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24 Domain decomposition and virtual control for fourth order problems

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Introduction

In this paper we consider domain decomposition strategies for fourth order operators featuring a dominant second order component. More specifically, given an open and bounded domain $\Omega \subset \mathbb{R}^2$ with continuous and Lipschitz boundary $\partial\Omega$, the fourth order problem we consider reads:

$$\begin{cases} \sigma^2\Delta^2 u - \Delta u = f & \text{in } \Omega \\ u = g, \qquad \mathbf{n}\cdot\nabla u = h & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\sigma = const.$ and the functions f, g and h are assigned with sufficient regularity, while $\mathbf{n}$ is the unit outward normal vector on $\partial\Omega$.

We will partition Ω into several subdomains (overlapping or not) and consider different ways to formulate (1) at the subdomain level. In particular, we are looking for suitable control problems, the control variables being faced on the subdomain interfaces. Furthermore, we address the so-called heterogeneous case, i.e. a situation in which the coefficient σ is set to zero on a subregion of Ω. Our control approach is then devised in order to handle the coupling between the original fourth order problem and the second order one that is obtained when dropping σ out. A similar heterogeneous coupling has been previously investigated for a second-order advection diffusion problem with dominant advection (see [GLQ00]).

An outline of the paper is as follows. First the overlapping decomposition and the heterogeneous coupling are considered: a natural choice for the cost functional is introduced and it has been proved that its minimization leads to a unique solution for the coupled problem. After, the non overlapping decomposition is taken into account and both homogeneous and heterogeneous coupling are considered. Numerical results are shown for both overlapping and non-overlapping decompositions.

The overlapping situation

For the sake of exposition we consider the case of decompositions by two subdomains Ω_1 and Ω_2, which satisfy

$$\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2, \quad \Omega_1 \cap \Omega_2 \neq \emptyset, \quad \Gamma = \partial\Omega.$$

We define $\Gamma_i = \partial\Omega_i \cap \Gamma$ and $S_i = \partial\Omega_i \setminus \Gamma_i$, for $i = 1, 2$. Then $\Gamma = \Gamma_1 \cup \Gamma_2$. Further we define the differential operators

$$L_1 := -\Delta, \quad L_2 := \sigma^2\Delta^2 - \Delta.$$

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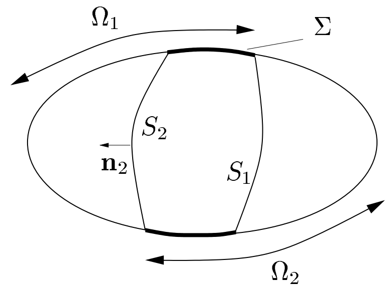

Figure 1: An overlapping decomposition of Ω in two subdomains.

The heterogeneous coupling by means of virtual control is formulated as follows:

$$\begin{cases} L_1 u_1 = f & \text{in } \Omega_1 \\ u_1 = g & \text{on } \Gamma_1 \\ u_1 = \lambda_1 & \text{on } S_1 \end{cases} \qquad \begin{cases} L_2 u_2 = f & & \text{in } \Omega_2 \\ u_2 = g, & \mathbf{n}\cdot\nabla u_2 = h & \text{on } \Gamma_2 \\ u_2 = \lambda_2, & \mathbf{n}_2\cdot\nabla u_2 = \mu_2 & \text{on } S_2 \end{cases} \tag{2}$$

where $\partial\Omega_i = S_i \cup \Gamma_i$, for $i = 1, 2$ (see Figure 1) and $\mathbf{n}_2$ is the unit outward normal vector on S_2.

The functions λ_1, λ_2 and μ_2 are the *virtual controls*. They are chosen in such a way that u_1 and u_2 "adjust" in the best possible way on the overlap $\Omega_1 \cap \Omega_2$. To this aim we introduce the cost functional

$$J(\lambda_1, \lambda_2, \mu_2) = \frac{1}{2}\int_{\Omega_1\cap\Omega_2} (u_1(\lambda_1) - u_2(\lambda_2, \mu_2))^2 \, d\Omega,$$

and consider the minimization problem:

$$\inf_{\lambda_1,\lambda_2,\mu_2} J(\lambda_1, \lambda_2, \mu_2). \tag{3}$$

This problem has a unique solution. Indeed, let us rewrite the solutions u_1 and u_2 of (2) as

$$u_1 = u_1^0 + v_1, \quad u_2 = u_2^0 + v_2,$$

where u_1^0 depends on the data f and g, u_2^0 depends on f, g and h, v_1 depends on λ_1, v_2 depends on λ_2 and μ_2, and satisfy:

$$\begin{array}{llllll} L_1 u_1^0 = f & \text{in } \Omega_1, & u_1^0 = g & \text{on } \Gamma_1, & u_1^0 = 0 & \text{on } S_1, \\ L_1 v_1 = 0 & \text{in } \Omega_1, & v_1 = 0 & \text{on } \Gamma_1, & v_1 = \lambda_1 & \text{on } S_1, \end{array} \tag{4}$$

and

$$\begin{array}{lllll} L_2 u_2^0 = f & \text{in } \Omega_2, & u_2^0 = g, & \mathbf{n}\cdot\nabla u_2^0 = h & \text{on } \Gamma_2, \\ & & u_2^0 = 0, & \mathbf{n}_2\cdot\nabla u_2^0 = 0 & \text{on } S_2, \\ L_2 v_2 = 0 & \text{in } \Omega_2, & v_2 = 0, & \mathbf{n}\cdot\nabla v_2 = 0 & \text{on } \Gamma_2, \\ & & v_2 = \lambda_2 & \mathbf{n}_2\cdot\nabla v_2 = \mu_2 & \text{on } S_2. \end{array} \tag{5}$$

Then

$$J(\lambda_1, \lambda_2, \mu_2) = \frac{1}{2} Q(\lambda_1, \lambda_2, \mu_2) + \mathcal{L}(\lambda_1, \lambda_2, \mu_2),$$

where the quadratic functional Q is given by

$$Q(\lambda_1, \lambda_2, \mu_2) = \int_{\Omega_1 \cap \Omega_2} (v_1 - v_2)^2 \, d\Omega,$$

while $\mathcal{L}$ is an affine functional. Consequently, if the functions λ_i and μ_2 are smooth enough, one can define a semi-norm

$$|||\{\lambda_1, \lambda_2, \mu_2\}||| = (Q(\lambda_1, \lambda_2, \mu_2))^{1/2}, \tag{6}$$

on the space of $\{\lambda_1, \lambda_2, \mu_2\}$.
Actually, this is a *norm*. Indeed if $Q(\lambda_1, \lambda_2, \mu_2) = 0$, then $v_1 = v_2 = v$ in $\Omega_1 \cap \Omega_2$. ¿From (4) we know that $\Delta v = 0$ in $\Omega_1 \cap \Omega_2$, and $v = 0$ on $\Sigma = \partial(\Omega_1 \cap \Omega_2) \cap \partial\Omega$. Moreover, from (5) we obtain that $\mathbf{n} \cdot \nabla v = 0$ on Σ too. Thus by the continuation theorem it follows that $v \equiv 0$ in $\Omega_1 \cap \Omega_2$. Then $\lambda_1 = \lambda_2 = \mu_2 = 0$ which leads to the conclusion that (6) is a norm.

Therefore, if all data are smooth enough, $\inf J(\lambda_1, \lambda_2, \mu_2)$ admits a solution in the space of $\{\lambda_1, \lambda_2, \mu_2\}$ obtained by completion for the norm (6).

Numerical results for the overlapping heterogeneous decomposition

In order to approximate the fourth order problem by Galerkin method with Lagrangian polynomials, we consider a mixed formulation of problem (1). For the sake of simplicity we consider homogeneous boundary data, that is $g \equiv 0$ and $h \equiv 0$. The mixed formulation we have adopted reads as follows. Given $f \in L^2(\Omega)$, find $(u, w) \in V := H_0^1(\Omega) \times H^1(\Omega)$:

$$\begin{cases} (\nabla u, \nabla z)_\Omega - \sigma(\nabla w, \nabla z)_\Omega = (f, z)_\Omega & \forall z \in H_0^1(\Omega) \\ \sigma(\nabla u, \nabla v)_\Omega + (w, v)_\Omega = 0 & \forall v \in H^1(\Omega), \end{cases} \tag{7}$$

where $(\cdot, \cdot)_\Omega$ denotes the L_2 inner product in Ω.

Remark 1 *Let us set*

$$\mathcal{A}(u, w; z, v) = (\nabla u, \nabla z)_\Omega - \sigma(\nabla w, \nabla z)_\Omega + \sigma(\nabla u, \nabla v)_\Omega + (w, v)_\Omega.$$

$\mathcal{A}$ is continuous over the space V and is positive over the space $H_0^1(\Omega) \times L^2(\Omega)$. In fact $\mathcal{A}(u, w; u, w) = \|\nabla u\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}$. Then, if the solution of (7) exists, it is unique. On the other hand, the weak form of problem (1) reads: find $u \in H_0^2(\Omega)$ such that:

$$\sigma^2(\Delta u, \Delta v)_\Omega + (\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H_0^2(\Omega).$$

Existence and uniqueness of u follows by Lax-Milgram Lemma. Moreover, $u \in H^4(\Omega)$ (if Ω is regular enough) and the couple $(u, w = \sigma\Delta u)$ is a solution to problem (7).

In order to formulate the mixed heterogeneous problem we define:
$\mathring{V}_2 = H_0^1(\Omega_2) \times H^1(\Omega_2)$, $\mathring{W}_1 = H_0^1(\Omega_1)$, $V_2 = H^1_{\Gamma_2}(\Omega_2) \times H^1(\Omega_2)$, $W_1 = H^1_{\Gamma_1}(\Omega_1)$ where

$H^1_{\Gamma_i}(\Omega_i) = \{v \in H^1(\Omega_i) : \ v_{|\Gamma_i} = 0\}$. Then we solve the minimization problem (3) where $u_1 \in W_1$, $(u_2, w_2) \in V_2$ are the solutions to the following problem

$$\begin{cases} (\nabla u_2, \nabla z)_{\Omega_2} - \sigma(\nabla w_2, \nabla z)_{\Omega_2} = (f, z)_{\Omega_2} & \forall z \in H^1_0(\Omega_2) \\ \sigma(\nabla u_2, \nabla v)_{\Omega_2} + (w_2, v)_{\Omega_2} = \sigma \int_{S_2} \mu_2 v ds & \forall v \in H^1(\Omega_2) \\ (\nabla u_1, \nabla z)_{\Omega_1} = (f, z)_{\Omega_1} & \forall z \in H^1_0(\Omega_1) \\ u_1 = \lambda_1 \text{ on } S_1, \qquad u_2 = \lambda_2 \text{ on } S_2 & \end{cases} \tag{8}$$

The minimization problem (3) is solved by the BFGS Quasi-Newton method with a mixed quadratic and cubic line search procedure ([JS96]), while we use a Galerkin approximation by conformal spectral elements to solve the associated problem (8).

We have considered the following domain and its decomposition:

$$\Omega = (-1, 1)^2, \ \Omega_1 = (-1, .5) \times (-1, 1), \ \Omega_2 = (0, 1) \times (-1, 1).$$

The right-hand side and the boundary data are chosen so that the analytical solution is $u(x, y) = (x^2 - 1)e^y + (y^2 - 1)e^x$.

In Ω_1 we have considered 3×2 equal spectral elements, while in Ω_2 2×2 equal spectral elements. If not otherwise specified, the polynomial degree has been set $N = 4$.

In order to assess numerically the above theory, we consider the following error terms, that we show in Table 1. The minimum value attained by the functional $J(\lambda_1, \lambda_2, \mu_2)$: $\hat{J}$; the maximum interface errors and the H^2-norm errors for $i = 1, 2$:

$$s_i := \|u_1 - u_2\|_{L^\infty(S_i)}, \quad \mathcal{E}(u)_i = \frac{\|u_i - u\|_{H^2(\Omega_i)}}{\|u\|_{H^2(\Omega_i)}}, \quad \mathcal{E}(u_N)_i = \frac{\|u_i - u_N\|_{H^2(\Omega_i)}}{\|u_N\|_{H^2(\Omega_i)}}, \tag{9}$$

where u is the analytical solution of the global fourth-order problem (1), u_i are the numerical solutions of the virtual control problem (3) and u_N is the spectral element solution of the discretized global fourth order problem (1).

σ	s_1	s_2	$\hat{J}$	$\mathcal{E}(u)_1$	$\mathcal{E}(u)_2$	$\mathcal{E}(u_N)_1$	$\mathcal{E}(u_N)_2$
1	1.90e-1	9.92e-2	6.58e-4	1.95	2.02	1.96	2.02
10^{-2}	3.64e-4	2.97e-3	1.47e-7	1.04e-3	3.08e-2	2.74e-4	3.08e-2
10^{-4}	1.28e-6	1.23e-6	6.36e-14	1.02e-3	6.96e-4	3.33e-6	3.62e-6
10^{-6}	1.25e-6	1.13e-6	6.18e-14	1.02e-3	6.96e-4	3.33e-6	1.06e-6

Table 1: Numerical results for the heterogeneous coupling with overlap.

We note that the minimum value attained by the functional J_1 tends to zero when the coefficient σ tends to zero, as well as the jumps of the solution across the interfaces. The H^2-norm errors are bounded from below by the discretization error, which depends on the spectral polynomial degree N.

The non overlapping situation

We consider now a decomposition by two disjoint subdomains Ω_1 and Ω_2 and a unique interface $S = \partial\Omega_1 \cap \partial\Omega_2$. Again, $\Gamma_i = \partial\Omega_i \cap \partial\Omega$ for $i = 1, 2$ (see Figure 2).

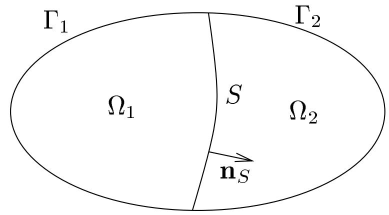

Figure 2: A partition of Ω in two disjoint subdomains.

The *homogeneous coupling* for the fourth order problem (1) would read as follows: we look for λ, μ on S which solve the minimization problem

$$\inf_{\lambda,\mu} J(u_1(\lambda,\mu), u_2(\lambda,\mu)) \tag{10}$$

where u_1 and u_2 satisfy:

$$\begin{cases} L_2 u_1 = f & & \text{in } \Omega_1 \\ u_1 = g, & \mathbf{n}\cdot\nabla u_1 = h & \text{on } \Gamma_1 \\ u_1 = \lambda, & \mathbf{n}_S\cdot\nabla u_1 = \mu & \text{on } S \end{cases} \qquad \begin{cases} L_2 u_2 = f & & \text{in } \Omega_2 \\ u_2 = g, & \mathbf{n}\cdot\nabla u_2 = h & \text{on } \Gamma_2 \\ u_2 = \lambda, & \mathbf{n}_S\cdot\nabla u_2 = \mu & \text{on } S, \end{cases} \tag{11}$$

and $\mathbf{n}_S$ is the unit normal vector on S directed from Ω_1 to Ω_2.

The most natural choice of the cost functional is

$$\begin{aligned} J_1(\lambda,\mu) = \quad & \frac{1}{2}\int_S \Big[(u_1-u_2)^2 + \left(\frac{\partial u_1}{\partial n_S} - \frac{\partial u_2}{\partial n_S}\right)^2 \\ & + (\Delta u_1 - \Delta u_2)^2 + \left(\frac{\partial \Delta u_1}{\partial n_S} - \frac{\partial \Delta u_2}{\partial n_S}\right)^2\Big] ds \end{aligned} \tag{12}$$

where both u_1 and u_2 depend on the virtual controls λ and μ and $\partial/\partial n_S$ stands for $\mathbf{n}_S\cdot\nabla$.

Remark 2 *The choice of the functional J_1 is justified by the fact that the global solution of problem (7), which annihilates the right hand side of (12), is looked for in $H^1_0(\Omega)\times H^1(\Omega)$.*

Another possible choice for the cost functional is obtained by looking at the mixed formulation of problem (11) that we are going to introduce. For $i = 1, 2$ we define $\mathring{V}_i = H^1_0(\Omega_i)\times H^1(\Omega_i)$ and $V_i = H^1_{\Gamma_i}(\Omega_i)\times H^1(\Omega_i)$. The mixed approach for the homogeneous coupled problem (11) reads: find $(u_i, w_i)\in V_i$ for $i = 1, 2$ such that:

$$(\nabla u_1, \nabla z_1)_{\Omega_1} - \sigma(\nabla w_1, \nabla z_1)_{\Omega_1} = (f, z_1)_{\Omega_1} \qquad \forall z_1 \in H^1_0(\Omega_1) \tag{13}$$

$$\sigma(\nabla u_1, \nabla v_1)_{\Omega_1} + (w_1, v_1)_{\Omega_1} = \sigma\int_S \mu v_1 \qquad \forall v_1 \in H^1(\Omega_1) \tag{14}$$

$$(\nabla u_2, \nabla z_2)_{\Omega_2} - \sigma(\nabla w_2, \nabla z_2)_{\Omega_2} = (f, z_2)_{\Omega_2} \qquad \forall z_2 \in H^1_0(\Omega_2) \tag{15}$$

$$\sigma(\nabla u_2, \nabla v_2)_{\Omega_2} + (w_2, v_2)_{\Omega_2} = -\sigma\int_S \mu v_2 \qquad \forall v_2 \in H^1(\Omega_2) \tag{16}$$

$$u_1 = u_2 = \lambda \qquad \text{on } S, \tag{17}$$

and the virtual controls λ and μ are determined by the minimization problem (10).
The choice of the functional is made based on the following observation. Taking z and $v \in C_0^\infty(\Omega)$ in (7) we obtain by integration by parts

$$-\Delta u + \sigma \Delta w = f \quad x - \text{a.e. in } \Omega \tag{18}$$

$$-\sigma \Delta u + w = 0 \quad x - \text{a.e. in } \Omega. \tag{19}$$

To be more general, let us assume that σ takes two different values σ_1 in Ω_1 and σ_2 in Ω_2. Then let $\varphi \in H_{00}^{1/2}(S)$ and denote by $\tilde{\varphi}_i$ an extension of φ in Ω_i such that $\tilde{\varphi}_i \in H^1(\Omega_i)$, $\tilde{\varphi}_{i|\Gamma_i} = 0$, $\tilde{\varphi}_{i|S} = \varphi$, $i = 1, 2$. Then, taking

$$z = \begin{cases} \tilde{\varphi}_1 & \text{in } \Omega_1 \\ \tilde{\varphi}_2 & \text{in } \Omega_2 \end{cases}$$

in (7) and using (18) we deduce that

$$\int_S \left(\frac{\partial u_1}{\partial n_S} - \sigma_1 \frac{\partial w_1}{\partial n_S} \right) \varphi - \int_S \left(\frac{\partial u_2}{\partial n_S} - \sigma_2 \frac{\partial w_2}{\partial n_S} \right) \varphi = 0 \quad \forall \varphi \in H_{00}^{1/2}(S). \tag{20}$$

Proceeding in a similar way in the second equation of (7), this time using (19), we obtain that

$$\int_S \left(\sigma_1 \frac{\partial u_1}{\partial n_S} - \sigma_2 \frac{\partial u_2}{\partial n_S} \right) \varphi = 0 \quad \forall \varphi \in H_{00}^{1/2}(S). \tag{21}$$

This latter condition is implicitly guaranteed by having chosen the same multiplier μ in (14) and (16). On the other hand, since problem (13)-(17) guarantees neither the continuity of w across the interface nor the transmission condition (20), we look for these properties by choosing the following cost functional

$$J_2(\lambda, \mu) = \frac{1}{2} \int_S \left[(w_1 - w_2)^2 + \left(\left(\frac{\partial u_1}{\partial n_S} - \sigma \frac{\partial w_1}{\partial n_S} \right) - \left(\frac{\partial u_2}{\partial n_S} - \sigma \frac{\partial w_2}{\partial n_S} \right) \right)^2 \right]$$

In Table 2 we show the numerical results obtained by the minimization of functional J_1, versus the coefficient σ. The quantities s_{du}, s_w and s_{dw} stand for the maximum norm of the jumps of $\partial u / \partial n_S$, w and $\partial w / \partial n_S$ on S, respectively, while $\hat{J}_1$ is the minimum value achieved by the cost functional J_1. Moreover $\mathcal{E}(u)_i$ and $\mathcal{E}(u_N)_i$ (for $i = 1, 2$) are the errors defined in (9). The jump of u on S is not reported since it is always of the same order of the machine precision.

σ	s_{du}	s_w	s_{dw}	$\hat{J}_1$	$\mathcal{E}(u)_1$	$\mathcal{E}(u)_2$	$\mathcal{E}(u_N)_1$	$\mathcal{E}(u_N)_2$
1.	3.71e-5	2.85e-5	4.73e-4	5.88e-08	9.80e-4	6.97e-4	6.21e-6	3.80e-6
10^{-2}	9.81e-7	2.72e-5	1.54e-5	2.22e-10	9.79e-4	6.96e-4	1.55e-7	1.20e-7
10^{-4}	2.03e-8	5.12e-7	2.26e-6	1.01e-12	9.79e-4	6.96e-4	1.20e-6	8.44e-7
10^{-6}	8.00e-5	1.47e-7	2.26e-8	3.41e-09	9.79e-4	6.96e-4	1.17e-6	8.23e-7

Table 2: Numerical results for the homogeneous coupling without overlap. Minimization of the functional J_1.

σ	s_{du}	s_w	s_{dw}	$\hat{J}_2$	$\mathcal{E}(u)_1$	$\mathcal{E}(u)_2$	$\mathcal{E}(u_N)_1$	$\mathcal{E}(u_N)_2$
1.	2.15e-5	2.83e-5	2.10e-4	1.20e-08	9.79e-4	6.96e-4	2.49e-6	2.47e-6
10^{-2}	3.31e-6	3.98e-6	2.07e-4	7.51e-12	9.79e-4	6.96e-4	1.31e-6	9.18e-7
10^{-4}	4.62e-6	5.32e-6	2.23e-6	1.12e-11	9.79e-4	6.96e-4	1.07e-6	7.49e-7
10^{-6}	8.00e-5	2.92e-7	2.27e-8	3.41e-09	9.79e-4	6.96e-4	1.18e-6	8.27e-7

Table 3: Numerical results for the homogeneous coupling without overlap. Minimization of the functional J_2.

σ	s_{du}	s_ϕ	$\hat{J}_3$	$\mathcal{E}(u)_1$	$\mathcal{E}(u)_2$	$\mathcal{E}(u_N)_1$	$\mathcal{E}(u_N)_2$
1.	8.28	2.47e-4	1.76e-02	1.82	1.78e-1	1.82	1.78e-1
10^{-2}	1.50e-1	8.62e-5	5.89e-05	1.13e-3	3.08e-2	5.63e-4	3.08e-2
10^{-4}	1.60e-5	3.08e-7	6.83e-09	9.79e-4	6.96e-4	1.13e-6	3.41e-6
10^{-6}	2.68e-7	2.69e-7	7.05e-13	9.79e-4	6.96e-4	1.19e-6	8.31e-7

Table 4: Numerical results for the heterogeneous coupling without overlap. Minimization of the functional J_3.

Remark 3 *When the functional J_1 is replaced by a simpler functional in which the terms depending on u_i are dropped, similar results to those of Table 2 are obtained.*

In Table 3 we show the numerical results obtained by the minimization of functional J_2.

The *heterogeneous coupling* for non overlapping situations reads as (8), where we use the virtual controls μ instead of μ_2 and a single control λ instead of λ_1 and λ_2 and then we solve the minimization problem (10). In this case we choose the following cost functional:

$$J_3(\lambda,\mu) = \quad \frac{1}{2}\int_S \Big[\left(\frac{\partial u_1}{\partial n_S} - \frac{\partial u_2}{\partial n_S} + \sigma \frac{\partial w_2}{\partial n_S} \right)^2 + \left(\sigma \frac{\partial u_2}{\partial n_S} \right)^2 \Big] ds.$$

Note that through the minimization of J_3 we are enforcing the fulfillment of the matching conditions (20) and (21) where, this time, we have taken $\sigma_1 = 0$.

In Table 4 we show the numerical results obtained by the minimization of functional J_3 on the heterogeneous coupling without overlap. In particular we define $s_\phi = \|\partial u_1/\partial n_S - \partial u_2/\partial n_S + \sigma \partial w_2/\partial n_S\|_{L^\infty(S)}$.

As for the overlapping case, we note that the minimum value attained by the functional J_3 tends to zero when the coefficient σ tends to zero, as well as the jump of the normal derivative of u across the interface S. Again, the H^2-norm errors are bounded from below by the discretization error, which depends on the spectral polynomial degree N.

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