathsf{A}\mathbf{v}(t) = \mathbf{F}(t) \\ \mathbf{u}(0) = \mathbf{u}^0,\ \mathbf{v}(0) = \mathbf{v}^0,\ \boldsymbol{\theta}(0) = \boldsymbol{\theta}^0\,, \end{cases} \tag{4.17}$$

where $\mathbf{u}^0$, $\mathbf{v}^0$ and $\boldsymbol{\theta}^0$ are the arrays in $\mathbb{R}^{N_\mathcal{H}}$ whose components are the coefficients of the expansions of $u_{\mathcal{H}0},\ u_{\mathcal{H}1}, \theta_{\mathcal{H}0}$ with respect to the Lagrange basis $\{\varphi_{\mathcal{H}i}\}_{i=1}^{N_\mathcal{H}}$, while $\mathbf{F}(t)$ is the array whose components are $F_i(t) = (\chi_\omega f(t), \varphi_{\mathcal{H}i})_{\mathcal{H},\Omega}$, for $i = 1,\ldots,N_\mathcal{H}$.

System (4.17) is a system of ordinary differential equations that we are going to discretize by Newmark and Crank–Nicolson schemes.[28]

4.3. *Time-advancing*

The Newmark method is a widely used method in structural mechanics to integrate systems of ordinary differential equations of second order in time. Its most important feature consists in the fact that it is a non-dissipative scheme, that is it does not introduce numerical damping. This fact is as much important as one looks for a null solution, as in the case of null controllability problems.

Let us consider the problem $y''(t) = g(t, y(t), y'(t)),\ t \in (t_0, T),\ y(t_0) = y_0$, $y'(t_0) = y_1$, where $g : [t_0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Given $\Delta t \in (0, T)$

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we set $t_0 = 0$ and $t_n = t_0 + n\Delta t$, with $n = 1, \dots, M$ and $M = \left[\frac{T-t_0}{\Delta t}\right]$. The Newmark method generates the following sequences:

$$\begin{cases} y^{n+1} = y^n + \Delta t z^n + (\Delta t)^2[\alpha g^{n+1} + (0.5-\alpha)g^n] \\ z^{n+1} = z^n + \Delta t[\beta g^{n+1} + (1-\beta)g^n] \end{cases} \tag{4.18}$$

for $n = 0, \dots, M$, where $y^0 = y_0$, $z^0 = y_1$, α and β are some non-negative parameters, $g^n = g^n(t_n, y^n, z^n)$ and z^n is an approximation of $y'(t_n)$.

For $\beta = 1/2$ and $\alpha = 1/4$, the Newmark method is second-order accurate in time and it is unconditionally stable. This popular choice is, however, unsuitable for long time integration, as the discrete solution may be affected by parasitic oscillations that are not damped as far as t increases. When this occurs, one can use $\alpha \geq (\beta + 1/2)^2/4$ for a suitable $\beta > 1/2$, although the method downgrades to a first order one.

The Crank–Nicolson scheme is used to approximate first-order ordinary differential equations like $y'(t) = g(t, y(t))$, $t > 0$, $y(t_0) = y_0$, where $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. It generates the sequence

$$y^{n+1} = y^n + \frac{\Delta t}{2}(g^n + g^{n+1}) \tag{4.19}$$

for $n = 0, \dots, M$, where $y^0 = y_0$ and $g^n = g(t_n)$.

The Crank–Nicolson scheme is second-order accurate in time and it is also unconditionally stable.

By approximating the second-order (resp. first-order) time derivative in (4.6) by the Newmark (resp. Crank–Nicolson) method we have: given $\mathbf{u}^0,\ \mathbf{v}^0,\ \boldsymbol{\theta}^0 \in \mathbb{R}^{N_\mathcal{H}}$ for $n \geq 1$ we look for the solution $\mathbf{u}^{n+1}, \mathbf{w}^{n+1},\ \mathbf{v}^{n+1},\ \boldsymbol{\theta}^{n+1} \in \mathbb{R}^{N_\mathcal{H}}$ of the linear system

$$\begin{cases} \mathsf{A}\mathbf{u}^{n+1} - \mathsf{M}\mathbf{w}^{n+1} = \mathbf{0}\,, \\[2ex] \dfrac{1}{\alpha(\Delta t)^2}\mathsf{M}\mathbf{u}^{n+1} + \mathsf{A}\mathbf{w}^{n+1} - \mathsf{A}\boldsymbol{\theta}^{n+1} = \dfrac{1}{\alpha(\Delta t)^2}\mathsf{M}(\mathbf{u}^n + \Delta t\mathbf{v}^n) + \dfrac{1-\alpha}{\alpha}\mathsf{A}(\boldsymbol{\theta}^n - \mathbf{w}^n)\,, \\[2ex] \mathsf{A}\mathbf{w}^{n+1} + \dfrac{1}{\beta\Delta t}\mathsf{M}\mathbf{v}^{n+1} - \mathsf{A}\boldsymbol{\theta}^{n+1} = \dfrac{1}{\beta\Delta t}\mathsf{M}\mathbf{v}^n + \dfrac{1-\beta}{\beta}\mathsf{A}(\boldsymbol{\theta}^n - \mathbf{w}^n)\,, \\[2ex] \mathsf{A}\mathbf{v}^{n+1} + \dfrac{2}{\Delta t}\mathsf{M}\boldsymbol{\theta}^{n+1} + \mathsf{A}\boldsymbol{\theta}^{n+1} = \mathbf{F}^{n+1} + \mathbf{F}^n + \dfrac{2}{\Delta t}\mathsf{M}\boldsymbol{\theta}^n - \mathsf{A}(\mathbf{v}^n + \boldsymbol{\theta}^n)\,. \end{cases}$$

The matrix of this linear system is sparse, unsymmetric and independent of time. Since we have to solve the system many times along the CG-algorithm (we remember that every evaluation of type (4.1) involves both a backward and a forward in time resolution), it is preferable to factorize the matrix at the beginning of the process and to solve the triangular systems at each time step, instead of solving the system by an iterative method.

Following the same notation as introduced in (4.14), the approximation of the

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solution $[u,\ w,\ v,\ \theta]^\top$ of (4.6) is, for any $n = 1, \ldots, M$,

$$\begin{aligned} u_{\mathcal{H}}^{\Delta t}(t_n) &= \sum_{i=1}^{N_{\mathcal{H}}} u_i^n \varphi_{\mathcal{H}i},\ w_{\mathcal{H}}^{\Delta t}(t_n) = \sum_{i=1}^{N_{\mathcal{H}}} w_i^n \varphi_{\mathcal{H}i} \\ v_{\mathcal{H}}^{\Delta t}(t_n) &= \sum_{i=1}^{N_{\mathcal{H}}} v_i^n \varphi_{\mathcal{H}i},\ \theta_{\mathcal{H}}^{\Delta t}(t_n) = \sum_{i=1}^{N_{\mathcal{H}}} \theta_i^n \varphi_{\mathcal{H}i}, \end{aligned} \tag{4.20}$$

where $\mathbf{u}^n = [u_i^n]_{i=1}^{N_{\mathcal{H}}}$, $\mathbf{v}^n = [v_i^n]_{i=1}^{N_{\mathcal{H}}}$, $\mathbf{w}^n = [w_i^n]_{i=1}^{N_{\mathcal{H}}}$, $\boldsymbol{\theta}^n = [\theta_i^n]_{i=1}^{N_{\mathcal{H}}}$, and then we set $\mathbf{z}_{\mathcal{H}}^{\Delta t}(t_n) = [u_{\mathcal{H}}^{\Delta t}(t_n),\ v_{\mathcal{H}}^{\Delta t}(t_n),\ \theta_{\mathcal{H}}^{\Delta t}(t_n)]^\top$.

Remark 4.3. Recalling also that the control f is unknown for the controllability problem and that it is computed through the approximation of the optimality system (3.21)–(3.22), we will denote the approximation of $f(t_n)$ by $f_{\mathcal{H}}^{\Delta t}(t_n)$ for $n = 1, \ldots, M$.

5. Numerical Results

First of all we present some numerical results attesting to the high accuracy in both space and time of the approximation described in Sec. 4. To this aim, given a function u on Ω and its approximation $u_{\mathcal{H}}^{\Delta t}$, we define the relative error in the discrete L^∞-norm, at time T, as

$$E_u(T) = \frac{\|u(T) - u_{\mathcal{H}}^{\Delta t}(T)\|_{\infty,\mathcal{H},\Omega}}{\|u(T)\|_{\infty,\mathcal{H},\Omega}}, \tag{5.1}$$

where $\|u(T)\|_{\infty,\mathcal{H},\Omega} := \max_{\mathbf{x}_i, i=1,\ldots,N_{\mathcal{H}}} |u(\mathbf{x}_i, T)|$ and $\mathbf{x}_i,\ i = 1, \ldots, N_{\mathcal{H}}$ are the GLL quadrature nodes in Ω.

We consider problem (1.1) on the computational domain $\Omega = (0, 0.5)^2$, $\omega \equiv \Omega$ $T = 1$, with the initial data:

$$\begin{aligned} u_0(x,y) &= \sin(2\pi x)\sin(2\pi y)\,, \\ u_1(x,y) &= -\sin(2\pi x)\sin(2\pi y)\,, \\ \theta_0(x,y) &= \frac{1+64\pi^4}{8\pi^2}\sin(2\pi x)\sin(2\pi y) \end{aligned} \tag{5.2}$$

and right-hand side

$$f(x,y,t) = \left(-\frac{1+64\pi^4}{8\pi^2} + 1 + 64\pi^4 - 8\pi^2\right)\sin(2\pi x)\sin(2\pi y)e^{-t}\,.$$

The corresponding exact solution is

$$\begin{aligned} u(x,y,t) &= \sin(2\pi x)\sin(2\pi y)e^{-t}\,, \\ \theta(x,y,t) &= \frac{1+64\pi^4}{8\pi^2}\sin(2\pi x)\sin(2\pi y)e^{-t}. \end{aligned} \tag{5.3}$$

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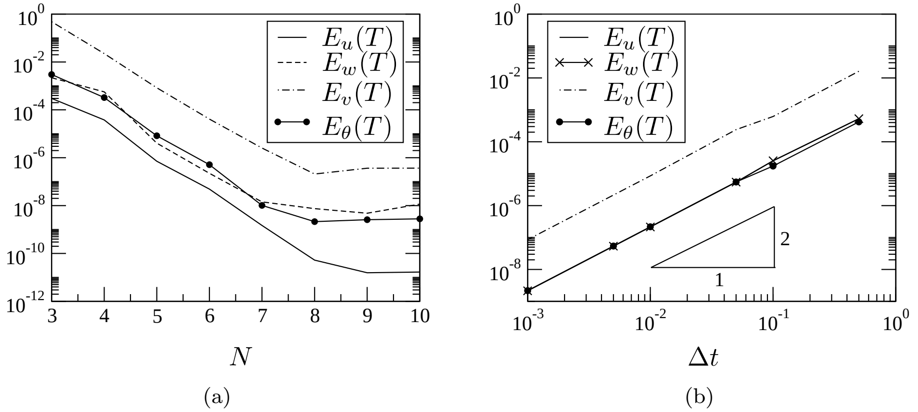

Fig. 1. (a) The relative errors in the discrete L^∞-norm at time $T = 10^{-3}$. The time-step is $\Delta t = 10^{-5}$. (b) The relative errors in L^∞-norm at time $T = 1$. The polynomial degree in each spectral element is $N = 12$.

We take a partition of Ω in 2×2 squared elements whose side is $h = 0.25$ and we choose $\alpha = 1/4$, $\beta = 1/2$ as parameters of the Newmark methods, so that it is second-order accurate in time.

In Fig. 1(a) we show the space approximation errors on all the component of the solution, which decays with exponential rate with respect to the polynomial degree N, for fixed $\Delta t = 10^{-5}$. We note that, for these data, for $N \geq 9$ the time-approximation error prevails over the space-approximation error. In Fig. 1(b) we show the second-order accuracy of the Newmark/Crank–Nicolson time-advancing scheme with polynomial degree $N = 12$.

5.1. *Test case #1: $\omega \equiv \Omega$*

We consider the computational domain $\Omega = (0,1)^2$ and $w \equiv \Omega$. The initial data will be (5.2) or the following:

$$\begin{aligned} u_0(x,y) &= [x(x-1)y(y-1)]^4\,, \\ u_1(x,y) &= -[x(x-1)y(y-1)]^4\,, \\ \theta_0(x,y) &= [x(x-1)y(y-1)]^4\,. \end{aligned} \tag{5.4}$$

In Fig. 2 (resp. Fig. 3) we show the numerical solution of (4.7) without control f and with initial data (5.2) (resp. (5.4)). For this test case we consider a discretization in 2×2 spectral element of size $h = 0.5$, polynomial degree $N = 9$ and we choose the parameters of the Newmark scheme $\alpha = 1/4$ and $\beta = 1/2$. For initial data (5.2) we set $\Delta t = 10^{-2}$, while for initial data (5.4) we set $\Delta t = 10^{-3}$ in order to avoid spurious oscillations due to the choice of $\alpha = 1/4$ and $\beta = 1/2$. The choice $\Delta t = 10^{-2}$ for initial data (5.2) and $\Delta t = 10^{-3}$ for initial data (5.4) will be taken in this section.

Next, we compare the solution of problem (1.1) with $f \equiv 0$, that is without control, with the solution of the controllability problem (1.7) and (1.8), obtained

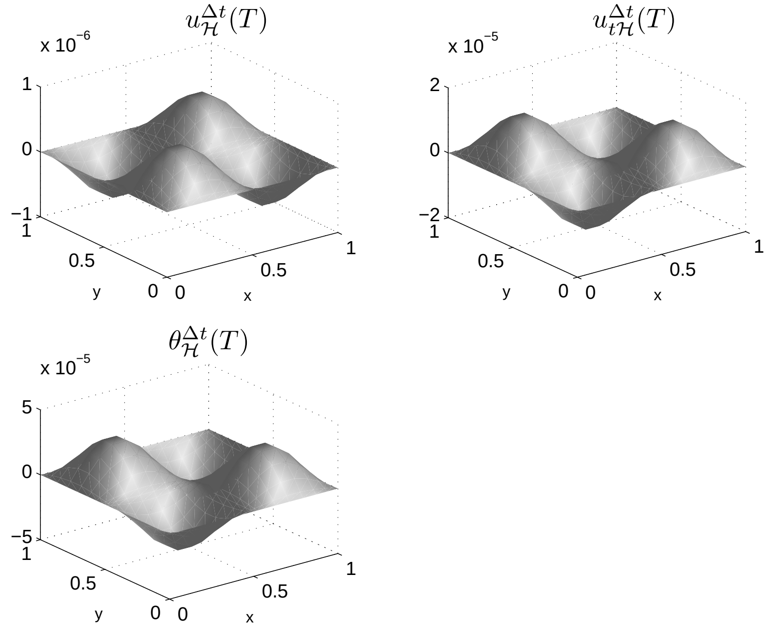

Fig. 2. Test case #1. The approximate solution without control in $T = 1$ for $k = 10^6$ and initial data (5.2).

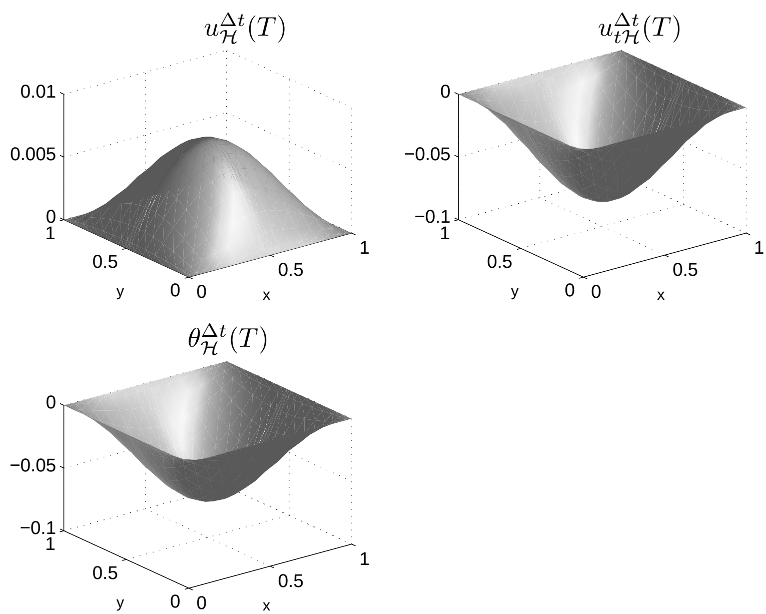

Fig. 3. Test case #1. The approximate solution without control in $T = 1$ for $k = 10^6$ and initial data (5.4).

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Table 1. Test case #1. Norm of the approximate solutions of (1.1) with and without control. We show the norm (5.5) of the solution at time $T = 1$ for the problem with initial data (5.2) (resp. (5.4)). The discretization parameters are $N = 9$, 2×2 spectral elements with side $h = 0.5$ and $\Delta t = 10^{-2}$ (resp. $\Delta t = 10^{-3}$).

	Initial data (5.2)		Initial data (5.4)	
	$\Vert\mathbf{z}_{\mathcal{H}}^{\Delta t}(T)\Vert_{H,\mathcal{H}}$	CG it.	$\Vert\mathbf{z}_{\mathcal{H}}^{\Delta t}(T)\Vert_{H,\mathcal{H}}$	CG it.
Without control	2.98E-05	–	8.59E-02	–
With control and $k = 10^2$	1.83E-05	2	2.43E-02	10
With control and $k = 10^4$	4.64E-07	6	3.38E-04	33
With control and $k = 10^6$	4.71E-09	9	3.39E-06	50
With control and $k = 10^8$	4.71E-11	9	3.39E-08	54

with various penalization parameters k. To this aim we introduce the discrete counterpart of $\Vert\mathbf{z}_{\mathcal{H}}^{\Delta t}(t_n)\Vert_H$ as

$$\Vert\mathbf{z}_{\mathcal{H}}^{\Delta t}(t_n)\Vert_{H,\mathcal{H}} := \left(\Vert w_{\mathcal{H}}^{\Delta t}(t_n)\Vert_{\mathcal{H},\Omega}^2 + \Vert v_{\mathcal{H}}^{\Delta t}(t_n)\Vert_{\mathcal{H},\Omega}^2 + \Vert \theta_{\mathcal{H}}^{\Delta t}(t_n)\Vert_{\mathcal{H},\Omega}^2\right)^{1/2}. \tag{5.5}$$

In Table 1 we show the norm (5.5) for both the solution depending on the initial data (5.2) and the solution depending on the initial data (5.4). From (3.18) it holds that there exists a positive constant C independent of k such that $\Vert\mathbf{z}_k(T)\Vert_H \le Ck^{-1}$. From Table 1 we can infer that the same relation holds for the approximate solution $\mathbf{z}_{\mathcal{H}}^{\Delta t}$. We note that the order of magnitude of the error strongly depends on the initial data, for the two sets of initial data used there is a difference of about three orders of magnitude in correspondence of the same penalization parameter k. In the same table we report the number of CG iterations needed to satisfy the stopping criterium $\Vert\mathbf{r}^{(n)}\Vert/\Vert\mathbf{r}^{(0)}\Vert < 10^{-10}$, being $\mathbf{r}^{(n)}$ the residual of equation (3.28) at the nth iteration. The number of CG iterations grows like the logarithm of the penalization parameter k.

In Figs. 4 and 5 the norm $\Vert f_{\mathcal{H}}^{\Delta t}(t)\Vert_{\mathcal{H}}$ of the approximation of the control f is shown. We observe that the functions $\Vert f_{\mathcal{H}}^{\Delta t}(t)\Vert_{\mathcal{H}}$ tend to assume the same behavior in $[0, T]$ when k increases. The discretization parameters and the initial data are those used for the results of Table 1.

In Fig. 6 we show the approximation of the quantity

$$\mathbb{E}_{\mathcal{H}}^{\Delta t}(T; \mathbf{z}_0) = \left(\int_0^T \Vert f_{\mathcal{H}}^{\Delta t}(t)\Vert_{L^2(\Omega)}^2\right)^{1/2} \Big/ \Vert\mathbf{z}_0\Vert_H$$

for different values of T, by suitable quadrature formulas. We observe that the behavior of $\mathbb{E}_{\mathcal{H}}^{\Delta t}(T; \mathbf{z}_0)$ obey the theorem given in both Avalos and Lasiecka[3] and Triggiani[31], which states that, with reference to the null controllability problem for the thermoelastic system (1.1) with $\omega \equiv \Omega$, it holds

$$\mathbb{E}(T) := \sup_{\Vert\mathbf{z}_0\Vert_H = 1} \left(\int_0^T \Vert f(t; \mathbf{z}_0)\Vert_{L^2(\Omega)}^2\right)^{1/2} = \mathcal{O}(T^{-5/2}), \tag{5.6}$$

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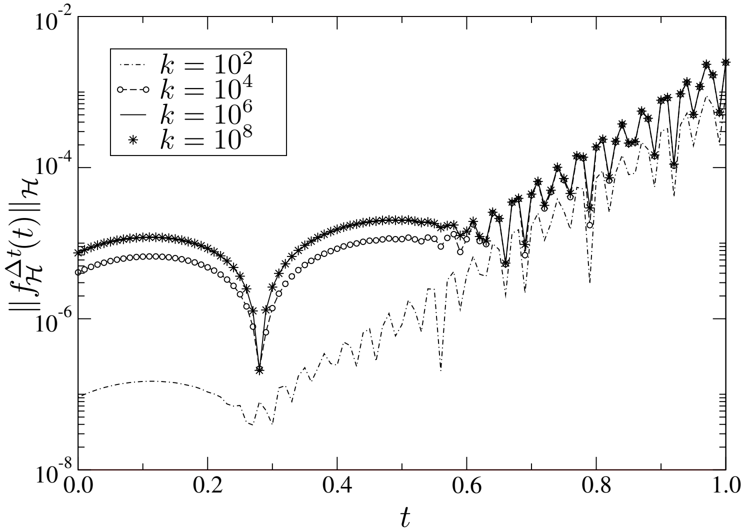

Fig. 4. Test case #1. The norm $\|f_{\mathcal{H}}^{\Delta t}(t)\|_{\mathcal{H}}$ vs. time t for the initial data (5.2) and $\omega \equiv \Omega$. The discretization parameters are those used for the results of Table 1.

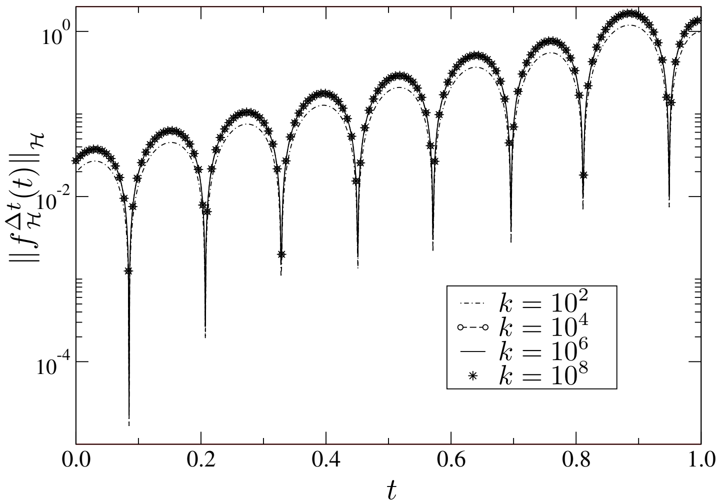

Fig. 5. Test case #1. The norm $\|f_{\mathcal{H}}^{\Delta t}(t)\|_{\mathcal{H}}$ vs. time t for the initial data (5.4) and $\omega \equiv \Omega$. The discretization parameters are those used for the results of Table 1.

where $f(t; \mathbf{z}_0)$ denotes the control obtained by solving (1.1) with initial condition $\mathbf{z}_0$. The results reported in Fig. 6 refer to a penalization parameter $k = 10^6$ and a discretization with 2×2 squared elements with $h = 0.5$ and $N = 9$ and $\Delta t = 10^{-2}$ (resp. $\Delta t = 10^{-3}$) for initial data (5.2) (resp. (5.4)).

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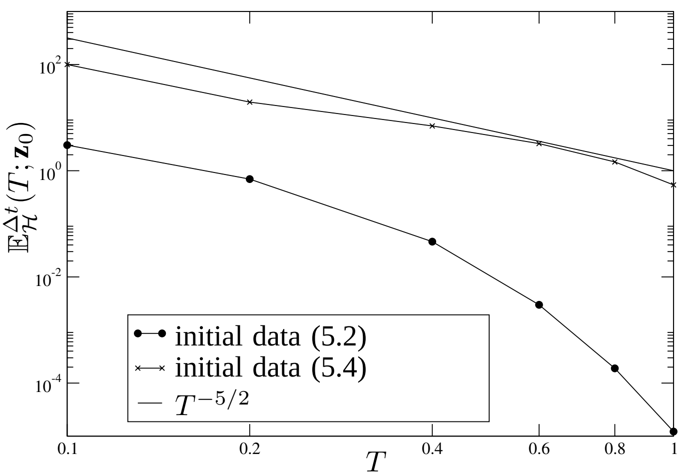

Fig. 6. Test case #1. The function $\mathbb{E}_{\mathcal{H}}^{\Delta t}(T;\mathbf{z}_0)$ vs. the final time T, compared with the function $T^{-5/2}$.

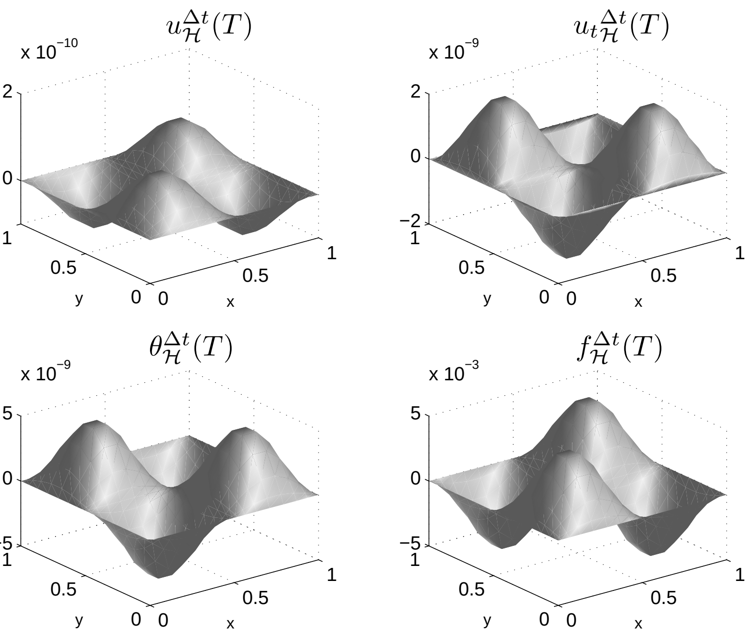

Fig. 7. Test case #1. The approximate solution and control in $T = 1$, for initial data (5.2), $\omega \equiv \Omega$, $k = 10^6$. The discretization parameters are those used for the results of Table 1.

Lastly, in Fig. 7 (resp. Fig. 8) we show the approximate solution

$$[u_{\mathcal{H}}^{\Delta t}(T),\ v_{\mathcal{H}}^{\Delta t}(T),\ \theta_{\mathcal{H}}^{\Delta t}(T)]^\top$$

and the control $f_{\mathcal{H}}^{\Delta t}(T)$ of the penalized controllability problem with initial data (5.2) (resp. (5.4)), at time $T = 1$, for $k = 10^6$.

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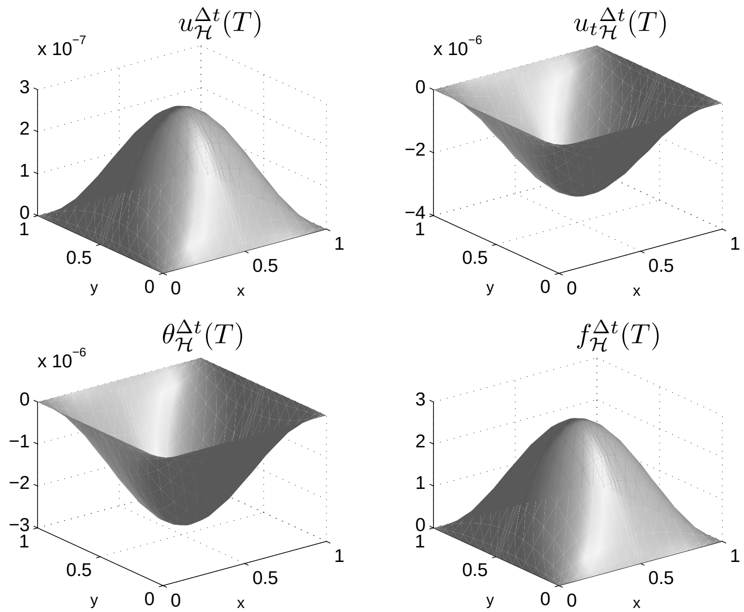

Fig. 8. Test case #1. The approximate solution and control in $T = 1$, for initial data (5.4) and $\omega \equiv \Omega$, $k = 10^6$. The discretization parameters are those used for the results of Table 1.

5.2. *Test case #2: $\omega \subset \Omega$*

We show here some numerical results when the domain $\omega \subset \Omega$. We consider the domain $\Omega = (0,1)^2$, $\omega = (0.3, 0.7)^2$ (resp. $\omega = (0.2, 0.5)^2$) and initial data (5.2) (resp. (5.4)). We built a $C^2(\Omega)$ regularization of the characteristic function χ_ω, in order to avoid numerical oscillations and very poor numerical solutions.

Also for this test case we consider a discretization in 2×2 spectral element of size $h = 0.5$, polynomial degree $N = 9$ and we choose the parameters of the Newmark scheme $\alpha = 1/4$ and $\beta = 1/2$. For initial data (5.2) we set $\Delta t = 10^{-2}$, while for initial data (5.4) we set $\Delta t = 10^{-3}$ in order to avoid spurious oscillations due to the choice of $\alpha = 1/4$ and $\beta = 1/2$.

As done for the previous test case we compare the solution of problem (1.1) with $f \equiv 0$, that is without control, with the solution of the controllability problem (1.7) and (1.8), obtained with various penalization parameters k. In Table 2 we show the norm (5.5) and the number of CG iterations needed to satisfy the stopping criterium $\|\mathbf{r}^{(n)}\|/\|\mathbf{r}^{(0)}\| \leq 10^{-10}$. First of all we note that, for k fixed, the norm $\|\mathbf{z}_{\mathcal{H}}^{\Delta t}(T)\|_{H,\mathcal{H}}$ is greater for $\omega \subset \Omega$ than for $\omega \equiv \Omega$ and the difference is as much as k is bigger. Even if the theory ensures that there exists a control on $\omega \subset \Omega$ that put the final state to zero in all the domain Ω, in practice the numerical solution strongly depends on both the form and amplitude of ω.

We also note that the convergence of CG algorithm is slower when $\omega \subset \Omega$ than when $\omega \equiv \Omega$. In particular the number of CG iterations needed to obtain

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Table 2. Test case #2. Comparison among the solution with and without control. We show the norm (5.5) of the solution at time $T = 1$. The discretization parameters are $N = 9$, 2×2 spectral elements with side $h = 0.5$ and $\Delta t = 10^{-2}$ (resp. $\Delta t = 10^{-3}$) for solving with initial data (5.2) (resp. (5.4)) and $\omega = (0.3, 0.7)^2$ (resp. $\omega = (0.2, 0.5)^2$).

	Initial data (5.2) $\omega = (0.3, 0.7)^2$		Initial data (5.4) $\omega = (0.2, 0.5)^2$	
	$\Vert\mathbf{z}_{\mathcal{H}}^{\Delta t}(T)\Vert_{H,\mathcal{H}}$	CG it.	$\Vert\mathbf{z}_{\mathcal{H}}^{\Delta t}(T)\Vert_{H,\mathcal{H}}$	CG it.
Without control	2.98E-05	–	8.59E-02	–
With control and $k = 10^2$	2.81E-05	7	5.80E-02	10
With control and $k = 10^4$	6.29E-06	46	5.60E-03	50
With control and $k = 10^6$	3.03E-07	316	2.90E-04	428

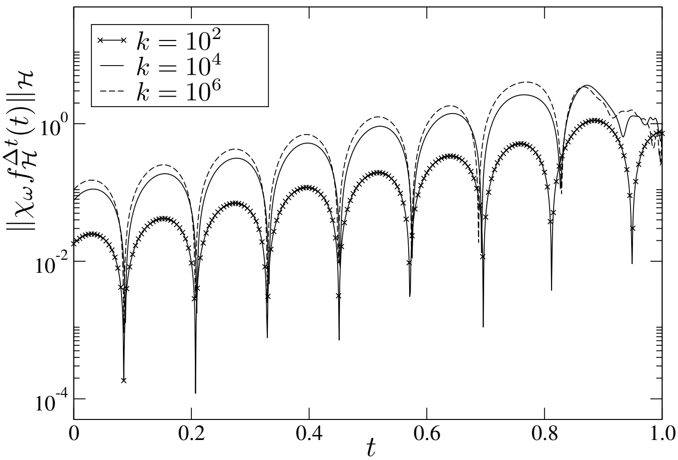

Fig. 9. Test case #2. The norm $\Vert\chi_\omega f_{\mathcal{H}}^{\Delta t}(t)\Vert_{\mathcal{H}}$ vs. time t for initial data (5.4) and $\omega = (0.2, 0.5)^2$. The discretization parameters and the initial data are those used for the results of Table 2.

convergence varies like $\sqrt{k}$.

In Fig. 9 the norm $\Vert\chi_\omega f_{\mathcal{H}}^{\Delta t}(t)\Vert_{\mathcal{H}}$ of the approximation of the control f is shown. We observe that different from the case $\omega \equiv \Omega$, here the functions $\Vert\chi_\omega f_{\mathcal{H}}^{\Delta t}(t)\Vert_{H,\mathcal{H}}$ do not tend to assume the same behavior in $[0, T]$ when k increases.

Remark 5.1. At the end of this subsection we want to analyze the norm $\Vert\mathbf{z}_{\mathcal{H}}^{\Delta t}(T)\Vert_{H,\mathcal{H}}$ versus both the choice of the domain ω and the penalization parameter k. We consider the initial data (5.4), $k = 10^2$, 10^4, 10^6, 10^8 and $\omega = \omega_0 \equiv \Omega$, $\omega = \omega_1 = (0.1, 0.9)^2$, $\omega = \omega_2 = (0.2, 0.8)^2$, $\omega = \omega_3 = (0.3, 0.7)^2$, $\omega = \omega_4 = (0.4, 0.6)^2$. We fix the stopping criterium of the CG algorithm and the discretization parameters $N = 9$, $h = 0.5$ and $\Delta t = 10^{-3}$. From Table 3 we observe that, when we fix k, the norm $\Vert\mathbf{z}_{\mathcal{H}}^{\Delta t}(T)\Vert_{H,\mathcal{H}}$ increases as meas(ω) decreases, and the increase speed of $\Vert\mathbf{z}_{\mathcal{H}}^{\Delta t}(T)\Vert_{H,\mathcal{H}}$ is greater for high values of k.

Table 3. The norm $\|\mathbf{z}_{\mathcal{H}}^{\Delta t}(T)\|_{H,\mathcal{H}}$ vs. both the choice of the domain ω and the penalization parameter k. Between brackets the number of CG-iterations needed to obtain convergence is shown.

	$k = 10^2$	$k = 10^4$	$k = 10^6$	$k = 10^8$
ω_0	2.43E-02 (10)	3.38E-04 (33)	3.39E-06 (50)	3.39E-08 (52)
ω_1	2.50E-02 (12)	3.78E-04 (62)	9.32E-06 (427)	1.42E-07 (1661)
ω_2	2.87E-02 (11)	1.02E-03 (60)	5.85E-05 (451)	3.50E-06 (5369)
ω_3	3.94E-02 (10)	2.47E-03 (47)	1.71E-04 (370)	9.28E-06 (5146)
ω_4	6.32E-02 (9)	6.13E-03 (32)	4.42E-04 (231)	1.71E-05 (3916)

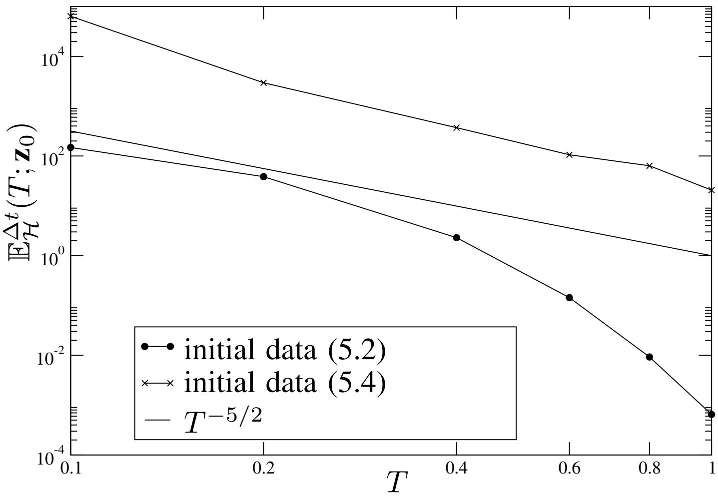

Fig. 10. Test case #2. The function $\mathbb{E}_{\mathcal{H}}^{\Delta t}(T;\mathbf{z}_0)$ vs. the final time T, compared with the function $T^{-5/2}$, for $\omega \subset \Omega$.

In Fig. 10 we show the approximation of the quantity $\mathbb{E}_{\mathcal{H}}^{\Delta t}(T;\mathbf{z}_0)$, by suitable quadrature formulas. For $\omega \subset \Omega$ no theoretical results are known. Here we compare the numerical values with the function $T^{-5/2}$ and we note that, for the polynomial initial data (5.4), $\mathbb{E}_{\mathcal{H}}^{\Delta t}(T;\mathbf{z}_0)$ grows more than $T^{-5/2}$. In Fig. 11 we compare the norm of the components $u_{\mathcal{H}}^{\Delta t}$ and $\theta_{\mathcal{H}}^{\Delta t}$ of the solution with the norm of the control, for initial data (5.4), $\omega = (0.2, 0.5)^2$, $k = 10^6$. Lastly, in Fig. 12 (resp.13) we show the solution of the penalized controllability problem at time $T = 1$ with initial data (5.2) (resp. (5.4)), for $k = 10^6$.

5.3. *Test case #3: An example of control of trajectories*

We take $\Omega = \{(x,y) \in \mathbb{R}^2\colon x^2 + y^2 < 1\}$, $\omega = \{(x,y) \in \mathbb{R}^2 : (x-0.2)^2/0.36 + (y-0.2)^2/0.16 < 1\}$ and $T = 1$. We look for the numerical solution of the problem of controllability of trajectories (see Sec. 1.1) with the initial data and the right-hand

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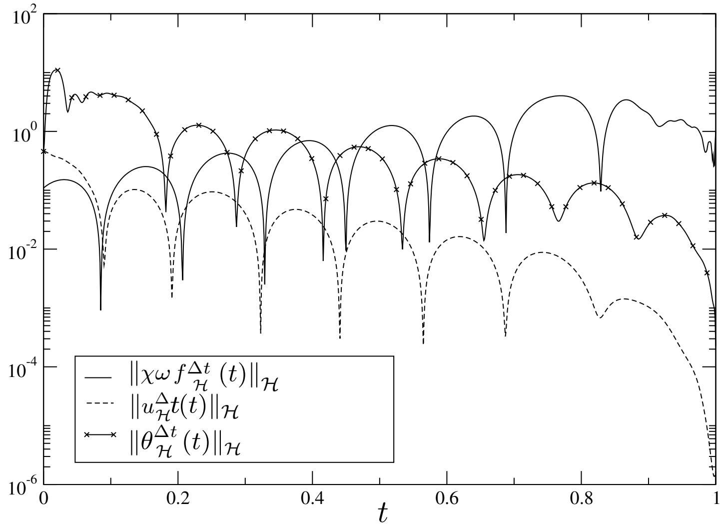

Fig. 11. Test case #2. Comparison between the norm of the component $u^{\Delta t}_{\mathcal{H}}$ and $\theta^{\Delta t}_{\mathcal{H}}$ of the solution with the norm of the control, for initial data (5.4), $\omega = (0.2, 0.5)^2$, $k = 10^6$.

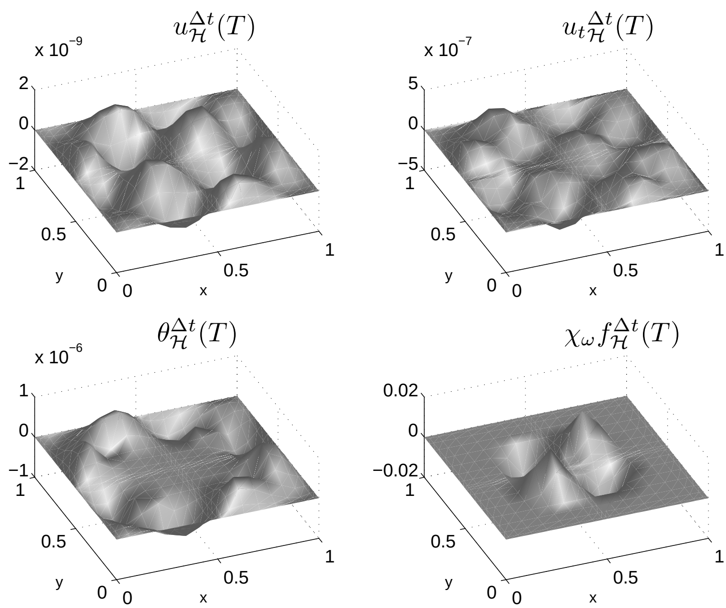

Fig. 12. Test case #2. The approximate solution and control in $T = 1$ for $k = 10^6$ and initial data (5.2).

side $\hat{\mathbf{z}}^0$, $\hat{f}$, $\tilde{\mathbf{z}}^0$:

$$\begin{aligned}
&\hat{\mathbf{z}}^0 = [\phi_1(\rho),\ -\phi_1(\rho),\ \phi_1(\rho)]^\top,\ \phi_1(\rho) = 250(\rho^2-\rho)^4,\ \rho = \sqrt{x^2+y^2}\,,\\
&\hat{f} = 100\chi_\omega\,,\\
&\tilde{\mathbf{z}}^0 = [\phi_2(\rho),\ -\phi_2(\rho),\ \phi_2(\rho)]^\top,\ \phi_2(\rho) = (\rho^2-1)^4\,.
\end{aligned} \tag{5.7}$$

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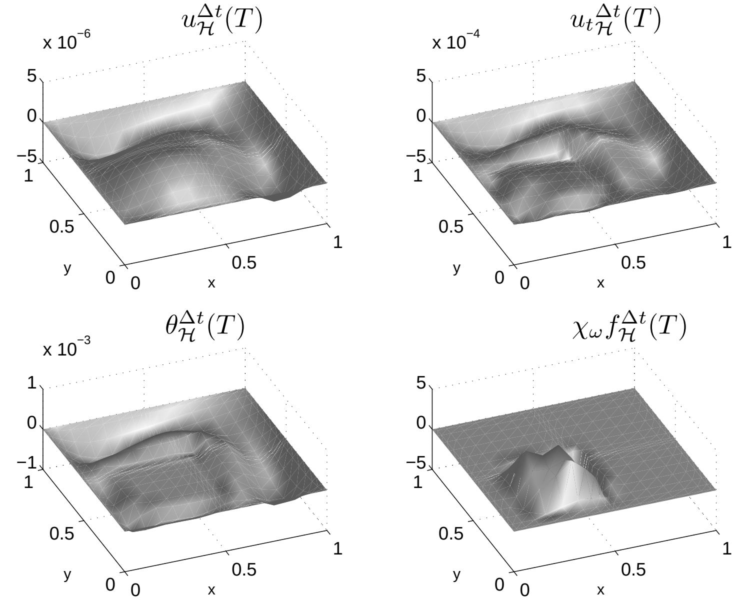

Fig. 13. Test case #2. The approximate solution and control in $T = 1$ for $k = 10^6$ and initial data (5.4).

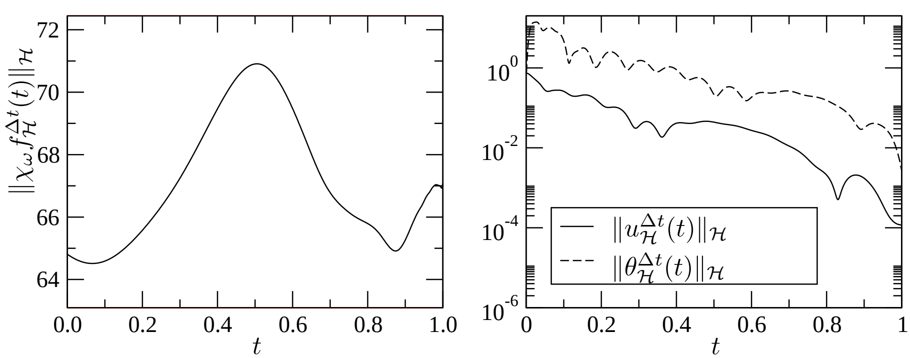

Fig. 14. Test case #3. The norm $\|\chi_\omega f_{\mathcal{H}}^{\Delta t}(t)\|_{\mathcal{H}}$ vs. time t (a) and the norms $\|u_{\mathcal{H}}^{\Delta t}(t)\|_{\mathcal{H}}$, $\|\theta_{\mathcal{H}}^{\Delta t}(t)\|_{\mathcal{H}}$ (b) for $k = 10^4$.

We consider a discretization of Ω in 12 spectral elements with $N = 8$, we take $\alpha = 1/4$, $\beta = 1/2$ for the Newmark scheme and $\Delta t = 10^{-3}$. The stopping criterium for the CG algorithm, used to look for the solution and the control of the null controllability problem, is $\|\mathbf{r}^{(n)}\|/\|\mathbf{r}^{(0)}\| < 10^{-8}$. For $k = 10^4$ the CG algorithm converges in 223 iterations and, following the same notations used in Sec. 1.1, the discrete norm of the approximation of the solution $\mathbf{z}(T; \mathbf{z}^0, f)$ is $\|\mathbf{z}_{\mathcal{H}}^{\Delta t}(T)\|_{H,\mathcal{H}} = 4.09E$-03, while for $k = 10^6$ the convergence is attained in 2250 iterations and $\|\mathbf{z}_{\mathcal{H}}^{\Delta t}(T)\|_{H,\mathcal{H}} = 4.27E$-04. In Fig. 14 the norm $\|\chi_\omega f_{\mathcal{H}}^{\Delta t}(t)\|_{\mathcal{H}}$ of the approximated

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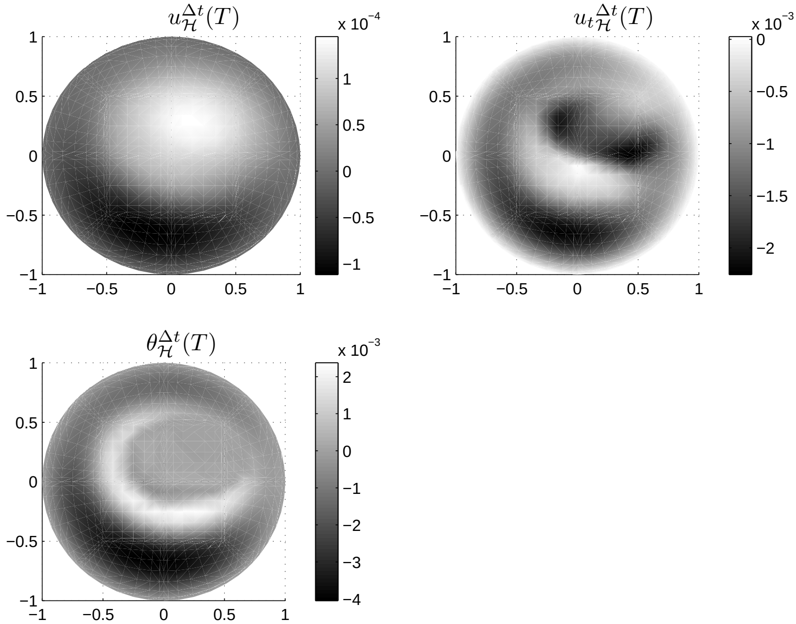

Fig. 15. Test case #3. The approximation of the solution $\mathbf{z}$ in $T = 1$ for $k = 10^6$.

control is shown versus time t, while in Fig. 15 the approximation of the solution $\mathbf{z}(T; \mathbf{z}^0, f)$ is shown at time T.

6. Conclusions

We have used penalization and duality arguments to construct a cost functional in order to solve the null controllability problem and the controllability of trajectories for a thermoelastic plate. Then, by applying the Conjugate Gradient algorithm, and classical approximation methods for partial differential equations, like spectral element methods and finite difference schemes, we have computed the numerical solution and the numerical control. For $\omega \equiv \Omega$ our numerical results observe theoretical estimates given in Avalos and Lasiecka[3] and Triggiani[31] (see (5.6): $\mathbb{E}(T) = \mathcal{O}(T^{-5/2})$); while for $\omega \subset \Omega$, for which theoretical results are absent, we see that $\mathbb{E}_{\mathcal{H}}^{\Delta t}(T; \mathbf{z}_0)$ grows more than $T^{-5/2}$ when T tends to zero.

When $\omega \equiv \Omega$ and the penalization parameter k tends to infinity, the numerical control forces very well the solution to the null target at time T.

When $\omega \subset \Omega$, the norm of the numerical solution at time T tends again to zero when k tends to infinity, but more slowly. On the other hand, from a practical point of view the use of large k is prohibitive, since the number of Conjugate Gradient iterations needed to obtain convergence depends on $\sqrt{k}$.

Our future work will be about both the numerical analysis of the approximation used and improvement of the computational algorithms efficiency.

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Acknowledgment

The authors wish to thank Assia Benabdallah for helpful discussion and pertinent comments. This work has been performed under the auspices of G.N.F.M.–C.N.R. and partially supported by Italian M.U.R.S.T. through the Research Program "Modelli Matematici per la Scienza dei Materiali".

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