Mortar coupling for heterogeneous partial differential equations

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1 Introduction

We are interested in the approximation of 2D elliptic equations with dominated advection and featuring boundary layers. In order to reduce the computational complexity, the domain is split into two subregions, the first one far from the layer, where we can neglect the viscosity effects, and the second one next to the layer. In the latter domain the original elliptic equation is solved, while in the former one, the pure convection equation obtained by the original one by dropping the diffusive term is approximated. The interface coupling is enforced by the non-conforming mortar method. We consider two different sets of interface conditions and we compare them for what concerns both computational efficiency and stability. One of the two sets of interface conditions turns out to be very effective, especially for very small viscosity when the mortar formulation of the original elliptic problem on the global domain can fail.

2 The heterogeneous problem

We consider an open bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, split into two open subsets Ω_1 and Ω_2 such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, $\Omega_1 \cap \Omega_2 = \emptyset$. Then, we denote by $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$, the interface between the sub domains and we assume that Γ is of class $C^{1,1}$. Given $f \in L^2(\Omega)$, $b_0 \in L^\infty(\Omega)$, $\nu \in L^\infty(\Omega_2 \cup \Gamma)$ and $\mathbf{b} \in [W^{1,\infty}(\Omega)]^2$ satisfying the following inequalities:

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$$\exists \nu_0 \in \mathbb{R} \text{ such that } \nu(\mathbf{x}) \geq \nu_0 > 0, \forall \mathbf{x} \in \Omega_2 \cup \Gamma,$$
$$\exists \sigma_0 \in \mathbb{R} \text{ such that } b_0(\mathbf{x}) + \tfrac{1}{2}\text{div}\mathbf{b}(\mathbf{x}) \geq \sigma_0 > 0, \forall \mathbf{x} \in \Omega,$$

we look for two functions u_1 and u_2 (defined in $\overline{\Omega}_1$ and $\overline{\Omega}_2$, respectively) solutions of the *heterogeneous problem*

$$\begin{cases} \text{div}(\mathbf{b}u_1) + b_0 u_1 = f & \text{in } \Omega_1, \\ \text{div}(-\nu\nabla u_2 + \mathbf{b}u_2) + b_0 u_2 = f & \text{in } \Omega_2, \\ u_1 = 0 & \text{on } (\partial\Omega_1 \setminus \Gamma)^{in} \\ u_2 = 0 & \text{on } \partial\Omega_2 \setminus \Gamma \end{cases} \tag{1}$$

and satisfying the interface conditions

$$u_1 = u_2 \text{ on } \Gamma^{in}, \qquad \mathbf{b}\cdot\mathbf{n}_\Gamma u_1 + \nu\frac{\partial u_2}{\partial \mathbf{n}_\Gamma} - \mathbf{b}\cdot\mathbf{n}_\Gamma u_2 = 0, \text{ on } \Gamma. \tag{2}$$

$\mathbf{n}_\Gamma$ denotes the normal unit vector to Γ oriented from Ω_1 to Ω_2, while for any non-empty subset $S \subseteq \partial\Omega_1$, $S^{in} = \{\mathbf{x} \in S : \mathbf{b}(\mathbf{x})\cdot\mathbf{n}_1(\mathbf{x}) < 0\}$ and $S^{out} = \{\mathbf{x} \in S : \mathbf{b}(\mathbf{x})\cdot\mathbf{n}_1(\mathbf{x}) \geq 0\}$ are the *inflow* and the *outflow* parts of S, respectively.

Equations (2) (named IC1) express the continuity of the velocity field across the *inflow* part of the interface and the continuity of the fluxes across the whole interface. They can be equivalently expressed as (named IC2):

$$u_1 = u_2, \; \nu\frac{\partial u_2}{\partial \mathbf{n}_\Gamma} = 0 \text{ on } \Gamma^{in}, \quad -\mathbf{b}\cdot\mathbf{n}_\Gamma u_1 = \nu\frac{\partial u_2}{\partial \mathbf{n}_\Gamma} - \mathbf{b}\cdot\mathbf{n}_\Gamma u_2 \text{ on } \Gamma^{out}. \tag{3}$$

Problem (1) with either interface conditions (2) or (3) is well-posed, see [5].

The heterogeneous problem (1), with either interface conditions IC1 or IC2, can formally be written as an interface problem by means of Steklov-Poincaré operators (see, e.g., [5, 3]). Let us define the trace spaces $\Lambda_1 = L_\mathbf{b}^2(\Gamma^{in}) = \{v : \Gamma^{in} \to \mathbb{R} : \sqrt{|\mathbf{b}\cdot\mathbf{n}_\Gamma|}v \in L^2(\Gamma^{in})\}$ and $\Lambda_2 = H_{00}^{1/2}(\Gamma^{in}) = \{v : L^2(\Gamma^{in}) : \exists \tilde{v} \in H^{1/2}(\partial\Omega_2) : \tilde{v}|_{\Gamma^{in}} = v, \tilde{v}|_{\partial\Omega_2 \setminus \Gamma^{in}} = 0\}$.

Solving (1),(2) is equivalent to seeking $\lambda_k \in \Lambda_k$ for $k = 1, 2$, such that

$$\begin{cases} \mathcal{S}_1\lambda_1 + \mathcal{S}_2\lambda_2 = \chi_1 + \chi_2 & \text{in } \Lambda_2', \\ \lambda_1 = \lambda_2|_{\Gamma^{in}} & \text{in } \Lambda_2, \end{cases} \tag{4}$$

where

$$\mathcal{S}_1\lambda_1 = -\mathbf{b}\cdot\mathbf{n}_1 u_1^{\lambda_1}, \quad \mathcal{S}_2\lambda_2 = \nu\frac{\partial u_2^{\lambda_2}}{\partial \mathbf{n}_2} - \mathbf{b}\cdot\mathbf{n}_2 u_2^{\lambda_2}, \quad \text{on } \Gamma, \tag{5}$$

are the local Steklov-Poincaré operators, while $u_1^{\lambda_1}$ and u_2^{λ} are the solution of

$$\begin{cases} \text{div}(\mathbf{b}u_1^{\lambda_1}) + b_0 u_1^{\lambda_1} = 0 & \text{in } \Omega_1, \\ u_1^{\lambda_1} = 0 \text{ on } (\partial\Omega_1 \setminus \Gamma)^{in}, \quad u_1^{\lambda_1} = \lambda \text{ on } \Gamma^{in}, \end{cases} \tag{6}$$

and

$$\begin{cases} \operatorname{div}(-\nu\nabla u_2^{\lambda_2} + \mathbf{b}u_2^{\lambda_2}) + b_0 u_2^{\lambda_2} = 0 \text{ in } \Omega_2 \\ u_2^{\lambda_2} = 0 \text{ on } \partial\Omega_2 \setminus \Gamma, \quad u_2^{\lambda_2} = \lambda_2 \text{ on } \Gamma, \end{cases} \tag{7}$$

respectively. Finally,

$$\chi_1 = \mathbf{b}\cdot\mathbf{n}_1 u_1^f, \quad \chi_2 = -\nu\frac{\partial u_2^f}{\partial\mathbf{n}_2} + \mathbf{b}\cdot\mathbf{n}_2 u_2^f = -\nu\frac{\partial u_2^f}{\partial\mathbf{n}_2}, \tag{8}$$

where u_1^f and u_2^f are the solutions of problems like (6) and (7), respectively, with null trace on the interface and external load f. Note that $\chi_1|_{\Gamma^{in}} = 0$.

If interface conditions IC2 are considered instead of IC1, the resulting Steklov-Poincaré equation reads: seek $\lambda_k \in \Lambda_k$, for $k = 1, 2$ such that

$$\begin{cases} \mathcal{S}_1^0\lambda_1 + \mathcal{S}_2^0\lambda_2 = \chi_1 + \chi_2 \text{ in } \Lambda_2' \\ \lambda_1 = \lambda_2|_{\Gamma^{in}} \qquad\qquad\quad \text{in } \Lambda_2 \end{cases} \tag{9}$$

where

$$\mathcal{S}_1^0\lambda_1 = \begin{cases} 0 & \text{on } \Gamma^{in} \\ -\mathbf{b}\cdot\mathbf{n}_1 u_1^{\lambda_1} & \text{on } \Gamma^{out}, \end{cases} \quad \mathcal{S}_2^0\lambda_2 = \begin{cases} \nu\dfrac{\partial u_2^{\lambda_2}}{\partial\mathbf{n}_2} & \text{on } \Gamma^{in} \\ \nu\dfrac{\partial u_2^{\lambda_2}}{\partial\mathbf{n}_2} - \mathbf{b}\cdot\mathbf{n}_2 u_2^{\lambda_2} & \text{on } \Gamma^{out}. \end{cases} \tag{10}$$

Remark 1. It is straightforward to prove that the operator $\mathcal{S}_2^0$ is always coercive on Λ_2, whereas $\mathcal{S}_2$ is coercive only if smallness assumption on $\mathbf{b}$ is assumed. If, e.g.,

$$\|\mathbf{b}\|_{L^\infty(\Gamma)} \le \varepsilon_0, \text{ with } 0 \le \varepsilon_0 \le 2\min\{\nu_0, \sigma_0\}/C_*^2, \tag{11}$$

(where C_* is the constant of the trace inequality $\|v\|_{L^2(\partial\Omega_2)} \le C_*\|v\|_{H^1(\Omega_2)}$) is satisfied then $\mathcal{S}_2$ is coercive on Λ_2. For this reason, the solution of problem (4) may produce oscillations around Γ^{in} when advection dominates (i.e. the global Péclet number is large), as will be shown later in our numerical results.

3 Mortar coupling for Spectral Element discretization

The discretization of the differential equation within each sub domain is performed by the quadrilateral conforming Spectral Element Method (SEM). We refer to [4] for a detailed description of this method. For $k = 1, 2$, let $\mathcal{T}_k = \{T_{k,m}\}_{m=1}^{M_k}$ be a partition of the computational domain $\Omega_k \subset \mathbb{R}^2$. The SEM finite dimensional space on $\overline{\Omega}_k$ is denoted by X_{k,δ_k} and it is the set of functions in $C^0(\overline{\Omega}_k)$ whose restriction to $T_{k,m}$ is a polynomial of degree N_k in each direction. δ_k is an abridged notation for "discrete", that accounts for

the local geometric sizes $h_{k,m}$ of $T_{k,m}$ and the local polynomial degrees N_k along each direction. Both geometric and polynomial conformity is guaranteed inside $\overline{\Omega}_k$.

The finite dimensional spaces in which we look for the SEM solution of either (4) or (9) are: $\Lambda_{1,\delta_1} \subset \Lambda_1$ and $\Lambda_{2,\delta_2} \subset \Lambda_2$. Their elements are globally continuous functions on Γ^{in} and Γ, respectively, and local polynomials of degree N_k on each edge induced by the partition $\mathcal{T}_k$.

For $k = 1, 2$, we denote by $\mathcal{N}_{k,\Gamma}$ the set of nodes of $\mathcal{T}_k \cap \Gamma$ whose cardinality is $N_{k,\Gamma}$. Similar notations are used for the nodes lying on either Γ^{in} or Γ^{out}.

The finite dimensional basis $\{\mu_1^{(i)}\}_{i=1}^{N_{1,\Gamma^{in}}}$ of Λ_{1,δ_1} ($\{\mu_2^{(i)}\}_{i=1}^{N_{2,\Gamma}}$ of Λ_{2,δ_2}, resp.) is composed by the characteristic Lagrange polynomials in Ω_1 (Ω_2, resp.) associated to the Legendre-Gauss-Lobatto (LGL) nodes of $\mathcal{N}_{1,\Gamma^{in}}$ ($\mathcal{N}_{2,\Gamma}$, resp.). Then we set $(S_{2,\delta_2})_{ij} = \int_\Gamma \mathcal{S}_2 \mu_2^{(j)} \mu_2^{(i)} d\Gamma$ for $i, j = 0, \dots, N_{2,\Gamma}$, and analogous notations are used to define matrices S_{2,δ_2}^0, S_{1,δ_1} and S_{1,δ_1}^0. Because of the high cost to compute integrals exactly, all integrals are approximated by Legendre-Gauss-Lobatto (LGL) quadrature rules.

We consider *non-conforming couplings*, i.e. we suppose that either the two partitions $\mathcal{T}_1$ and $\mathcal{T}_2$ do not share the same edges on Γ and/or the polynomial degrees do not coincide in the hyperbolic domain Ω_1 and in the elliptic one Ω_2. We adopt mortar methods (see, e.g., [2]) to glue non-conforming discretization across Γ.

The endpoints of the edges of $\mathcal{T}_1 \cap \Gamma^{in}$ are denoted by $v_1^{(i)}$, for $i = 1, \dots, N_{1,v}$. $\tilde{\Lambda}_{1,\delta_1}$ is a suitable finite dimensional space of functions living on Γ^{in} and its basis functions ψ_l are characterized by being L^2 functions on Γ^{in} and local polynomials of degree $N_1 - 2$ on each edge of $\mathcal{T}_1 \cap \Gamma^{in}$. Therefore, the dimension of $\tilde{\Lambda}_{1,\delta_1}$ is $N_{\tilde{\Lambda}_1} = N_{1,\Gamma^{in}} - N_{1,v}$. By choosing Ω_2 as the master domain and Ω_1 as the slave, the continuity constraint $\lambda_1 = \lambda_2|_{\Gamma^{in}}$ is imposed weakly, i.e. by requiring that

$$\int_{\Gamma^{in}} (\lambda_{1,\delta_1} - \lambda_{2,\delta_2}) \psi_l d\Gamma = 0 \qquad \forall \psi_l \in \tilde{\Lambda}_{1,\delta_1}, \tag{12}$$

jointly with the strong continuity at the nodes $v_1^{(i)}$ of $\mathcal{T}_1 \cap \Gamma^{in}$, for $i = 1, \dots, N_{1,v}$. This leads us to define a new set of *mortar* functions in Λ_{1,δ_1}, which are denoted by $\tilde{\mu}_1^{(k)}$ (for $k = 1, \dots, N_{2,\Gamma^{in}}$) and satisfy the constraints:

$$\begin{cases} \tilde{\mu}_1^{(k)}(v_1^{(i)}) = \mu_2^{(k)}(v_1^{(i)}), & i = 1, \dots, N_{1,v} \text{ and } v_1^{(i)} \text{ being endpoint} \\ & \text{of at least one edge of } \mathcal{T}_1 \cap \Gamma^{in} \\ \int_{\Gamma^{in}} (\tilde{\mu}_1^{(k)} - \mu_2^{(k)}) \psi_l d\Gamma = 0, & l = 1, \dots, N_{\tilde{\Lambda}_1} \text{ and for all } \psi_l \in \tilde{\Lambda}_{1,\delta_1}. \end{cases} \tag{13}$$

Remark 2. We choose Ω_2 as the master domain because the nature of the heterogeneous problem requires to work with the trace of the elliptic solution on the whole interface and with the trace of the hyperbolic one only on Γ^{in}.

Therefore it is more convenient to have the master trace at disposal on the whole Γ, instead of on a part of it.

The matrix form of system (13) reads

$$P\Xi = \Phi, \tag{14}$$

where $\Xi = [\xi_{jk}] \in \mathbb{R}^{N_{1,\Gamma^{in}} \times N_{2,\Gamma^{in}}}$ is defined by the relations

$$\tilde{\mu}_1^{(k)} = \sum_{j=1}^{N_{1,\Gamma^{in}}} \xi_{jk} \mu_1^{(j)}, \qquad k = 1, \ldots, N_{2,\Gamma^{in}}, \tag{15}$$

while $P \in \mathbb{R}^{N_{1,\Gamma^{in}} \times N_{1,\Gamma^{in}}}$ and $\Phi \in \mathbb{R}^{N_{1,\Gamma^{in}} \times N_{2,\Gamma^{in}}}$, are defined starting from (13). The matrix P is non-singular in view of the inf-sup condition for $\mathbb{Q}_N - \mathbb{Q}_{N-2}$ ([2]). Once the discretization in Ω_1 and Ω_2 has been chosen, the matrix Ξ can be explicitly computed by solving (14).

The matrix Ξ enforces the gluing between degrees of freedom defined on $\mathcal{N}_{2,\Gamma^{in}}$ and $\mathcal{N}_{1,\Gamma^{in}}$. Therefore, Steklov-Poincaré equations (4) and (9) can be written in a nonconforming setting, by the use of matrix Ξ.

On Γ^{out} no continuity constraint, neither strong nor weak, is imposed, since the continuity of fluxes is a natural consequence of the interface equation. Nevertheless, on Γ^{out} we have to compute integrals of basis functions associated to two different meshes. To this aim we introduce the matrix $Q \in \mathbb{R}^{N_{2,\Gamma^{out}} \times N_{1,\Gamma^{out}}}$ for the evaluations of functions of Λ_{1,δ_1} at the nodes of $\mathcal{T}_2 \cap \Gamma$, and the matrix $D = M_{2,\delta_2}^{out} Q (M_{1,\delta_1}^{out})^{-1}$, where M_{k,δ_k}^{out} are the mass matrices induced by the LGL quadrature formulas on Γ^{out}, for $k = 1, 2$.

The nonconforming finite dimensional counterpart of (4) reads: find $\lambda_{k,\delta_k} \in \Lambda_{k,\delta_k}$ for $k = 1, 2$, such that

$$\begin{cases} \underbrace{\left(S_{2,\delta_2} + \begin{bmatrix} \Xi^T S_{1,\delta_1}^{in} \Xi & 0 \\ DS_{1,\delta_1}^{out} \Xi & 0 \end{bmatrix} \right)}_{S_\delta} \begin{bmatrix} \lambda_{2,\delta_2}^{in} \\ \lambda_{2,\delta_2}^{out} \end{bmatrix} = \begin{bmatrix} M_{2,\delta_2}^{in} \chi_{2,\delta_2}^{in} \\ M_{2,\delta_2}^{out} \chi_{2,\delta_2}^{out} + D\chi_{1,\delta_1}^{out} \end{bmatrix} \\ \lambda_{1,\delta} = \Xi \lambda_{2,\delta_2}^{in} \end{cases} \tag{16}$$

whereas that of (9) becomes: find $\lambda_{k,\delta_k} \in \Lambda_{k,\delta_k}$ for $k = 1, 2$, such that

$$\begin{cases} \underbrace{\left(S_{2,\delta_2}^0 + \begin{bmatrix} 0 & 0 \\ DS_{1,\delta_1}^{out} \Xi & 0 \end{bmatrix} \right)}_{S_\delta^0} \begin{bmatrix} \lambda_{2,\delta_2}^{in} \\ \lambda_{2,\delta_2}^{out} \end{bmatrix} = \begin{bmatrix} M_{2,\delta_2}^{in} \chi_{2,\delta_2}^{in} \\ M_{2,\delta_2}^{out} \chi_{2,\delta_2}^{out} + D\chi_{1,\delta_1}^{out} \end{bmatrix} \\ \lambda_{1,\delta} = \Xi \lambda_{2,\delta_2}^{in}. \end{cases} \tag{17}$$

The upper scripts in and out denote the restriction to Γ^{in} and Γ^{out}, resp.

The numerical solutions of these linear systems is carried out by preconditioned Bi-CGStab iterations (see, [6]).

When conforming discretization is used across the interface (i.e. $\delta_1 = \delta_2$), matrix Ξ reduces to the identity matrix. In this situation, it is well known (see, e.g. [5]) that S_{2,δ_2}^0 is an optimal preconditioner for the matrix S_δ^0, i.e. $\exists C_0 > 0$ independent of δ such that its spectral condition number $\mathcal{K}((S_{2,\delta_2}^0)^{-1} S_\delta^0)$ is bounded by C_0. When $\delta_1 = \delta_2$, S_{2,δ_2}^0 is an optimal preconditioner also for S_δ (see [3]), i.e. there exists $C_1 > 0$ independent of δ such that $\mathcal{K}((S_{2,\delta_2}^0)^{-1} S_\delta) \leq C_1$, and numerical results show that $C_0 \leq C_1$.

We extend here the use of the preconditioner $S_{2,\delta}^0$ to the non-conforming case.

4 Numerical results

Test case: the computational domain $\Omega = (-1, 1)^2$ is split in $\Omega_1 = (-1, 0.8) \times (-1, 1)$ and $\Omega_2 = (0.8, 1) \times (-1, 1)$. The interface is $\Gamma = \{0.8\} \times (-1, 1)$. The data of the problem are: $\mathbf{b} = [5y, 1 - x]^t$, $b_0 = 1$, $f = 1$ and the inflow interface is $\Gamma^{in} = \{0.8\} \times (-1, 0)$. The imposed Dirichlet boundary conditions are: $u_1 = 1$ on $((-1, 0.8) \times \{-1\}) \cup (\{-1\} \times (0, 1))$, $u_2 = 0$ on $\{1\} \times (-1, 1)$, $u_2 = 1$ on $(0.8, 1) \times \{-1\}$, while the homogeneous Neumann condition $\frac{\partial u_2}{\partial \mathbf{n}_2} = 0$ is imposed on $(0.8, 1) \times \{1\}$.

Because of the presence of a boundary layer near the right vertical side, the mesh is refined there (without losing the conformity inside Ω_2) to prevent the numerical solution to be affected by spurious oscillations.

In Fig. 1 the number of Preconditioned Bi-CGStab (PBi-CGStab) iterations (with preconditioner S_{2,δ_2}) required to reduce the relative norm of the residual of 12 orders of magnitude is plotted versus the polynomial degrees N_1 and N_2 of the mortar discretization. These results refer to $\nu = 10^{-2}$ and show that the Steklov-Poincaré formulation (9) performs better than (4). The analysis of this and other test cases leads us to conjecture that $\mathcal{K}((S_{2,\delta_2}^0)^{-1} S_\delta^0) \leq C_0$ still holds for non-conforming coupling ($\delta_1 \neq \delta_2$), while

$$\mathcal{K}((S_{2,\delta_2}^0)^{-1} S_\delta) \simeq C_1 \mathcal{K}(\Xi \Xi^T) \simeq C_1 \begin{cases} (N_2 - N_1 + 1)^{3/2} & \text{if } N_1 < N_2 \\ C_2 & \text{if } N_1 \geq N_2, \end{cases} \quad (18)$$

where C_1 is the constant defined in the previous section, and C_2 is another positive constant independent of δ.

Therefore, formulation (17) corresponding to IC2 is optimally preconditioned by S_{2,δ_2}^0 and it is better than (16) (corresponding to IC1) for what concerns the computational efficiency.

Moreover, when the viscosity vanishes (see Table 1), the performance of the SP0 approach (17) does not downgrade, as the number of PBi-CGStab iterations keeps bounded: 3 or 4 iterations are enough to satisfy the stopping test independently of both viscosity and discretization parameters.
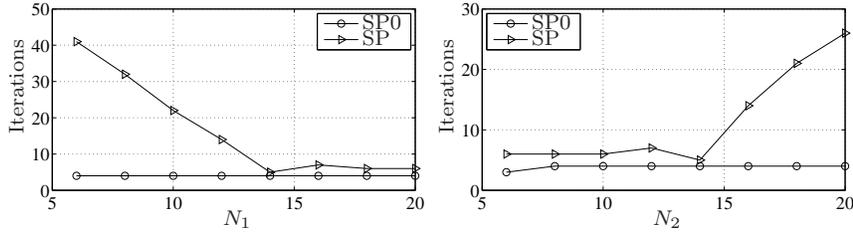
Fig. 1 Preconditioned Bi-CGStab iterations. The viscosity is $\nu = 10^{-2}$. *At left,* $N_2 = 14$ is fixed, *at right,* $N_1 = 14$ is fixed. 4×4 equal spectral elements are taken in each Ω_k

On the contrary, the number of PBi-CGStab iterations required by SP approach (16) noticeably grows up when $\nu \to 0$ and behaves like $(N_2 - N_1 + 1)^{3/4}$ when $N_1 < N_2$, in agreement with (18).

The large number of PBi-CGStab iterations required by SP is due to the presence of instabilities across Γ^{in} which develop when advection dominates and the larger $N_2 - N_1$ is, the more they are pronounced.

We verified that the same instability occurs when mortar methods are applied to solve the pure elliptic-elliptic couplings with dominated advection and interface condition $\nu \frac{\partial u_1}{\partial \mathbf{n}_\Gamma} - \mathbf{b} \cdot \mathbf{n}_\Gamma u_1 = \nu \frac{\partial u_2}{\partial \mathbf{n}_\Gamma} - \mathbf{b} \cdot \mathbf{n}_\Gamma u_2$ on the whole interface Γ. Indeed, the local Steklov-Poincaré operators associated to the latter interface condition behaves like operator $\mathcal{S}_2$ introduced in (5), and they can lose the coercivity when $\|\mathbf{b}\|_{L^\infty(\Omega)}$ is large. This is the subject of a work in progress. (See also[1].)

In conclusion, the heterogeneous approach (1) with interface conditions IC2 and non-conforming mortar coupling turns out to be the most efficient and accurate one for vanishing viscosity and it is also a valid way to overcome instabilities arising from the mortar discretization of elliptic equations with dominated advection.

In Fig. 2 the heterogeneous solutions obtained by solving both (17) and (16) with $\nu = 10^{-4}$, $N_1 = 8$ and $N_2 = 24$ are shown. The elliptic solution u_2 provided by (16) (Fig. 2, right) exhibits non-trivial oscillations, while that provided by (17) (Fig. 2, left) does not.

Table 1 PBi-CGStab iterations to solve systems SP0 (17) and SP (16) with $P = S_{2,\delta_2}^0$ versus the viscosity. *At left,* $N_1 = 8$, *at right,* $N_1 = 20$, $N_2 = 24$. 4×4 equal spectral elements are taken in each Ω_k. $N_2 = 64$ along x-direction in the elements next to the layer

ν	10^{-1}	10^{-2}	10^{-3}	10^{-4}
SP0	3	4	3	3
SP	10	45	262	587

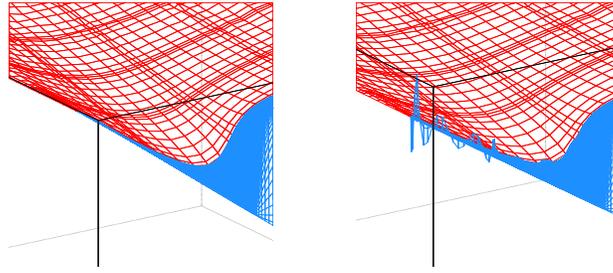| ν	10^{-1}	10^{-2}	10^{-3}	10^{-4}
SP0	3	3	3	4
SP	7	17	35	86

Fig. 2 Zoom on the numerical solution for $\nu = 10^{-3}$ and: (9) (left), (4) (right) with $N_1 = 8$ and $N_2 = 24$. The elliptic solution u_2 is in front, while the hyperbolic one u_1 is behind.

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