The Interface Control Domain Decomposition (ICDD) method for the Stokes problem

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We study the Interface Control Domain Decomposition (ICDD) for the Stokes equation. We reformulate this problem introducing auxiliary control variables that represent either the traces of the fluid velocity or the normal stress across subdomain interfaces. Then, we characterize suitable cost functionals whose minimization permits to recover the solution of the original problem. We analyze the well-posedness of the optimal control problems associated to the different choices of the cost functionals, and we propose a discretization of the problem based on hp finite elements. The effectiveness of the proposed methods is illustrated through several numerical tests.

Keywords: Stokes equations, Domain Decomposition Methods, Optimal control, hp-Finite Elements, ICDD

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1. INTRODUCTION

The Interface Control Domain Decomposition (ICDD) method was introduced in [4, 5] as a solution strategy for boundary value problems governed by elliptic partial differential equations. In this paper we extend this methodology to the Stokes equations and we study its effectiveness in computing the solution of this linear model for laminar incompressible flows.

The ICDD method, which shares some similarities with the classic overlapping Schwarz method [17–19] and with the Least Square Conjugate Gradient [10] and the Virtual Control [13] methods, is characterized by a decomposition of the original domain into overlapping regions and by the introduction of new auxiliary variables on the subdomain interfaces. In the case of the Stokes problem, these variables may represent either the trace of the fluid velocity or the normal stress across the interfaces. In either case, they play the role of control variables that can be determined as solution of an optimal control problem that imposes the minimization of a suitably defined cost functional involving the solutions of well-posed local subproblem.

The ICDD method can thus be regarded as a novel domain decomposition method whose interest lies in the fact that, at least in the case of two subdomains, it may show convergence rates independent of the computational grid, of the polynomial degree used for the numerical approximation and, for a particular choice of the cost functional, also independence on the size of the overlapping. The choice of the cost functional is crucial to ensure the uniqueness of the solution on the overlapping area. In particular, we show that, for the Stokes problem, the cost functionals must account for both the velocity and the pressure across the interfaces to ensure the matching of these two variables in the overlapping regions.

What makes the ICDD method even more attractive is also its capability of handling differential problems of heterogeneous type, i.e., governed by different type of equations in different subregions of the computational domain. Some examples of such application of the method were provided in [4, 5] in the case of advection/advection-diffusion problems. Another interesting problem with many significant applications is the coupling of Stokes and Darcy equation to model filtration processes (see [3, 4, 6, 14]).

The outline of the paper is as follows. In section 2 we write the Stokes problem in a bounded domain and we reformulate it in equivalent ways after splitting the original domain into two overlapping regions. In section 3, after introducing a discretization of the problem using hp finite elements, we present the ICDD method considering the cases of Dirichlet, Neumann and mixed control variables. In each case we write the corresponding optimality system with its algebraic counterpart. In section 4 we present several numerical results aimed at studying the convergence behavior of the proposed ICDD methods with respect to the grid size, the polynomial degree, and the size of the overlapping region. Finally, section 5 is devoted to the theoretical analysis of the different methods.

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2. PROBLEM SETTING

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be an open bounded domain with Lipschitz boundary $\partial\Omega$. We assume that $\overline{\partial\Omega} = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$ and that $\Gamma_D \neq \emptyset$ while Γ_N might be empty. We consider the Stokes problem:

Problem $\mathcal{P}_\Omega$*:*

$$\begin{array}{rcll} -\mathbf{div}\,\mathbf{T}(\mathbf{u}, p) &=& \mathbf{f} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} &=& 0 & \text{in } \Omega \\ \mathbf{u} &=& \boldsymbol{\phi}_D & \text{on } \Gamma_D \\ \mathbf{T}(\mathbf{u}, p)\cdot\mathbf{n} &=& \boldsymbol{\phi}_N & \text{on } \Gamma_N \end{array} \tag{1}$$

describing the motion of a steady, viscous, incompressible fluid confined in the region Ω. Here, $\mathbf{T}(\mathbf{u}, p) = 2\nu\nabla^s\mathbf{u} - p\mathbf{I}$ is the Cauchy stress tensor being $\nabla^s\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, $\nu > 0$ is the fluid viscosity, $\mathbf{u}$ its velocity and p its pressure and $\mathbf{n}$ is the unit normal vector to $\partial\Omega$ directed outwards the domain Ω. We assume that $\mathbf{f} \in [L^2(\Omega)]^d$, $\boldsymbol{\phi}_D \in [H^{1/2}(\Gamma_D)]^d$ and $\boldsymbol{\phi}_N \in [H^{-1/2}(\Gamma_N)]^d$ are assigned functions. If $\partial\Omega \equiv \Gamma_D$ (i.e., $\Gamma_N = \emptyset$), the compatibility condition $\int_{\partial\Omega} \boldsymbol{\phi}_D \cdot \mathbf{n} = 0$ must hold, and a further condition on p, e.g.,

$$\int_\Omega p = 0$$

must be enforced to guarantee the well-posedness of problem (1).

The weak form of problem (1) is: find $\mathbf{u} \in [H^1(\Omega)]^d$, $\mathbf{u} = \boldsymbol{\phi}_D$ on Γ_D, and $p \in L^2(\Omega)$ such that, for all $\mathbf{v} \in [H^1(\Omega)]^d$, $\mathbf{v} = \mathbf{0}$ on Γ_D, $q \in L^2(\Omega)$,

$$\begin{array}{rcl} a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) &=& \displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v} + \int_{\Gamma_N} \boldsymbol{\phi}_N \cdot \mathbf{v} \\ b(q, \mathbf{u}) &=& 0, \end{array}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) : \nabla\mathbf{v} \tag{2}$$

and

$$b(q, \mathbf{v}) = -\int_\Omega q \operatorname{div} \mathbf{v}. \tag{3}$$

For simplicity of exposition, in the rest of the paper we will often use the strong form of the Stokes problem, but it must be understood that the analysis is carried out in the weak setting.

We consider an overlapping decomposition of the domain Ω in two subdomains Ω_1 and Ω_2: $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$. We denote the overlapping region by $\Omega_{12} = \Omega_1 \cap \Omega_2$ and let $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$. Moreover, let $\Gamma_D^i = \Gamma_D \cap \partial\Omega_i$ and $\Gamma_N^i = \Gamma_N \cap \partial\Omega_i$ (see figure 1).

We reformulate the Stokes problem (1) on the split domain in the following possible ways.

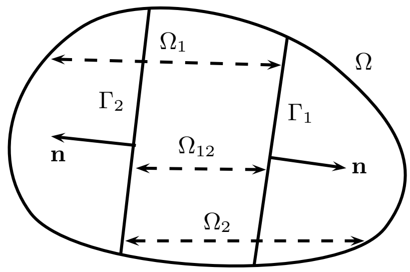

FIG. 1: Representation of the computational domain Ω and of its overlapping splitting.

Problem $\mathcal{P}_{\Gamma,t}$*:*

$$\begin{array}{rcll} -\mathbf{div}\,\mathbf{T}(\mathbf{u}_i, p_i) &=& \mathbf{f} & \text{in } \Omega_i,\ i = 1, 2, \\ \operatorname{div} \mathbf{u}_i &=& 0 & \text{in } \Omega_i,\ i = 1, 2, \\ \mathbf{u}_i &=& \boldsymbol{\phi}_D & \text{on } \Gamma_D^i,\ i = 1, 2, \\ \mathbf{T}(\mathbf{u}_i, p_i)\cdot\mathbf{n} &=& \boldsymbol{\phi}_N & \text{on } \Gamma_N^i,\ i = 1, 2, \\ \mathbf{u}_1 &=& \mathbf{u}_2 & \text{on } \Gamma_1 \cup \Gamma_2. \end{array} \tag{4}$$

In case $\Gamma_N^i = \emptyset$ for some i, we would supplement (4) with the condition

$$\int_{\Omega_i} p_i = 0$$

to ensure the well-posedness of the corresponding local problem.

Problem $\mathcal{P}_{\Gamma,f}$*:*

$$\begin{array}{rcll} -\mathbf{div}\,\mathbf{T}(\mathbf{u}_i, p_i) &=& \mathbf{f} & \text{in } \Omega_i,\ i = 1, 2, \\ \operatorname{div} \mathbf{u}_i &=& 0 & \text{in } \Omega_i,\ i = 1, 2, \\ \mathbf{u}_i &=& \boldsymbol{\phi}_D & \text{on } \Gamma_D^i,\ i = 1, 2, \\ \mathbf{T}(\mathbf{u}_i, p_i)\cdot\mathbf{n} &=& \boldsymbol{\phi}_N & \text{on } \Gamma_N^i,\ i = 1, 2, \\ \mathbf{T}(\mathbf{u}_1, p_1)\cdot\mathbf{n} &=& \mathbf{T}(\mathbf{u}_2, p_2)\cdot\mathbf{n} & \text{on } \Gamma_1 \cup \Gamma_2. \end{array} \tag{5}$$

Condition $(5)_5$ on Γ_1 should be understood as follows. The normal vector $\mathbf{n}$ on Γ_1 is directed outward of Ω_1 and the normal component of the tensor $\mathbf{T}(\mathbf{u}_2, p_2)$ is computed upon restricting it to Ω_{12}. On the other hand, on Γ_2 the normal vector $\mathbf{n}$ is directed outward of Ω_2 and the normal component of the tensor $\mathbf{T}(\mathbf{u}_1, p_1)$ is taken upon restricting it to Ω_{12}.

Moreover, we consider the problem:

Problem $\mathcal{P}_{\Gamma,tf}$*:*

$$\begin{array}{rcll} -\mathbf{div}\,\mathbf{T}(\mathbf{u}_i, p_i) &=& \mathbf{f} & \text{in } \Omega_i,\ i = 1, 2, \\ \operatorname{div} \mathbf{u}_i &=& 0 & \text{in } \Omega_i,\ i = 1, 2, \\ \mathbf{u}_i &=& \boldsymbol{\phi}_D & \text{on } \Gamma_D^i,\ i = 1, 2, \\ \mathbf{T}(\mathbf{u}_i, p_i)\cdot\mathbf{n} &=& \boldsymbol{\phi}_N & \text{on } \Gamma_N^i,\ i = 1, 2, \\ \mathbf{u}_1 &=& \mathbf{u}_2 & \text{on } \Gamma_1, \\ \mathbf{T}(\mathbf{u}_1, p_1)\cdot\mathbf{n} &=& \mathbf{T}(\mathbf{u}_2, p_2)\cdot\mathbf{n} & \text{on } \Gamma_2. \end{array} \tag{6}$$

If $\Gamma_N^1 = \emptyset$, we should impose

$$\int_{\Omega_1} p_1 = 0$$

to guarantee the well-posedness of the Stokes problem in Ω_1.

Let us introduce the following spaces

$$\begin{array}{l} \boldsymbol{V} = [H^1(\Omega)]^d, \quad \boldsymbol{V}_i = [H^1(\Omega_i)]^d, \quad i=1,2 \\ Q = L^2(\Omega), \quad Q_0 = \{q \in Q \,:\, \int_\Omega q = 0\} \\ Q_i = L^2(\Omega_i), \quad Q_{i,0} = \{q \in Q_i \,:\, \int_{\Omega_i} q = 0\} \quad i=1,2 \end{array} \tag{7}$$

and the following affine manifolds

$$\begin{array}{l} \boldsymbol{V}_{\phi_D} = \{\mathbf{v} \in [H^1(\Omega)]^d \,:\, \mathbf{v} = \boldsymbol{\phi}_D \text{ on } \Gamma_D\} \\ \boldsymbol{V}_{i,\phi_D} = \{\mathbf{v} \in [H^1(\Omega_i)]^d \,:\, \mathbf{v} = \boldsymbol{\phi}_D \text{ on } \Gamma_D^i\}, \quad i=1,2. \end{array} \tag{8}$$

Finally, we set

$$\boldsymbol{V}_{i,0} = \{\mathbf{v} \in [H^1(\Omega_i)]^d \,:\, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D^i\}, \quad i=1,2. \tag{9}$$

To prove that the Stokes problem (1) is equivalent to either (4), or (5), or (6), we will denote $\mathbf{w} = \mathbf{u}_{1|\Omega_{12}} - \mathbf{u}_{2|\Omega_{12}}$ and $q = p_{1|\Omega_{12}} - p_{2|\Omega_{12}}$ the difference in Ω_{12} between the local solutions. Note that $(\mathbf{w}, q)$ satisfies the Stokes equations:

$$\begin{array}{rcll} -\mathbf{div}\,\mathbf{T}(\mathbf{w}, q) &=& \mathbf{0} & \text{in } \Omega_{12} \\ \operatorname{div}\mathbf{w} &=& 0 & \text{in } \Omega_{12}. \end{array} \tag{10}$$

The boundary conditions fulfilled by $\mathbf{w}$ and q on $\partial\Omega_{12}$, as well as the spaces to which these functions belong will be specified case by case.

Assumption 2.1 *We suppose that one of the following assumptions is verified:* $\Gamma_N = \emptyset$; $\Gamma_N \neq \emptyset$ *and* $\Gamma_N \cap \partial\Omega_{12} \neq \emptyset$; $\Gamma_N \cap \partial\Omega_{12} = \emptyset$ *with* $\Gamma_N \neq \emptyset$ *connected.*

Proposition 2.1 (Equivalence between $\mathcal{P}_\Omega$ and $\mathcal{P}_{\Gamma,t}$) *The Stokes problems $\mathcal{P}_\Omega$ and $\mathcal{P}_{\Gamma,t}$ are equivalent if Assumption 2.1 holds. Equivalence holds in the sense that if $(\mathbf{u}, p)$ and $(\mathbf{u}_i, p_i)$ $(i = 1,2)$ are the unique solutions of $\mathcal{P}_\Omega$ and $\mathcal{P}_{\Gamma,t}$, respectively, there exist two uniquely determined constants $C_1, C_2 \in \mathbb{R}$, possibly null, such that, for $i = 1,2$, $\mathbf{u}_{|\Omega_i} = \mathbf{u}_i$ and $p_{|\Omega_i} = p_i + C_i$.*

Proof. We treat the different cases separately.

1. Assume first that $\Gamma_N \cap \partial\Omega_{12} \neq \emptyset$. Then, problem (1) is well-posed in $(\mathbf{u}, p) \in \boldsymbol{V}_{\phi_D} \times Q$ and the restrictions of its solution to Ω_i satisfy (4) by construction.

   Viceversa, for $i = 1,2$, let $(\mathbf{u}_i, p_i) \in \boldsymbol{V}_{i,\phi_D} \times Q_i$ $(i = 1,2)$ be the solutions of the well-posed local problems

$$\begin{array}{rcll} -\mathbf{div}\,\mathbf{T}(\mathbf{u}_i, p_i) &=& \mathbf{f} & \text{in } \Omega_i \\ \operatorname{div}\mathbf{u}_i &=& 0 & \text{in } \Omega_i \\ \mathbf{u}_i &=& \boldsymbol{\phi}_D & \text{on } \Gamma_D^i \\ \mathbf{T}(\mathbf{u}_i, p_i)\cdot\mathbf{n} &=& \boldsymbol{\phi}_N & \text{on } \Gamma_N^i \\ \mathbf{u}_i &=& \mathbf{u}_j & \text{on } \Gamma_i,\ j = 3-i. \end{array}$$

   By construction, the functions $\mathbf{w}$ and q satisfy problem (10) with boundary conditions

$$\begin{array}{rcll} \mathbf{T}(\mathbf{w}, q)\cdot\mathbf{n} &=& \mathbf{0} & \text{on } \partial\Omega_{12} \cap \Gamma_N \\ \mathbf{w} &=& \mathbf{0} & \text{on } \partial\Omega_{12} \setminus \Gamma_N. \end{array}$$

   This problem is well-posed and admits the unique solution $\mathbf{w} = \mathbf{0}$ and $q = 0$, hence $\mathbf{u}_1 = \mathbf{u}_2$ and $p_1 = p_2$ in Ω_{12}. Thus, we can set

$$\mathbf{u} = \begin{cases} \mathbf{u}_1 & \text{in } \Omega_1 \setminus \Omega_{12} \\ \mathbf{u}_1 = \mathbf{u}_2 & \text{in } \Omega_{12} \\ \mathbf{u}_2 & \text{in } \Omega_2 \setminus \Omega_{12}, \end{cases} \tag{11}$$

   and

$$p = \begin{cases} p_1 & \text{in } \Omega_1 \setminus \Omega_{12} \\ p_1 = p_2 & \text{in } \Omega_{12} \\ p_2 & \text{in } \Omega_2 \setminus \Omega_{12}. \end{cases} \tag{12}$$

   By construction, functions $\mathbf{u}$ and p belong to $\boldsymbol{V}_{\phi_D} \times Q$ and they satisfy problem (1). In this case $C_1 = C_2 = 0$.

2. Let now $\Gamma_N \cap \partial\Omega_{12} = \emptyset$ and assume that Γ_N is connected. In this case, either $\Gamma_N^1 = \emptyset$ or $\Gamma_N^2 = \emptyset$. We consider the latter case; the former can be treated analogously.

   If $(\mathbf{u}, p) \in \boldsymbol{V}_{\phi_D} \times Q$ is the solution of $\mathcal{P}_\Omega$, if we set $\mathbf{u}_i = \mathbf{u}_{|\Omega_i}$ $(i = 1,2)$, $p_1 = p_{|\Omega_1}$,

$$p_2 = p_{|\Omega_2} - \frac{1}{|\Omega_2|}\int_{\Omega_2} p_{|\Omega_2},$$

   we can immediately verify that $(\mathbf{u}_i, p_i) \in \boldsymbol{V}_{i,\phi_D} \times Q_i$ $(i = 1,2)$ are solutions of $\mathcal{P}_{\Gamma,t}$ with $\int_{\Omega_2} p_2 = 0$. Thus, $C_1 = 0$ and $C_2 = -\frac{1}{|\Omega_2|}\int_{\Omega_2} p_{|\Omega_2}$.

   Viceversa, let $(\mathbf{u}_1, p_1) \in \boldsymbol{V}_{1,\phi_D} \times Q_1$, $(\mathbf{u}_2, p_2) \in \boldsymbol{V}_{2,\phi_D} \times Q_{2,0}$ be the solutions of $\mathcal{P}_{\Gamma,t}$. The functions $(\mathbf{w}, q)$ satisfy (10) with $\mathbf{w} = \mathbf{0}$ on $\partial\Omega_{12}$. Then, $\mathbf{w} = \mathbf{0}$ and $q = const$ in Ω_{12}. The function q is uniquely determined by $\int_{\Omega_{12}} q = \int_{\Omega_{12}} (p_1 - p_2)$ which implies

$$q = \frac{1}{|\Omega_{12}|}\int_{\Omega_{12}} (p_1 - p_2).$$

   If we take $\mathbf{u}$ as in (11) and

$$p = \begin{cases} p_1 & \text{in } \Omega_1 \setminus \Omega_{12} \\ p_1 = p_2 + q & \text{in } \Omega_{12} \\ p_2 + q & \text{in } \Omega_2 \setminus \Omega_{12}, \end{cases}$$

   then $(\mathbf{u}, p)$ satisfy $\mathcal{P}_\Omega$ and the thesis follows with $C_1 = 0$ and $C_2 = q$.

3. Let $(\mathbf{u}, p) \in \boldsymbol{V}_{\phi_D} \times Q_0$ be the solution of $\mathcal{P}_\Omega$. Then, for $i = 1,2$, the functions

$$\mathbf{u}_i = \mathbf{u}_{|\Omega_i}, \quad p_i = p_{|\Omega_i} - \frac{1}{|\Omega_i|}\int_{\Omega_i} p_{|\Omega_i}$$

belong to $\boldsymbol{V}_{i,\phi_D} \times Q_{i,0}$ and they satisfy $\mathcal{P}_{\Gamma,t}$. Thus, $C_1 = 0$ and $C_2 = -\frac{1}{|\Omega_i|}\int_{\Omega_i} p_{|\Omega_i}$.

Viceversa, let $(\mathbf{u}_i, p_i) \in \boldsymbol{V}_{i,\phi_D} \times Q_{i,0}$ be solutions of $\mathcal{P}_{\Gamma,t}$. Then, the functions $\mathbf{w}$ and q satisfy (10) with boundary condition $\mathbf{w} = \mathbf{0}$ on $\partial\Omega_{12}$. Then, $\mathbf{w} = \mathbf{0}$ in Ω_{12} and $q = const$ in Ω_{12}. The constant q is uniquely determined by

$$\int_{\Omega_{12}} q = \int_{\Omega_{12}} (p_1 - p_2)$$

that is

$$q = \frac{1}{|\Omega_{12}|}\int_{\Omega_{12}} (p_1 - p_2).$$

If we define the constants

$$C_1 = \frac{1}{|\Omega|}\left(\int_{\Omega_{12}} p_2 - |\Omega \setminus \Omega_1| q\right)$$

and

$$C_2 = \frac{1}{|\Omega|}\left(\int_{\Omega_{12}} p_2 + |\Omega_1| q\right),$$

since $C_2 - C_1 = q$, then $p_1 + C_1 = p_2 + C_2$ in Ω_{12}. Thus, we can easily verify that the functions $\mathbf{u}$ and p defined respectively as in (11) and as

$$p = \begin{cases} p_1 + C_1 & \text{in } \Omega_1 \setminus \Omega_{12} \\ p_1 + C_1 = p_2 + C_2 & \text{in } \Omega_{12} \\ p_2 + C_2 & \text{in } \Omega_2 \setminus \Omega_{12} \end{cases} \tag{13}$$

are solutions of $\mathcal{P}_\Omega$ with $\int_\Omega p = 0$.

□

Remark 2.1 *If $\partial\Omega_{12} \cap \Gamma_N = \emptyset$ and $\Gamma_N^i \neq \emptyset$ $(i = 1, 2)$, problems $\mathcal{P}_\Omega$ and $\mathcal{P}_{\Gamma,t}$ are not equivalent.*

In fact, if $(\mathbf{u}_i, p_i)$ are the solutions of $\mathcal{P}_{\Gamma_t}$, the functions $\mathbf{w}$ and q satisfy (10) with boundary condition $\mathbf{w} = \mathbf{0}$ on $\partial\Omega_{12}$. Then, $\mathbf{w} = \mathbf{0}$ and $q = const$ in Ω_{12} with q uniquely given by

$$q = \frac{1}{|\Omega_{12}|}\int_{\Omega_{12}} (p_1 - p_2).$$

Then, proceeding similarly to the third case of the proof of Proposition 2.1, there exist two unique constants C_1, C_2 with $q = C_2 - C_1$ so that we can define $\mathbf{u}$ and p as in (11) and (13), respectively. The Neumann boundary conditions in $\mathcal{P}_{\Gamma,t}$ imply

$$\mathbf{T}(\mathbf{u}_i, p_i) \cdot \mathbf{n} = \boldsymbol{\phi}_N \quad \text{on } \Gamma_N^i$$

and, by definition of $\mathbf{u}$ and p, we have

$$\mathbf{T}(\mathbf{u}, p) \cdot \mathbf{n} = \boldsymbol{\phi}_N + C_i \mathbf{n} \quad \text{on } \Gamma_N^i.$$

Thus, $(\mathbf{u}, p)$ satisfy problem $\mathcal{P}_\Omega$ if and only if $C_1 = C_2 = 0$, but we cannot guarantee that this condition is fulfilled.

Proposition 2.2 (Equivalence between $\mathcal{P}_\Omega$ and $\mathcal{P}_{\Gamma,f}$) *If $\partial\Omega_{12} \cap \Gamma_D \neq \emptyset$, the Stokes problems $\mathcal{P}_\Omega$ and $\mathcal{P}_{\Gamma,f}$ are equivalent in the sense that there exist unique constants $C_1, C_2 \in \mathbb{R}$ such that $\mathbf{u}_{|\Omega_i} = \mathbf{u}_i$ and $p_{|\Omega_i} = p_i + C_i$, $(\mathbf{u}, p)$ and $(\mathbf{u}_i, p_i)$ $(i = 1, 2)$ being, respectively, the unique solutions of $\mathcal{P}_\Omega$ and $\mathcal{P}_{\Gamma,f}$.*

Proof. The proof goes along the same arguments used for Proposition 2.1 so that we only define the constants in the cases $\Gamma_N \neq \emptyset$ or $\Gamma_N = \emptyset$.

In the first case it is straightforward to see that the equivalence holds with $C_1 = C_2 = 0$. On the other hand, if $\Gamma_N = \emptyset$, the functions $\mathbf{w}$ and q satisfy the problem (10) with boundary conditions

$$\begin{aligned} \mathbf{w} &= \mathbf{0} && \text{on } \partial\Omega_{12} \cap \partial\Omega \\ \mathbf{T}(\mathbf{w}, q) \cdot \mathbf{n} &= \mathbf{0} && \text{on } \Gamma_1 \cup \Gamma_2. \end{aligned}$$

This problem is well-posed and its solution is $\mathbf{w} = \mathbf{0}$ and $q = 0$. Thus, $\mathbf{u}_1 = \mathbf{u}_2$ and $p_1 = p_2$ in Ω_{12} and we can define velocity $\mathbf{u}$ and a pressure $\tilde{p}$ analogously to (11) and (12). However, the function $\tilde{p}$ would belong to Q but not to Q_0, so that we define

$$C_1 = C_2 = -\frac{1}{|\Omega|}\int_\Omega \tilde{p}$$

and $p = \tilde{p} + C_1$ to recover the null average. □

Remark 2.2 *Problems $\mathcal{P}_\Gamma$ and $\mathcal{P}_{\Gamma,f}$ are not equivalent if $\partial\Omega_{12} \cap \Gamma_D = \emptyset$. In fact, in this case problem (10) in Ω_{12} would be supplemented with the boundary condition $\mathbf{T}(\mathbf{w}, q) \cdot \mathbf{n} = \mathbf{0}$ on $\partial\Omega_{12}$ which has infinite non-trivial solutions that may differ one from another not only by a constant.*

Proposition 2.3 (Equivalence between $\mathcal{P}_\Omega$ and $\mathcal{P}_{\Gamma,tf}$) *The Stokes problems $\mathcal{P}_\Omega$ and $\mathcal{P}_{\Gamma,tf}$ are equivalent if either $\Gamma_N = \emptyset$, or $\Gamma_N \cap \partial\Omega_{12} \neq \emptyset$, or $\Gamma_N \cap \partial\Omega_{12} = \emptyset$ and $\Gamma_N^1 \neq \emptyset$. Equivalence holds in the sense that if $(\mathbf{u}, p)$ and $(\mathbf{u}_i, p_i)$ $(i = 1, 2)$ are the unique solutions of $\mathcal{P}_\Omega$ and $\mathcal{P}_{\Gamma,t}$, respectively, there exist two uniquely determined constants $C_1, C_2 \in \mathbb{R}$, possibly null, such that, for $i = 1, 2$, $\mathbf{u}_{|\Omega_i} = \mathbf{u}_i$ and $p_{|\Omega_i} = p_i + C_i$.*

Proof. The proof develops along the lines of the previous propositions. Let us only point out that the equivalence holds with $C_1 = C_2 = 0$ if $\Gamma_N \neq \emptyset$. Otherwise, if $\Gamma_N = \emptyset$, if $(\mathbf{u}, p) \in \boldsymbol{V}_{\phi_D} \times Q_0$ is the solution of $\mathcal{P}_\Omega$, then $\mathbf{u}_i = \mathbf{u}_{|\Omega_i}$, $p_2 = p_{|\Omega_2}$ and

$$p_1 = p_{|\Omega_1} - \frac{1}{|\Omega_1|}\int_{\Omega_1} p_{|\Omega_1}$$

are the solutions of $\mathcal{P}_{\Gamma,tf}$.

Viceversa, if $(\mathbf{u}_1, p_1) \in \boldsymbol{V}_{1,\phi_D} \times Q_{1,0}$ and $(\mathbf{u}_2, p_2) \in \boldsymbol{V}_{2,\phi_D} \times Q_2$ are the solutions of $\mathcal{P}_{\Gamma,tf}$, then we need to set

$$C_1 = C_2 = -\frac{1}{|\Omega|}\int_{\Omega_2 \setminus \Omega_{12}} p_2.$$

□

Remark 2.3 *Problems $\mathcal{P}_\Gamma$ and $\mathcal{P}_{\Gamma,tf}$ are not equivalent if $\partial\Omega_{12} \cap \Gamma_N = \emptyset$, $\Gamma_N^1 = \emptyset$, and $\Gamma_N^2 \neq \emptyset$. In fact, if $(\mathbf{u}_1, p_1) \in \boldsymbol{V}_{1,\phi_D} \times Q_{1,0}$ and $(\mathbf{u}_2, p_2) \in \boldsymbol{V}_{2,\phi_D} \times Q_2$ are the solutions of $\mathcal{P}_{\Gamma,tf}$, then $(\mathbf{w}, q)$ satisfy problem* (10) *in Ω_{12} with boundary condition $\mathbf{T}(\mathbf{w}, q)\cdot\mathbf{n} = \mathbf{0}$ on Γ_2 and $\mathbf{w} = \mathbf{0}$ on $\partial\Omega_{12} \setminus \Gamma_2$. The solution of this problem in Ω_{12} is identically null. However, since $\int_{\Omega_{12}} q = \int_{\Omega_{12}}(p_1 - p_2)$ with $p_1 \in Q_{1,0}$ and p_2 uniquely determined by the Neumann boundary condition on Γ_N^2, we cannot guarantee that $q = 0$.*

Notice that a result similar to Proposition 2.3 could be obtained by switching the role of the interface conditions $(6)_5$ and $(6)_6$, i.e., considering

Problem $\mathcal{P}_{\Gamma,ft}$:

$$\begin{array}{rcll} -\mathbf{div}\,\mathbf{T}(\mathbf{u}_i, p_i) &=& \mathbf{f} & \text{in } \Omega_i,\ i = 1,2,\\ \operatorname{div}\mathbf{u}_i &=& 0 & \text{in } \Omega_i,\ i = 1,2,\\ \mathbf{u}_i &=& \boldsymbol{\phi}_D & \text{on } \Gamma_D^i,\ i = 1,2,\\ \mathbf{T}(\mathbf{u}_i, p_i)\cdot\mathbf{n} &=& \boldsymbol{\phi}_N & \text{on } \Gamma_N^i,\ i = 1,2,\\ \mathbf{T}(\mathbf{u}_1, p_1)\cdot\mathbf{n} &=& \mathbf{T}(\mathbf{u}_2, p_2)\cdot\mathbf{n} & \text{on } \Gamma_1\\ \mathbf{u}_1 &=& \mathbf{u}_2 & \text{on } \Gamma_2. \end{array} \tag{14}$$

3. FORMULATION OF THE ICDD METHOD FOR THE STOKES PROBLEM

For the sake of simplicity we will consider *homogeneous* boundary conditions, i.e., we will set $\boldsymbol{\phi}_D = \mathbf{0}$ on Γ_D and $\boldsymbol{\phi}_N = \mathbf{0}$ on Γ_N. Moreover, since we will be interested in computing a finite dimensional approximation of the solution of the Stokes problem, we introduce the ICDD method directly at the discrete level.

3.1. hp-FEM discretization

We introduce two regular computational grids $\mathcal{T}_1$ and $\mathcal{T}_2$ in Ω_1 and Ω_2 made by either simplices or quadrilaterals/hexahedra. We suppose that each element $T \in \mathcal{T}_i$ is obtained by a C^1 diffeomorphism $\mathbf{F}_T$ of the reference element $\hat{T}$ and we suppose that two adjacent elements of $\mathcal{T}_i$ share either a common vertex or a complete edge or a complete face (when $d = 3$). Moreover, we assume that they coincide in Ω_{12} and that both interfaces Γ_1 and Γ_2 do not cross any element of Ω_1 or Ω_2. We discretize both primal and dual problems in each subdomain by hp finite element methods (hp-FEM). Because of the difficulty to compute integrals exactly for large p, typically when quadrilaterals are used, Legendre-Gauss-Lobatto quadrature formulas are employed to approximate the bilinear forms $a_{|\Omega_i}$ and $b_{|\Omega_i}$ (see (2)-(3)) as well as the L^2-inner products in Ω_i and on the interfaces. This leads to the so called *Galerkin approach with Numerical Integration* (G-NI) [1, 2] and to the Spectral Element Method with Numerical Integration (SEM-NI). In particular, we consider either inf-sup stable finite dimensional spaces or stabilized couples of spaces of the same degree (see [7, 8, 11, 15]) to approximate the velocity and the pressure and we assume that the polynomials used for the pressure are continuous (see, e.g., [9, 16]). More precisely, given an integer $p \geq 1$, let $\mathbb{P}_p$ be the space of polynomials whose global degree is less than or equal to p in the variables $x_1, \ldots, x_d$ and $\mathbb{Q}_p$ be the space of polynomials that are of degree less than or equal to p with respect to each variable $x_1, \ldots, x_d$. The space $\mathbb{P}_p$ is associated with simplicial partitions, while $\mathbb{Q}_p$ to quadrilateral ones. We introduce the finite dimensional space on $\overline{\Omega}_i$ defined by

$$X_{i,h}^p = \{v \in C^0(\overline{\Omega}_i) \,:\, v_{|T} \in \mathbb{P}_p,\ \forall T \in \mathcal{T}_i\}$$

in the simplicial case, and by

$$X_{i,h}^p = \{v \in C^0(\overline{\Omega}_i) \,:\, v_{|T} \circ \mathbf{F}_T \in \mathbb{Q}_p,\ \forall T \in \mathcal{T}_i\}$$

for quadrilaterals. Then, the finite dimensional spaces for velocity and pressure are, respectively,

$$\boldsymbol{V}_{i,h} = [X_{i,h}^p]^d \cap \boldsymbol{V}_{i,0}, \quad Q_{i,h} = X_{i,h}^r \tag{15}$$

for suitable polynomial degrees p and r.

3.2. ICDD method with Dirichlet controls

Assume, for simplicity, that $\partial\Omega_{12} \cap \Gamma_N \neq \emptyset$ and $\Gamma_D \neq \emptyset$. (We will discuss this issue more in detail in section 5.) We define the space of discrete Dirichlet controls as

$$\boldsymbol{\Lambda}_{i,h}^D = \{\boldsymbol{\lambda}_{i,h} \in C^0(\Gamma_i) \,:\, \exists \mathbf{v}_{i,h} \in \boldsymbol{V}_{i,h} \text{ with } \boldsymbol{\lambda}_{i,h} = \mathbf{v}_{i,h|\Gamma_i}\}.$$

and let

$$\boldsymbol{\Lambda}_h^D = \boldsymbol{\Lambda}_{1,h}^D \times \boldsymbol{\Lambda}_{2,h}^D.$$

For $i = 1,2$, we consider two control functions $\boldsymbol{\lambda}_{i,h} \in \boldsymbol{\Lambda}_{i,h}^D$ and the state problems: find $(\mathbf{u}_{i,h}, p_{i,h}) \in \boldsymbol{V}_{i,h} \times Q_{i,h}$ such that, for all $(\mathbf{v}_{i,h}, q_{i,h}) \in \boldsymbol{V}_{i,h} \times Q_i$, $\mathbf{v}_{i,h} = \mathbf{0}$ on Γ_i,

$$\begin{array}{l} a_i(\mathbf{u}_{i,h}, \mathbf{v}_{i,h}) + b_i(p_{i,h}, \mathbf{v}_{i,h}) = \int_{\Omega_i} \mathbf{f}\cdot\mathbf{v}_{i,h}\\ b_i(q_{i,h}, \mathbf{u}_{i,h}) = 0\\ \mathbf{u}_{i,h} = \boldsymbol{\lambda}_{i,h} \text{ on } \Gamma_i \end{array} \tag{16}$$

where a_i and b_i denote the restriction of the bilinear forms (2) and (3) to Ω_i. In fact, $(\mathbf{u}_{i,h}, p_{i,h})$ depends on both $\boldsymbol{\lambda}_{i,h}$ and $\mathbf{f}$, however such dependence will be understood for the sake of notation.

The unknown controls on the interface are obtained by solving a minimization problem for a cost functional suitably depending on the difference between $\mathbf{u}_{1,h}$ and $\mathbf{u}_{2,h}$ on the interfaces Γ_1 and Γ_2. More precisely, inspired by $(4)_5$, we look for

$$\inf_{\underline{\boldsymbol{\lambda}}_h = (\boldsymbol{\lambda}_{1,h}, \boldsymbol{\lambda}_{2,h})} \left[J_t(\underline{\boldsymbol{\lambda}}_h) := \frac{1}{2}\sum_{i=1}^{2} \|\mathbf{u}_{1,h} - \mathbf{u}_{2,h}\|^2_{L^2(\Gamma_i)} \right]. \tag{17}$$

To the minimization problem (17) we can associate the following optimality system: find $\underline{\boldsymbol{\lambda}}_h = (\boldsymbol{\lambda}_{1,h}, \boldsymbol{\lambda}_{2,h}) \in \boldsymbol{\Lambda}_h^D$ and, for $i = 1, 2$, $(\mathbf{u}_{i,h}, p_{i,h}) \in \boldsymbol{V}_{i,h} \times Q_{i,h}$, $(\mathbf{w}_{i,h}, q_{i,h}) \in \boldsymbol{V}_{i,h} \times Q_{i,h}$ such that, for all $(\mathbf{v}_{i,h}, \psi_{i,h}) \in \boldsymbol{V}_{i,h} \times Q_{i,h}$ with $\mathbf{v}_{i,h} = \mathbf{0}$ on Γ_i,

$$\begin{array}{l} a_i(\mathbf{u}_{i,h}, \mathbf{v}_{i,h}) + b_i(p_{i,h}, \mathbf{v}_{i,h}) = \int_{\Omega_i} \mathbf{f} \cdot \mathbf{v}_{i,h} \\ b_i(\psi_{i,h}, \mathbf{u}_{i,h}) = 0 \\ \mathbf{u}_{i,h} = \boldsymbol{\lambda}_{i,h} \text{ on } \Gamma_i \end{array} \tag{18}$$

$$\begin{array}{l} a_i(\mathbf{w}_{i,h}, \mathbf{v}_{i,h}) + b_i(q_{i,h}, \mathbf{v}_{i,h}) = 0 \\ b_i(\psi_{i,h}, \mathbf{w}_{i,h}) = 0 \\ \mathbf{w}_{i,h} = (-1)^{i+1}(\mathbf{u}_{1,h} - \mathbf{u}_{2,h}) \text{ on } \Gamma_i, \end{array} \tag{19}$$

and, for all $(\boldsymbol{\mu}_{1,h}, \boldsymbol{\mu}_{2,h}) \in \boldsymbol{\Lambda}_h^D$,

$$\begin{array}{l} \displaystyle\int_{\Gamma_1} ((\mathbf{u}_{1,h} - \mathbf{u}_{2,h}) + \mathbf{w}_{2,h})\boldsymbol{\mu}_{1,h}\, d\Gamma + \\ \quad \displaystyle\int_{\Gamma_2} (-(\mathbf{u}_{1,h} - \mathbf{u}_{2,h}) + \mathbf{w}_{1,h})\boldsymbol{\mu}_{2,h}\, d\Gamma = 0. \end{array} \tag{20}$$

3.3. Algebraic formulation of ICDD with Dirichlet controls

To the Stokes problem in subdomain Ω_i ($i = 1, 2$) we can associate the matrix

$$S_i = \begin{pmatrix} A_i & B_i^T \\ B_i & 0 \end{pmatrix}$$

where A_i corresponds to the finite dimensional approximation of the bilinear form $a_{|\Omega_i}$ (see (2)), while B_i corresponds to the discretization of $b_{|\Omega_i}$ (see (3)). When stabilization is used, the matrices S_i take the form

$$S_i = \begin{pmatrix} A_i & B_i^T \\ B_i & 0 \end{pmatrix} + \begin{pmatrix} \tilde{A}_i & \tilde{B}_i^T \\ \tilde{B}_i & \tilde{C}_i \end{pmatrix}$$

where $\tilde{A}_i$, $\tilde{B}_i$ and $\tilde{C}_i$ are assembled locally, element by element, and they take into account the integration of the differential Stokes operator.

In the following we will denote by the index I_i the degrees of freedom for the velocity and the pressure belonging to $\Omega_i \setminus \Gamma_i$, while the index Γ_i will refer to the degrees of freedom on the interface Γ_i. For the sake of exposition, we will reorder the nodes in Ω_i putting those associated with $\Omega_i \setminus \Gamma_i$ first followed by those on the interfaces. Correspondingly, with obvious choice of notation, we can rewrite the Stokes matrix S_i as

$$\begin{aligned} S_i &= \left(\begin{array}{cc|cc} A_{I_iI_i} & B_{I_iI_i}^T & A_{I_i\Gamma_i} & B_{I_i\Gamma_i}^T \\ B_{I_iI_i} & 0 & B_{I_i\Gamma_i} & 0 \\ \hline A_{\Gamma_iI_i} & B_{\Gamma_iI_i}^T & A_{\Gamma_i\Gamma_i} & B_{\Gamma_i\Gamma_i}^T \\ B_{\Gamma_iI_i} & 0 & B_{\Gamma_i\Gamma_i} & 0 \end{array}\right) \\ &= \begin{pmatrix} S_{I_iI_i} & S_{I_i\Gamma_i} \\ S_{\Gamma_iI_i} & S_{\Gamma_i\Gamma_i} \end{pmatrix}. \end{aligned}$$

Moreover, we will indicate by M_{Γ_i} the mass matrix on the interface Γ_i.

Finally, in the rest of the section, we will denote by $\mathbf{F}_i$ the right-hand side for the state problems in Ω_i, while $\mathbf{U}_i$ and $\mathbf{W}_i$ will be the vectors of unknown velocity and pressure in Ω_i for the state and the adjoint problems, respectively. $\boldsymbol{\lambda}_{\Gamma_i}$ is the vector of the unknown Dirichlet controls on Γ_i:

$$\begin{array}{l} \boldsymbol{\lambda}_{\Gamma_i} = \left((\boldsymbol{\lambda}_{\Gamma_i})_1, \ldots, (\boldsymbol{\lambda}_{\Gamma_i})_{N_{\Gamma_i}}\right), \\ (\boldsymbol{\lambda}_{\Gamma_i})_j = \boldsymbol{\lambda}_{i,h}(\mathbf{x}_j), \quad j \in \mathcal{G}_i, \end{array}$$

where $\mathcal{G}_i$ is the set of the N_{Γ_i} indices corresponding to the velocity degrees of freedom on the interface Γ_i and $\mathbf{x}_j$ is a node on Γ_i ($(\boldsymbol{\lambda}_{\Gamma_i})_j$ is the nodal value of the discrete control function $\boldsymbol{\lambda}_{i,h}$ at the node $\mathbf{x}_j$).

We consider now the optimality system associated with the functional J_t with Dirichlet controls that we introduced in section 5 5.1 5.1.1. If R_{ij} denotes the algebraic restriction operator of the velocity unknowns in Ω_j to the interface Γ_i ($i, j = 1, 2$), the algebraic counterpart of (18)-(20) reads:

$$\mathbf{S}_t \mathbf{y}_t = \mathbf{b}_t \tag{21}$$

where $\mathbf{y}_t = (\mathbf{U}_{I_1}, \mathbf{U}_{I_2}, \mathbf{W}_{I_1}, \mathbf{W}_{I_2}, \boldsymbol{\lambda}_{\Gamma_1}, \boldsymbol{\lambda}_{\Gamma_2})^T$, $\mathbf{b}_t = (\mathbf{F}_1, \mathbf{F}_2, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})^T$ and the matrix $\mathbf{S}_t$ is defined as

$$\left(\begin{array}{cc|cc|cc} S_{I_1I_1} & 0 & 0 & 0 & S_{I_1\Gamma_1} & 0 \\ 0 & S_{I_2I_2} & 0 & 0 & 0 & S_{I_2\Gamma_2} \\ \hline 0 & -S_{I_1\Gamma_1}R_{12} & S_{I_1I_1} & 0 & S_{I_1\Gamma_1} & 0 \\ -S_{I_2\Gamma_2}R_{21} & 0 & 0 & S_{I_2I_2} & 0 & S_{I_2\Gamma_2} \\ \hline 0 & -M_{\Gamma_1}R_{12} & 0 & M_{\Gamma_1}R_{12} & M_{\Gamma_1} & 0 \\ -M_{\Gamma_2}R_{21} & 0 & M_{\Gamma_2}R_{21} & 0 & 0 & M_{\Gamma_2} \end{array}\right)$$

For the numerical solution of the linear systems (21), we compute the Schur complement system with respect to the control variables $(\boldsymbol{\lambda}_{\Gamma_1}, \boldsymbol{\lambda}_{\Gamma_2})$ and solve them through an iterative method like, e.g., Bi-CGstab ([20]).

The Schur complement system reads

$$\boldsymbol{\Sigma}_t \begin{pmatrix} \boldsymbol{\lambda}_{\Gamma_1} \\ \boldsymbol{\lambda}_{\Gamma_2} \end{pmatrix} = \boldsymbol{\chi}_t \tag{22}$$

where

$$\boldsymbol{\Sigma}_t = \begin{pmatrix} M_{\Gamma_1}(I_{\Gamma_1} - (R_{12}S_{I_1I_1}^{-1}S_{I_1\Gamma_1})^2) \\ M_{\Gamma_2}(I_{\Gamma_2} - (R_{21}S_{I_2I_2}^{-1}S_{I_2\Gamma_2})^2) \end{pmatrix}$$

and

$$\boldsymbol{\chi}_t = \begin{pmatrix} M_{\Gamma_1}R_{12}(I_{\Gamma_1} - S_{I_1I_1}^{-1}S_{I_1\Gamma_1}R_{12})S_{I_1I_1}^{-1}\mathbf{F}_1 \\ M_{\Gamma_2}R_{21}(I_{\Gamma_2} - S_{I_2I_2}^{-1}S_{I_2\Gamma_2}R_{21})S_{I_2I_2}^{-1}\mathbf{F}_2 \end{pmatrix}$$

I_{Γ_i} is the identity matrix on the interface Γ_i.

3.4. ICDD method with Neumann and mixed controls

Let $\mathbf{\Lambda}_{i,h}^N$ denote the space of discrete Neumann controls on Γ_i. We require that $\mathbf{\Lambda}_{i,h}^N \subset L^2(\Gamma_i)$.

For $i=1,2$, given the control functions $\boldsymbol{\lambda}_{i,h} \in \mathbf{\Lambda}_{i,h}^N$, consider the *discrete* state problems: find $(\mathbf{u}_{i,h}, p_{i,h}) \in \mathbf{V}_{i,h} \times Q_{i,h}$, such that, for all $(\mathbf{v}_{i,h}, q_{i,h}) \in \mathbf{V}_{i,h} \times Q_{i,h}$,

$$\begin{aligned} & a_i(\mathbf{u}_{i,h}, \mathbf{v}_{i,h}) + b_i(p_{i,h}, \mathbf{v}_{i,h}) = \int_{\Gamma_i} \boldsymbol{\lambda}_{i,h} \cdot \mathbf{v}_{i,h} \\ & \qquad + \int_{\Omega_i} \mathbf{f} \cdot \mathbf{v}_{i,h} \\ & b_i(q_{i,h}, \mathbf{u}_{i,h}) = 0. \end{aligned} \tag{23}$$

Let $T_k \subset \Omega_i$ be a generic element in Ω_i; we introduce the set $\mathcal{E}_i = \{k : \text{meas}(\partial T_k \cap \Gamma_i) > 0\}$ and, for any $k \in \mathcal{E}_i$, the edges $e_{ik} = \partial T_k \cap \Gamma_i$. Thanks to the definition of $[X_{i,h}^p]^d$, for any $\mathbf{v}_{i,h} \in [X_{i,h}^p]^d$ and $q_{i,h} \in X_{i,h}^r$, it holds $\mathbf{v}_{i,h}|_{\overline{T}_k} \in [C^1(\overline{T}_k)]^d$ and $q_{i,h}|_{\overline{T}_k} \in C^0(\overline{T}_k)$, and then we define the *discrete normal stress*

$$\hat{\mathbf{\Phi}}_{i,i,h} = T(\mathbf{u}_{i,h}, p_{i,h}) \cdot \mathbf{n} \quad \text{on } \Gamma_i.$$

This definition makes sense in classic way on each $e_{ik} \subset \Gamma_i$, so that $\hat{\mathbf{\Phi}}_{i,i,h} \in [L^2(\Gamma_i)]^d$.

We are interested in evaluating the discrete normal stress associated with $(\mathbf{u}_{i,h}, p_{i,h})$ also on the interface Γ_j ($j = 3-i$), which is internal to Ω_i.

With this aim we first restrict $(\mathbf{u}_{i,h}, p_{i,h})$ to Ω_{12} and then extend it to Ω_j in such a way that such extension $(\tilde{\mathbf{u}}_{i,h}, \tilde{p}_{i,h})$ belongs to $\mathbf{V}_{j,h} \times Q_{j,h}$. Then we define

$$\hat{\mathbf{\Phi}}_{i,j,h} = T(\tilde{\mathbf{u}}_{i,h}, \tilde{p}_{i,h}) \cdot \mathbf{n} \qquad \text{on } \Gamma_j$$

and it holds $\hat{\mathbf{\Phi}}_{i,j,h} \in [L^2(\Gamma_j)]^d$

Following $(5)_5$, the discrete Neumann controls $\boldsymbol{\lambda}_{i,h}$ on the interface Γ_i are obtained as solution of the following minimization problem

$$\inf_{\boldsymbol{\lambda}_{1,h}, \boldsymbol{\lambda}_{2,h}} \left[J_f(\boldsymbol{\lambda}_{1,h}, \boldsymbol{\lambda}_{2,h}) = \frac{1}{2} \sum_{i=1}^{2} \|\hat{\mathbf{\Phi}}_{1,i,h} - \hat{\mathbf{\Phi}}_{2,i,h}\|^2_{L^2(\Gamma_i)} \right]. \tag{24}$$

In practice, the discrete normal stresses on the interfaces Γ_i are obtained as residuals of the first equation in (23), as we are going to show.

Let $\mathcal{I}_i^u$ and $\mathcal{I}_i^p$ be the sets of indices of the nodes of the meshes in Ω_i for the velocity and the pressure, respectively. Moreover, let $\mathcal{G}_i^u \subset \mathcal{I}_i^u$ be the subsets of indices of the nodes lying on Γ_i. We consider matching meshes on the overlap Ω_{12}. In $[X_{i,h}^p]^d$ we take the basis $\mathcal{B}_i^u$ of the characteristic Lagrange polynomials $\boldsymbol{\varphi}_{i,\ell}$ with $\ell \in \mathcal{I}_i^u$. Similarly, in $Q_{i,h}$ we consider the basis $\mathcal{B}_i^p$ of the characteristic Lagrange polynomials $\psi_{i,k}$, with $k \in \mathcal{I}_i^p$.

Now, let $(\mathbf{u}_{i,h}, p_{i,h})$ be the solution of (23). For any $\ell \in \mathcal{G}_i^u$, we define the vectors $(\mathbf{\Phi}_{i,\Gamma_i})_\ell \in \mathbb{R}^d$ of the weak discrete normal stresses on Γ_i associated with $(\mathbf{u}_{i,h}, p_{i,h})$ as

$$\begin{aligned} (\mathbf{\Phi}_{i,\Gamma_i})_\ell &= a_i(\mathbf{u}_{i,h}, \boldsymbol{\varphi}_{i,\ell}) + b_i(p_{i,h}, \boldsymbol{\varphi}_{i,\ell}) \\ &\quad - \int_{\Omega_i} \mathbf{f} \cdot \boldsymbol{\varphi}_{i,\ell}. \end{aligned} \tag{25}$$

Similarly, for any $\ell \in \mathcal{G}_j^u$ and $\boldsymbol{\varphi}_{j,\ell} \in \mathcal{B}_j^u$ we define the vectors $(\mathbf{\Phi}_{i,\Gamma_j})_\ell \in \mathbb{R}^d$ of the weak discrete normal stresses on Γ_j associated with $(\mathbf{u}_{i,h}, p_{i,h})$ as

$$\begin{aligned} (\mathbf{\Phi}_{i,\Gamma_j})_\ell &= a_j(\tilde{\mathbf{u}}_{i,h}, \boldsymbol{\varphi}_{j,\ell}) + b_j(\tilde{p}_{i,h}, \boldsymbol{\varphi}_{j,\ell}) \\ &\quad - \int_{\Omega_j} \mathbf{f} \cdot \boldsymbol{\varphi}_{j,\ell}. \end{aligned} \tag{26}$$

It holds

$$(\mathbf{\Phi}_{i,\Gamma_j})_\ell = \int_{\Gamma_j} \hat{\mathbf{\Phi}}_{i,j,h} \cdot \boldsymbol{\varphi}_{j,l} \qquad \forall \ell \in \mathcal{G}_j^u,\ i,j \in \{1,2\}.$$

To the minimization of problem (24) we can associate the following optimality system: find $\underline{\boldsymbol{\lambda}}_h = (\boldsymbol{\lambda}_{1,h}, \boldsymbol{\lambda}_{2,h}) \in \mathbf{\Lambda}_h^N$, and, for $i=1,2$, $(\mathbf{u}_{i,h}, p_{i,h})$, $(\mathbf{w}_{i,h}, q_{i,h}) \in \mathbf{V}_{i,h} \times Q_{i,h}$ such that

$$\begin{aligned} & a_i(\mathbf{u}_{i,h}, \boldsymbol{\varphi}_{i,\ell}) + b_i(p_{i,h}, \boldsymbol{\varphi}_{i,\ell}) = \int_{\Gamma_i} \boldsymbol{\lambda}_{i,h} \cdot \boldsymbol{\varphi}_{i,\ell} \\ & \qquad + \int_{\Omega_i} \mathbf{f} \cdot \boldsymbol{\varphi}_{i,\ell}, \qquad \forall \ell \in \mathcal{I}_i^u \\ & b_i(\psi_{i,k}, \mathbf{u}_{i,h}) = 0 \qquad \forall k \in \mathcal{I}_i^p, \end{aligned} \tag{27}$$

$$\begin{aligned} & a_i(\mathbf{w}_{i,h}, \boldsymbol{\varphi}_{i,\ell}) + b_i(q_{i,h}, \boldsymbol{\varphi}_{i,\ell}) = (\mathbf{\Phi}_{i,\Gamma_i})_\ell - (\mathbf{\Phi}_{j,\Gamma_i})_\ell \\ & \qquad \forall \ell \in \mathcal{I}_i^u \\ & b_i(\psi_{i,k}, \mathbf{w}_{i,h}) = 0 \qquad \forall k \in \mathcal{I}_i^p, \end{aligned} \tag{28}$$

and

$$\sum_{i=1}^{2} [(\mathbf{\Phi}_{i,\Gamma_i})_\ell - (\mathbf{\Phi}_{j,\Gamma_i})_\ell + (\mathbf{\Psi}_{j,\Gamma_i})_\ell] = 0 \qquad \forall \ell \in \mathcal{G}_i^u, \tag{29}$$

where $j = 3-i$ and

$$(\mathbf{\Psi}_{j,\Gamma_i})_\ell = a_j(\tilde{\mathbf{w}}_{i,h}, \boldsymbol{\varphi}_{j,\ell}) + b_j(\tilde{q}_{i,h}, \boldsymbol{\varphi}_{j,\ell})$$

is the weak representation of the discrete normal stress on Γ_j associated with the dual state solution $(\mathbf{w}_{i,h}, q_{i,h})$.

An alternative strategy consists in choosing mixed controls, e.g., a discrete Dirichlet control $\boldsymbol{\lambda}_{1,h} \in \mathbf{\Lambda}_{1,h}^D$ on Γ_1 and a Neumann control $\boldsymbol{\lambda}_{2,h} \in \mathbf{\Lambda}_{2,h}^N$ on Γ_2 and to minimize the difference between both interface velocities and interface normal stresses.

Following $(6)_5$ and $(6)_6$, the corresponding minimization problems would read:

$$\begin{aligned} \inf_{\boldsymbol{\lambda}_{1,h}, \boldsymbol{\lambda}_{2,h}} \Big[J_{tf}(\boldsymbol{\lambda}_{1,h}, \boldsymbol{\lambda}_{2,h}) &:= \frac{1}{2} \|\mathbf{u}_{1,h} - \mathbf{u}_{2,h}\|^2_{L^2(\Gamma_1)} \\ &\quad + \frac{1}{2} \|\hat{\mathbf{\Phi}}_{1,2,h} - \hat{\mathbf{\Phi}}_{2,2,h}\|^2_{L^2(\Gamma_2)} \Big] \end{aligned} \tag{30}$$

Alternatively, following $(14)_5$ and $(14)_6$, we could consider a discrete Neumann control on Γ_1 and a discrete Dirichlet control on Γ_2 and the corresponding minimization problem:

$$\inf_{\boldsymbol{\lambda}_{1,h},\boldsymbol{\lambda}_{2,h}} \Big[J_{ft}(\boldsymbol{\lambda}_{1,h},\boldsymbol{\lambda}_{2,h}) := \frac{1}{2}\|\hat{\boldsymbol{\Phi}}_{1,1,h} - \hat{\boldsymbol{\Phi}}_{2,1,h}\|^2_{L^2(\Gamma_1)} + \frac{1}{2}\|\mathbf{u}_{1,h} - \mathbf{u}_{2,h}\|^2_{L^2(\Gamma_2)} \Big]. \quad (31)$$

To the minimization problem (30) we associate the following optimality system: find $\underline{\boldsymbol{\lambda}}_h = (\boldsymbol{\lambda}_{1,h},\boldsymbol{\lambda}_{2,h}) \in \boldsymbol{\Lambda}^D_{1,h} \times \boldsymbol{\Lambda}^N_{2,h}$ and, for $i = 1, 2$, $(\mathbf{u}_{i,h}, p_{i,h}) \in \boldsymbol{V}_{i,h} \times Q_{i,h}$, $(\mathbf{w}_i, q_i) \in \boldsymbol{V}_{i,h} \times Q_{i,h}$ such that

$$\begin{aligned} & a_1(\mathbf{u}_{1,h},\boldsymbol{\varphi}_{1,\ell}) + b_1(p_{1,h},\boldsymbol{\varphi}_{1,\ell}) = \textstyle\int_{\Omega_1} \mathbf{f}\cdot\boldsymbol{\varphi}_{1,\ell}, \forall \ell \in \mathcal{I}^u_1 \\ & b_1(\psi_{1,k},\mathbf{u}_{1,h}) = 0 \qquad \forall k \in \mathcal{I}^p_1, \\ & \mathbf{u}_{1,h} = \boldsymbol{\lambda}_{1,h} \quad \text{on } \Gamma_1 \end{aligned} \quad (32)$$

$$\begin{aligned} & a_2(\mathbf{u}_{2,h},\boldsymbol{\varphi}_{2,\ell}) + b_2(p_{2,h},\boldsymbol{\varphi}_{2,\ell}) = \textstyle\int_{\Gamma_2} \boldsymbol{\lambda}_{2,h}\cdot\boldsymbol{\varphi}_{2,\ell} \\ & \quad + \textstyle\int_{\Omega_2} \mathbf{f}\cdot\boldsymbol{\varphi}_{2,\ell}, \qquad \forall \ell \in \mathcal{I}^u_2 \\ & b_2(\psi_{2,k},\mathbf{u}_{2,h}) = 0 \qquad \forall k \in \mathcal{I}^p_2, \end{aligned} \quad (33)$$

$$\begin{aligned} & a_1(\mathbf{w}_{1,h},\boldsymbol{\varphi}_{1,\ell}) + b_1(q_{1,h},\boldsymbol{\varphi}_{1,\ell}) = 0 \quad \forall \ell \in \mathcal{I}^u_1 \\ & b_1(\psi_{1,k},\mathbf{w}_{1,h}) = 0 \qquad \forall k \in \mathcal{I}^p_1, \\ & \mathbf{w}_{1,h} = \mathbf{u}_{1,h} - \mathbf{u}_{2,h} \quad \text{on } \Gamma_1 \end{aligned} \quad (34)$$

$$\begin{aligned} & a_2(\mathbf{w}_{2,h},\boldsymbol{\varphi}_{2,\ell}) + b_2(q_{2,h},\boldsymbol{\varphi}_{2,\ell}) = (\boldsymbol{\Phi}_{2,\Gamma_2})_\ell - (\boldsymbol{\Phi}_{1,\Gamma_2})_\ell \qquad \forall \ell \in \mathcal{I}^u_2 \\ & b_2(\psi_{2,k},\mathbf{w}_{2,h}) = 0 \qquad \forall k \in \mathcal{I}^p_2 \end{aligned} \quad (35)$$

and

$$\begin{aligned} & \big[(\mathbf{u}_{1,h|\Gamma_1})_\ell - (\mathbf{u}_{2,h|\Gamma_1})_\ell + (\mathbf{w}_{2,h|\Gamma_1})_\ell\big] \\ & + \big[-(\boldsymbol{\Phi}_{1,\Gamma_2})_\jmath + (\boldsymbol{\Phi}_{2,\Gamma_2})_\jmath + (\boldsymbol{\Psi}_{1,\Gamma_2})_\jmath\big] = 0 \\ & \qquad \forall \ell \in \mathcal{G}^u_1,\ \forall \jmath \in \mathcal{G}^u_2. \end{aligned} \quad (36)$$

To the minimization problem (31) we now associate the optimality system: find $\underline{\boldsymbol{\lambda}}_h = (\boldsymbol{\lambda}_{1,h},\boldsymbol{\lambda}_{2,h}) \in \boldsymbol{\Lambda}^N_{1,h} \times \boldsymbol{\Lambda}^D_{2,h}$ and, for $i = 1, 2$, $(\mathbf{u}_{i,h}, p_{i,h}) \in \boldsymbol{V}_{i,h} \times Q_{i,h}$, $(\mathbf{w}_i, q_i) \in \boldsymbol{V}_{i,h} \times Q_{i,h}$ such that

$$\begin{aligned} & a_1(\mathbf{u}_{1,h},\boldsymbol{\varphi}_{1,\ell}) + b_1(p_{1,h},\boldsymbol{\varphi}_{1,\ell}) = \textstyle\int_{\Gamma_1} \boldsymbol{\lambda}_{1,h}\cdot\boldsymbol{\varphi}_{1,\ell} \\ & \quad + \textstyle\int_{\Omega_1} \mathbf{f}\cdot\boldsymbol{\varphi}_{1,\ell}, \qquad \forall \ell \in \mathcal{I}^u_1 \\ & b_1(\psi_{1,k},\mathbf{u}_{1,h}) = 0 \qquad \forall k \in \mathcal{I}^p_1, \end{aligned} \quad (37)$$

$$\begin{aligned} & a_2(\mathbf{u}_{2,h},\boldsymbol{\varphi}_{2,\ell}) + b_2(p_{2,h},\boldsymbol{\varphi}_{2,\ell}) = \textstyle\int_{\Omega_2} \mathbf{f}\cdot\boldsymbol{\varphi}_{2,\ell}, \forall \ell \in \mathcal{I}^u_2 \\ & b_2(\psi_{2,k},\mathbf{u}_{2,h}) = 0 \qquad \forall k \in \mathcal{I}^p_2, \\ & \mathbf{u}_{2,h} = \boldsymbol{\lambda}_{2,h} \quad \text{on } \Gamma_2 \end{aligned} \quad (38)$$

$$\begin{aligned} & a_1(\mathbf{w}_{1,h},\boldsymbol{\varphi}_{1,\ell}) + b_1(q_{1,h},\boldsymbol{\varphi}_{1,\ell}) = (\boldsymbol{\Phi}_{1,\Gamma_1})_\ell - (\boldsymbol{\Phi}_{2,\Gamma_1})_\ell \qquad \forall \ell \in \mathcal{I}^u_1 \\ & b_1(\psi_{1,k},\mathbf{w}_{1,h}) = 0 \qquad \forall k \in \mathcal{I}^p_1 \end{aligned} \quad (39)$$

$$\begin{aligned} & a_2(\mathbf{w}_{2,h},\boldsymbol{\varphi}_{2,\ell}) + b_2(q_{2,h},\boldsymbol{\varphi}_{2,\ell}) = 0 \quad \forall \ell \in \mathcal{I}^u_2 \\ & b_2(\psi_{2,k},\mathbf{w}_{2,h}) = 0 \qquad \forall k \in \mathcal{I}^p_2, \\ & \mathbf{w}_{2,h} = \mathbf{u}_{1,h} - \mathbf{u}_{2,h} \quad \text{on } \Gamma_2 \end{aligned} \quad (40)$$

and

$$\begin{aligned} & \big[(\boldsymbol{\Phi}_{1,\Gamma_1})_\jmath - (\boldsymbol{\Phi}_{2,\Gamma_1})_\jmath + (\boldsymbol{\Psi}_{2,\Gamma_1})_\jmath\big] \\ & + \big[(\mathbf{u}_{1,h|\Gamma_2})_\ell - (\mathbf{u}_{2,h|\Gamma_2})_\ell + (\mathbf{w}_{1,h|\Gamma_2})_\ell\big] \\ & \qquad \forall \jmath \in \mathcal{G}^u_1,\ \forall \ell \in \mathcal{G}^u_2. \end{aligned} \quad (41)$$

3.5. Algebraic formulation of ICDD with Neumann and mixed controls

Using the previous notations, the discrete values of the Neumann controls are given by

$$(\boldsymbol{\lambda}_{i,h})_\ell = \sum_{k\in\mathcal{E}_i} \int_{e_{ik}} \boldsymbol{\lambda}_{i,h}\cdot\boldsymbol{\varphi}_{i,\ell} \quad \forall \ell \in \mathcal{G}^u_i.$$

Denoting by T_{ji} the finite dimensional counterpart of the operator that associates to the velocity and pressure in Ω_i the corresponding normal stress tensor on the interface Γ_j ($j = 1, 2$) (as in (25)), after discretization the optimality system (27)-(29) for the functional J_f with Neumann controls yields the following matrix:

$$\left(\begin{array}{cc|cc|cc} S_1 & 0 & 0 & 0 & -I_{\Gamma_1} & 0 \\ 0 & S_2 & 0 & 0 & 0 & -I_{\Gamma_2} \\ \hline 0 & T_{12} & S_1 & 0 & -I_{\Gamma_1} & 0 \\ T_{21} & 0 & 0 & S_2 & 0 & -I_{\Gamma_2} \\ \hline 0 & -T_{12} & 0 & T_{12} & I_{\Gamma_1} & 0 \\ -T_{21} & 0 & T_{21} & 0 & 0 & I_{\Gamma_2} \end{array}\right). \quad (42)$$

The corresponding Schur complement system becomes

$$\boldsymbol{\Sigma}_f \begin{pmatrix} \boldsymbol{\lambda}_{\Gamma_1} \\ \boldsymbol{\lambda}_{\Gamma_2} \end{pmatrix} = \boldsymbol{\chi}_f \quad (43)$$

where

$$\boldsymbol{\Sigma}_f = \begin{pmatrix} I_{\Gamma_1} - (T_{12}S_1^{-1})^2 \\ I_{\Gamma_2} - (T_{21}S_2^{-1})^2 \end{pmatrix}$$

and

$$\boldsymbol{\chi}_f = \begin{pmatrix} T_{12}S_1^{-1}(I_{\Gamma_1} + T_{12}S_1^{-1})\mathbf{F}_1 \\ T_{21}S_2^{-1}(I_{\Gamma_2} + T_{21}S_2^{-1})\mathbf{F}_2 \end{pmatrix}$$

Finally, the matrix associated with the optimality system (32)-(36) for the functional J_{tf} with mixed controls is:

$$\left(\begin{array}{cc|cc|cc} S_{I_1I_1} & 0 & 0 & 0 & S_{I_1\Gamma_1} & 0 \\ 0 & S_2 & 0 & 0 & 0 & -I_{\Gamma_2} \\ \hline 0 & -S_{I_1\Gamma_1}R_{12} & S_{I_1I_1} & 0 & S_{I_1\Gamma_1} & 0 \\ T_{21} & 0 & 0 & S_2 & 0 & -I_{\Gamma_2} \\ \hline 0 & -M_{\Gamma_1}R_{12} & 0 & M_{\Gamma_1}R_{12} & M_{\Gamma_1} & 0 \\ -T_{21} & 0 & T_{21} & 0 & 0 & I_{\Gamma_2} \end{array}\right).$$

Its corresponding Schur complement system becomes

$$\boldsymbol{\Sigma}_{tf}\begin{pmatrix}\boldsymbol{\lambda}_{\Gamma_1}\\ \boldsymbol{\lambda}_{\Gamma_2}\end{pmatrix} = \boldsymbol{\chi}_{tf} \tag{44}$$

where

$$\boldsymbol{\Sigma}_{tf} = \begin{pmatrix} M_{\Gamma_1}(I_{\Gamma_1} - (R_{12}S_{I_1I_1}^{-1}S_{I_1\Gamma_1})^2) \\ I_{\Gamma_2} - (T_{21}S_2^{-1})^2 \end{pmatrix}$$

and

$$\boldsymbol{\chi}_{tf} = \begin{pmatrix} M_{\Gamma_1}R_{12}S_{I_1I_1}^{-1}(I_{\Gamma_1} - S_{I_1\Gamma_1}R_{12}S_{I_1I_1}^{-1})\mathbf{F}_1 \\ T_{21}S_2^{-1}(I_{\Gamma_2} + T_{21}S_2^{-1})\mathbf{F}_2 \end{pmatrix}.$$

4. NUMERICAL RESULTS

4.1. Test cases with respect to an analytic solution

We consider the domain $\Omega = (0,1)\times(0,2)$ with $\Omega_1 = (0,1)\times(1-\delta/2,2)$ and $\Omega_2 = (0,1)\times(0,1+\delta/2)$, $\delta > 0$ being a suitable parameter characterizing the width of the overlapping region. The viscosity ν is set to 1, while the force $\mathbf{f}$ and the boundary conditions are chosen such that the Stokes problem admits the solution $\mathbf{u} = (\exp(y), -\exp(x))^T$ and $p = \exp(x)\sin(y)$. Concerning the boundary conditions, we impose Neumann conditions on the boundary $1\times(0,2)$ while Dirichlet boundary conditions are imposed on the remaining boundaries. We compute the solution of the optimality system using the Bi-CGStab method on the Schur complement (22) setting the tolerance to 10^{-9}.

First, we consider the case of an overlap with fixed width $\delta = 0.2$. We use both Taylor-Hood elements with three computational meshes characterized by $h = 2^{-2}, 2^{-3}, 2^{-4}$, and stabilized hp-FEM $\mathbb{Q}_p - \mathbb{Q}_p$ [8]. In the latter case, we consider 4×5 quad elements in each subdomain Ω_i, 4×1 elements in Ω_{12} and each quad element has sides of length $h = 2^{-2}$.

In tables I and II we report the number of iterations required to converge, the computed infimum of the cost functional J_t and the errors

$$e_1^u = \left(\|\mathbf{u}_1 - \mathbf{u}_{1,h}\|^2_{H^1(\Omega_1)} + \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|^2_{H^1(\Omega_2)}\right)^{1/2},$$

$$e_0^p = \left(\|p_1 - p_{1,h}\|^2_{L^2(\Omega_1)} + \|p_2 - p_{2,h}\|^2_{L^2(\Omega_2)}\right)^{1/2},$$

$e^u_{12,0} = \|\mathbf{u}_{1,h} - \mathbf{u}_{2,h}\|_{L^2(\Omega_{12})}$, $e^p_{12,0} = \|p_{1,h} - p_{2,h}\|_{L^2(\Omega_{12})}$, where $\mathbf{u}_{i,h} \in \boldsymbol{V}_{i,h}$ and $p_{i,h} \in Q_{i,h}$ are the solutions of (18)-(20).

The number of iterations is independent of both the grid size h and the polynomial degree p. Notice that the convergence order for the errors e_1^u and e_0^p in table I agrees with the expected optimal accuracy for the Taylor-Hood elements.

TABLE I: Test case with analytic solution. Results for the functional J_t with Taylor-Hood elements with respect to different values of h. Fixed overlap with $\delta = 0.2$.

h	#iter	inf J_t	e_1^u	e_0^p	$e^u_{12,0}$	$e^p_{12,0}$
2^{-2}	8	1.047e-16	1.146e-02	9.322e-03	8.081e-05	5.982e-03
2^{-3}	9	8.612e-20	2.835e-03	2.175e-03	6.033e-06	5.832e-04
2^{-4}	9	6.003e-20	7.088e-04	5.345e-04	5.432e-07	8.516e-05

TABLE II: Test case with analytic solution. Results for the functional J_t with stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with respect to different polynomial degrees p. Fixed overlap with $\delta = 0.2$.

p	#iter	inf J_t	e_1^u	e_0^p	$e^u_{12,0}$	$e^p_{12,0}$
2	10	1.759e-19	1.402e-03	2.676e-03	8.471e-06	1.543e-03
3	9	1.402e-19	6.719e-05	1.299e-04	6.220e-07	1.124e-04
4	9	3.766e-20	3.922e-07	3.157e-07	5.870e-10	8.652e-08
5	9	1.308e-20	8.740e-09	1.269e-08	7.617e-11	1.079e-08

Next, we study the case where the width of the overlap tends to zero on a fixed computational mesh. When using the Taylor-Hood elements, we set $h = 0.04$ and $\delta = 5h, \ldots, h$; the subdomains are defined as follows: for $\delta = 5h$, $\Omega_1 = (0,1)\times(0.92,2)$ and $\Omega_2 = (0,1)\times(0,1.12)$; for $\delta = 4h$, $\Omega_1 = (0,1)\times(0.92,2)$ and $\Omega_2 = (0,1)\times(0,1.08)$; for $\delta = 3h$, $\Omega_1 = (0,1)\times(0.96,2)$ and $\Omega_2 = (0,1)\times(0,1.08)$; for $\delta = 2h$, $\Omega_1 = (0,1)\times(0.96,2)$ and $\Omega_2 = (0,1)\times(0,1.04)$; for $\delta = h$, $\Omega_1 = (0,1)\times(0.96,2)$ and $\Omega_2 = (0,1)\times(0,1)$. For stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ approximations, we take $p = 4$ and we partition each subdomain in 4×5 quad elements; $\Omega_i\setminus\Omega_{12}$ is partitioned into 4×4 equal quad elements of size $h_x\cdot h_y$, $h_x = 0.25$ and $h_y = (1-\delta/2)/4$; Ω_{12} is partitioned in 1×5 quads of size $h_x\cdot\delta$; the value of δ ranges from 0.2 to 0.01. Results reported in tables III and IV show that the required number of iterations increases when δ decreases.

Finally, we carry out a convergence test with Taylor-Hood elements setting $\delta = h$ and letting $h\to 0$. Also in this case we can see that the number of iterations required to converge grows when h decreases. Results are reported in table V.

These numerical results show that the ICDD method is not very effective especially when considering small overlapping regions. This behavior may due to the fact that the functional J_t involves no information on the pressure fields

TABLE III: Test case with analytic solution. Results for the functional J_t with Taylor-Hood elements with $h = 0.04$ and $\delta \to 0$.

δ	#iter	inf J_t	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	9	9.281e-20	4.020e-04	3.271e-04	2.525e-07	3.270e-05
$4h$	10	1.565e-16	3.991e-04	3.242e-04	2.553e-07	3.611e-05
$3h$	13	8.237e-19	3.967e-04	3.213e-04	2.418e-07	4.456e-05
$2h$	18	7.275e-17	3.994e-04	3.345e-04	2.699e-07	1.362e-04
h	37	1.923e-14	4.780e-04	6.030e-04	1.688e-07	3.249e-04

TABLE IV: Test case with analytic solution. Results for the functional J_t with stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with respect to different polynomial degrees p for $\delta \to 0$. By * we denote that the method did not converge within 250 iterations.

δ	#iter	inf J_t	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	9	3.766e-20	3.922e-07	3.157e-07	5.870e-10	8.652e-08
0.1	15	2.391e-17	5.956e-07	7.930e-07	1.729e-09	4.741e-07
0.05	25	1.266e-17	2.806e-06	5.088e-06	1.333e-09	2.561e-06
0.02	71	3.369e-16	2.699e-05	4.974e-05	2.856e-09	1.571e-05
0.01	250*	4.208e-04	1.614e+01	2.909e+01	2.056e-03	7.755e+00

in the overlap, since it imposes only the continuity of velocities on the interfaces.

The number of iterations is independent of the mesh size h and of the polynomial degree p. However, a dependence on the size of the overlap can be estimated as

$$\#iter \sim C\delta^{-1},$$

for a suitable positive constant $C > 0$.

We consider now the case of Neumann and mixed controls.

First, we consider the case of an overlap with fixed width $\delta = 0.2$. The setting and the discretization are the same used before. In tables VI and VII we report the number of iterations and the computed errors for the case of the functional J_f using Taylor-Hood and stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ approximations, respectively, while in tables VIII and IX we report the results obtained for the functional J_{tf}.

Then, we consider the case where the width of the overlap tends to zero on a fixed computational mesh. Results are shown in tables X and XII for the Taylor-Hood elements with $h = 0.04$ and in tables XI and XIII for the stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with $p = 4$. Both functionals J_f and J_{tf} are used.

TABLE V: Test case with analytic solution. Results for the functional J_t with Taylor-Hood elements with $\delta = h$ and $\delta \to 0$.

$\delta = h$	#iter	inf J_t	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$1/3$	6	4.687e-18	2.615e-02	2.345e-02	5.390e-04	2.558e-02
$1/6$	10	1.309e-17	6.779e-03	5.792e-03	3.262e-05	3.035e-03
$2/25$	19	4.989e-19	1.626e-03	1.533e-03	2.240e-06	7.548e-04
$1/25$	37	1.923e-14	4.780e-04	6.030e-04	1.688e-07	3.249e-04

TABLE VI: Test case with analytic solution. Results for the functional J_f with Taylor-Hood elements with respect to different values of h. Fixed overlap with $\delta = 0.2$.

h	#iter	inf J_f	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
2^{-2}	9	1.904e-19	1.150e-02	9.065e-03	7.206e-04	1.878e-03
2^{-3}	9	2.070e-18	2.839e-03	2.154e-03	7.145e-05	2.391e-04
2^{-4}	9	1.233e-18	7.087e-04	5.319e-04	7.679e-06	3.352e-05

TABLE VII: Test case with analytic solution. Results for the functional J_f with stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with respect to different polynomial degrees p. Fixed overlap with $\delta = 0.2$.

p	#iter	inf J_f	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
2	10	2.249e-24	1.235e-03	2.370e-03	1.485e-05	5.159e-04
3	9	3.615e-18	2.141e-05	5.376e-05	4.256e-07	8.935e-06
4	9	2.438e-18	3.889e-07	3.031e-07	3.230e-09	2.777e-08
5	9	2.377e-18	6.996e-09	6.726e-09	7.773e-10	1.596e-09

Finally, we study the behavior of the ICDD method with functionals J_f and J_{tf} using Taylor-Hood elements setting $\delta = h$ and letting $h \to 0$. Results are reported in tables XIV and XV.

Differently from the case of Dirichlet controls with functional J_t, we can see that both functionals J_f and J_{tf} require a much lower number of iterations to converge. This shows that *controlling the pressure and not only the velocity on the interfaces is crucial for the Stokes problem.*

Moreover, we can see that the best convergence results are obtained with mixed controls and functional J_{tf}: as a matter of fact, in this case the number of iterations is independent from the mesh size h, from the degree p of polynomial used, and from the measure δ of the overlap.

Neumann controls with functional J_f also provide a number of iterations independent of the mesh size h and of the polynomial degree p. However, a dependence on the size of the overlap can be noticed as

$$\#iter \sim C\delta^{-1/2}$$

for a suitable positive constant $C > 0$.

TABLE VIII: Test case with analytic solution. Results for the functional J_{tf} with Taylor-Hood elements with respect to different values of h. Fixed overlap with $\delta = 0.2$.

h	#iter	inf J_{tf}	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
2^{-2}	6	1.211e-16	1.126e-02	8.778e-03	5.597e-05	2.285e-03
2^{-3}	6	2.516e-16	2.829e-03	2.143e-03	4.352e-06	2.885e-04
2^{-4}	6	1.445e-16	7.082e-04	5.314e-04	3.417e-07	3.938e-05

TABLE IX: Test case with analytic solution. Results for the functional J_{tf} with stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with respect to different polynomial degrees p. Fixed overlap with $\delta = 0.2$.

p	#iter	inf J_{tf}	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
2	6	4.003e-18	1.231e-03	2.391e-03	1.101e-05	6.378e-04
3	6	3.044e-18	2.147e-05	5.461e-05	1.894e-07	1.281e-05
4	6	2.334e-18	3.890e-07	3.038e-07	3.459e-10	3.191e-08
5	6	2.185e-18	6.358e-09	6.231e-09	8.761e-11	2.384e-09

TABLE X: Test case with analytic solution. Results for the functional J_f with Taylor-Hood elements with $h = 0.04$ and $\delta \to 0$.

δ	#iter	inf J_f	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	9	1.743e-18	4.019e-04	3.268e-04	3.778e-07	1.523e-05
$4h$	11	1.734e-21	3.990e-04	3.238e-04	4.201e-07	1.577e-05
$3h$	14	5.866e-21	3.964e-04	3.206e-04	4.590e-07	1.453e-05
$2h$	18	3.328e-15	3.935e-04	3.176e-04	1.011e-06	1.904e-05
h	35	8.742e-15	3.917e-04	3.160e-04	5.071e-06	9.156e-06

4.2. A test case without analytic solution

We consider the computational domain $\Omega = (0,1) \times (0,2)$ with $\Omega_1 = (0,1) \times (1-\delta/2, 2)$ and $\Omega_2 = (0,1) \times (0, 1+\delta/2)$, as represented schematically in Figure 2. The force is set to $\mathbf{f} = \mathbf{0}$ and the viscosity is $\nu = 2.e{-}3$. We impose homogeneous Neumann boundary conditions for the fluid normal stress on the edges l_4 and l_7. On the remaining boundaries, apart from the edge l_6, we impose homogeneous Dirichlet boundary conditions for the fluid velocity unless on $\{0\} \times (1.1, 2)$ where we set a parabolic profile with maximum equal to 1.

On the edge l_6 we may impose either homogeneous Neumann or Dirichlet boundary conditions to compare the behavior of the different methods that we have studied. In particular, we want to show that the functional J_t with Dirichlet controls will not provide a correct solution when l_6 is set as a Dirichlet boundary, since this case violates Assumption 2.1.

For this problem, besides the errors $e_{12,0}^u$ and $e_{12,0}^p$ on the overlap, we also compute

TABLE XI: Test case with analytic solution. Results for the functional J_f with stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with respect to different polynomial degrees p for $\delta \to 0$. By * we denote that the method did not converge within 250 iterations.

δ	#iter	inf J_f	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	9	2.438e-18	3.889e-07	3.031e-07	3.230e-09	2.777e-08
0.1	15	3.759e-17	4.635e-07	4.247e-07	8.579e-09	2.059e-08
0.05	25	4.783e-15	6.835e-07	7.114e-07	5.991e-08	2.168e-08
0.02	96	2.032e-16	6.653e-07	6.873e-07	3.315e-08	2.328e-08
0.01	250*	5.751e-04	8.371e-01	9.842e-01	5.206e-02	1.733e-03

TABLE XII: Test case with analytic solution. Results for the functional J_{tf} with Taylor-Hood elements with $h = 0.04$ and $\delta \to 0$.

δ	#iter	inf J_{tf}	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	6	1.861e-16	4.019e-04	3.268e-04	4.044e-07	2.324e-05
$4h$	6	2.339e-16	3.989e-04	3.239e-04	3.723e-07	2.348e-05
$3h$	7	1.441e-16	3.964e-04	3.206e-04	3.163e-07	2.153e-05
$2h$	7	4.691e-15	3.935e-04	3.176e-04	2.709e-07	2.561e-05
h	7	5.075e-15	3.907e-04	3.141e-04	1.596e-07	1.332e-05

TABLE XIII: Test case with analytic solution. Results for the functional J_{tf} with stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with respect to different polynomial degrees p for $\delta \to 0$.

δ	#iter	inf J_{tf}	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	6	2.334e-18	3.890e-07	3.038e-07	3.459e-10	3.191e-08
0.1	7	7.935e-16	4.621e-07	4.239e-07	8.769e-10	4.387e-08
0.05	7	6.422e-16	5.093e-07	4.877e-07	4.043e-10	3.179e-08
0.02	7	1.103e-14	5.421e-07	5.183e-07	9.533e-10	2.265e-08
0.01	8	4.678e-14	5.511e-07	5.318e-07	5.511e-10	4.550e-08

$$e_1^u = \left(\|\mathbf{U}_{1,h} - \mathbf{u}_{1,h}\|^2_{H^1(\Omega_1)} + \|\mathbf{U}_{2,h} - \mathbf{u}_{2,h}\|^2_{H^1(\Omega_2)}\right)^{1/2},$$
$$e_0^p = \left(\|P_{1,h} - p_{1,h}\|^2_{L^2(\Omega_1)} + \|P_{2,h} - p_{2,h}\|^2_{L^2(\Omega_2)}\right)^{1/2},$$

where $(\mathbf{U}_{i,h}, P_{i,h})$ is the restriction to the subdomain Ω_i of the solution computed on the same mesh but considering the domain as a whole without any splitting and solving (1).

First, we impose homogeneous Dirichlet boundary conditions on l_6. The results obtained in correspondence of the different functionals J_t, J_f and J_{tf} are reported in table XVI for Taylor-Hood elements and in table XVII for stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with $p = 6$.

As expected, the minimization of the functional J_t does not allow to recover the correct solution, whereas both J_f and J_{tf} converge to the correct solution. In Figure 3 we show the original solution, while in Figures 4 and 5 we show, respectively, the solutions obtained through minimization of the functional J_t and J_tf. We can see that the functional J_t has no control on the pressure, which therefore does not match on the overlap.

Now, we impose homogeneous Neumann boundary conditions on l_6. In this case, according to the theory, all functionals allow to correctly compute the single-domain solu-

TABLE XIV: Test case with analytic solution. Results for the functional J_f with Taylor-Hood elements with $\delta = h$ and $\delta \to 0$.

$\delta = h$	#iter	inf J_f	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$1/3$	6	2.610e-18	2.510e-02	2.051e-02	1.112e-03	5.119e-03
$1/6$	10	6.433e-16	6.723e-03	5.416e-03	1.038e-04	5.577e-04
$2/25$	18	7.986e-16	1.576e-03	1.280e-03	2.405e-05	7.098e-05
$1/25$	35	8.742e-15	3.917e-04	3.160e-04	5.071e-06	9.156e-06

TABLE XV: Test case with analytic solution. Results for the functional J_{tf} with Taylor-Hood elements with $\delta = h$ and $\delta \to 0$.

$\delta = h$	#iter	inf J_{tf}	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$1/3$	5	4.337e-18	2.487e-02	2.051e-02	7.757e-04	8.554e-03
$1/6$	6	1.596e-15	6.709e-03	5.411e-03	4.145e-05	8.970e-04
$2/25$	7	7.951e-15	1.572e-03	1.275e-03	2.503e-06	1.067e-04
$1/25$	7	5.075e-15	3.907e-04	3.141e-04	1.596e-07	1.332e-05

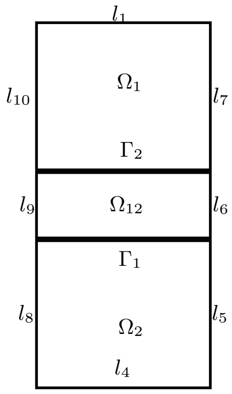

FIG. 2: Schematic representation of the computational domain.

tion and their behaviors are similar to those observed in the previous tests with analytic solution. The functional J_{tf} associated with mixed controls is the one that converges in the lowest number of iterations with a slight dependence on δ. Results are reported in table XVIII for Taylor-Hood elements and in table XIX for stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with $p = 6$.

In Figure 6 we show the single-domain solution, while in Figures 7 and 8 we show, respectively, the solutions obtained through minimization of the functional J_t and J_{tf}. We can see that, although the functional J_t has no control on the pressure, the Neumann boundary condition on the edge l_6 allows the pressure to match almost perfectly in the overlapping region. Notice that the difference shown in Fig. 7 is of the same order of the errors reported in tables XVIII and XIX.

Finally, let us consider a test case in which the interface is a piecewise linear curve (identified by element edges), as shown in Fig. 9. We compute the solution by imposing a Neumann boundary condition on the boundary l_6 considering stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with $p = 6$ and $\delta \to 0$. The iterations numbers shown in table XX behave similarly to those presented in the third block of table XIX: the algorithm is not strongly influenced by the shape of the interface.

To assess the robustness of the method with respect to the viscosity coefficient ν, we compute the solution of the problem with Neumann boundary condition on l_6 using the ICDD method associated with the functional J_{tf}, the one that provided the best results in the previous tests. We consider a discretization by Taylor-Hood elements on a mesh with fixed $h = 0.04$ and $\delta \to 0$ and we set the viscosity $\nu = 10^{-2}, 10^{-4}, 10^{-6}$. Numerical results are reported in tables XXI, XXII; clearly they show that the method is robust with respect to variations of the parameter ν.

TABLE XVI: Test case without analytic solution. Dirichlet boundary condition on l_6. Results for the functionals J_t (top), J_f (mid) and J_{tf} (bottom) with Taylor-Hood elements with fixed $h = 0.04$ and $\delta \to 0$.

δ	#iter	inf J_t	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	8	9.779e-23	4.556e-02	6.647e-04	8.754e-04	8.841e-04
$4h$	9	5.197e-21	2.805e-01	3.740e-03	9.122e-04	4.568e-03
$3h$	12	5.289e-23	2.967e-01	3.917e-03	1.002e-03	4.261e-03
$2h$	16	6.533e-21	2.618e-02	3.220e-04	8.911e-05	2.979e-04
h	31	8.042e-22	1.921e-01	2.231e-03	9.904e-05	1.510e-03

δ	#iter	inf J_f	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	7	9.209e-26	1.592e-03	8.534e-05	1.452e-03	1.192e-04
$4h$	8	2.376e-24	3.395e-03	7.345e-05	2.684e-03	8.409e-05
$3h$	10	3.043e-24	5.410e-03	9.049e-05	3.822e-03	8.284e-05
$2h$	15	2.700e-27	2.634e-03	3.620e-05	1.786e-03	2.044e-05
h	28	1.563e-25	4.216e-02	4.909e-04	1.975e-02	8.692e-05

δ	#iter	inf J_{tf}	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	5	2.989e-20	1.295e-03	8.531e-05	6.078e-04	1.206e-04
$4h$	6	1.843e-22	9.068e-04	6.122e-05	3.653e-04	8.674e-05
$3h$	6	1.311e-20	9.063e-04	6.157e-05	3.419e-04	8.737e-05
$2h$	6	1.540e-21	2.885e-04	2.160e-05	9.796e-05	3.046e-05
h	7	8.444e-20	1.208e-03	9.366e-05	2.619e-04	1.284e-04

5. ANALYSIS OF THE ICDD METHOD FOR THE STOKES PROBLEM

In this section we analyze the ICDD method that we have presented in the previous sections with the aim of guaranteeing the well-posedness of the minimization problem. We begin with the analysis in the continuous case with Dirichlet controls.

5.1. Analysis of the optimal control problem with Dirichlet controls

For $i = 1, 2$, we introduce the following spaces:

$$\begin{aligned} \mathbf{\Lambda}_i &= \{\boldsymbol{\mu} \in [H^{1/2}(\Gamma_i)]^d \,:\, \exists \mathbf{v} \in [H^1(\Omega_i)]^d, \\ &\qquad \mathbf{v} = \boldsymbol{\mu} \text{ on } \Gamma_i \text{ and } \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D^i\} \end{aligned} \tag{45}$$

$$\mathbf{\Lambda}_{i,0} = \{\boldsymbol{\mu} \in \mathbf{\Lambda}_i \,:\, \textstyle\int_{\Gamma_i} \boldsymbol{\mu} \cdot \mathbf{n} = 0\} \tag{46}$$

TABLE XVII: Test case without analytic solution. Dirichlet boundary condition on l_6. Results for the functionals J_t (top), J_f (mid) and J_{tf} (bottom) with stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with fixed $p = 6$ and $\delta \to 0$.

δ	#iter	inf J_t	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	8	1.582e-20	3.151e-02	1.239e-04	2.158e-05	1.176e-04
0.1	14	1.582e-24	2.884e-01	9.612e-04	3.424e-05	7.473e-04
0.05	25	1.698e-22	4.565e-01	1.389e-03	2.740e-05	8.364e-04
0.02	65	4.433e-22	2.349e+00	7.292e-03	4.594e-05	2.861e-03
0.01	214	2.483e-23	6.278e+00	2.021e-02	4.281e-05	5.798e-03

δ	#iter	inf J_f	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	6	1.955e-26	3.475e-03	3.354e-06	1.761e-04	1.605e-05
0.1	11	1.431e-25	4.699e-03	1.056e-05	4.011e-04	1.344e-05
0.05	22	1.154e-24	1.235e-02	3.908e-05	1.148e-03	1.213e-05
0.02	55	1.241e-24	1.037e-02	2.953e-05	6.474e-04	1.436e-05
0.01	165	2.021e-23	3.626e-02	1.025e-04	1.660e-03	2.036e-05

δ	#iter	inf J_{tf}	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	5	3.071e-23	3.360e-03	1.862e-06	5.640e-05	1.618e-05
0.1	6	2.434e-21	3.506e-03	2.626e-06	4.390e-05	1.396e-05
0.05	7	7.309e-22	3.656e-03	4.075e-06	3.731e-05	1.384e-05
0.02	7	5.923e-20	4.451e-03	2.545e-05	4.227e-05	2.417e-05
0.01	7	2.192e-20	5.731e-03	4.025e-05	2.936e-05	3.735e-05

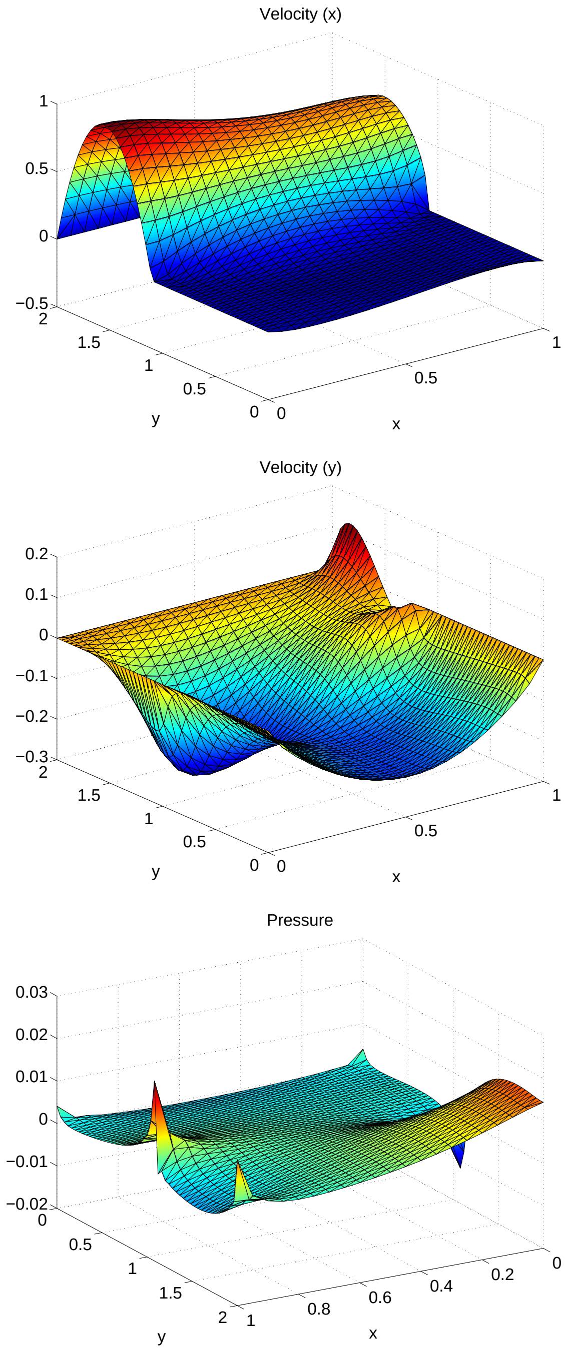

FIG. 3: Test case without analytic solution. Dirichlet boundary condition on l_6. Reference monodomain solution computed using Taylor-Hood finite elements.

We will denote by

$$\mathbf{\Lambda}_i^D = \begin{cases} \mathbf{\Lambda}_i & \text{if } \partial\Omega_i \cap \Gamma_N \neq \emptyset \\ \mathbf{\Lambda}_{i,0} & \text{if } \partial\Omega_i \cap \Gamma_N = \emptyset, \end{cases} \qquad i = 1, 2, \tag{47}$$

the spaces of *admissible Dirichlet controls*. Moreover, we will denote

$$\mathbf{\Lambda}^D = \mathbf{\Lambda}_1^D \times \mathbf{\Lambda}_2^D. \tag{48}$$

For $i = 1, 2$, we consider two unknown control functions $\boldsymbol{\lambda}_i \in \mathbf{\Lambda}_i^D$ and the associated state problems

$$\begin{aligned} -\mathbf{div}\, \mathbf{T}(\mathbf{u}_i^{\boldsymbol{\lambda}_i,\mathbf{f}}, p_i^{\boldsymbol{\lambda}_i,\mathbf{f}}) &= \mathbf{f} && \text{in } \Omega_i \\ \operatorname{div} \mathbf{u}_i^{\boldsymbol{\lambda}_i,\mathbf{f}} &= 0 && \text{in } \Omega_i \\ \mathbf{u}_i^{\boldsymbol{\lambda}_i,\mathbf{f}} &= \boldsymbol{\lambda}_i && \text{on } \Gamma_i \end{aligned} \tag{49}$$

with suitable homogeneous boundary conditions on $\partial\Omega_i \setminus \Gamma_i$. If $\Gamma_N^i = \emptyset$, we add the constraint $\int_{\Omega_i} p_i^{\boldsymbol{\lambda}_i,\mathbf{f}} = 0$. The unknown controls on the interface are obtained by solving the minimization problem

$$\inf_{\underline{\boldsymbol{\lambda}}=(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\in\mathbf{\Lambda}^D} \left[J_t(\underline{\boldsymbol{\lambda}}) := \frac{1}{2}\sum_{i=1}^{2} \|\mathbf{u}_1^{\boldsymbol{\lambda}_1,\mathbf{f}} - \mathbf{u}_2^{\boldsymbol{\lambda}_2,\mathbf{f}}\|^2_{L^2(\Gamma_i)} \right] \tag{50}$$

where, for the sake of simplicity, we adopt the same notation as in the discrete case.

This yields an optimal control problem where both the control functions and the observations are of boundary (interface) type.

Thanks to the linearity of the problem, we have $\mathbf{u}_i^{\boldsymbol{\lambda}_i,\mathbf{f}} = \mathbf{u}_i^{\boldsymbol{\lambda}_i,\mathbf{0}} + \mathbf{u}_i^{\mathbf{0},\mathbf{f}}$ and $p_i^{\boldsymbol{\lambda}_i,\mathbf{f}} = p_i^{\boldsymbol{\lambda}_i,\mathbf{0}} + p_i^{\mathbf{0},\mathbf{f}}$. For the sake of simplicity, we will indicate $\mathbf{u}_i^{\boldsymbol{\lambda}_i} = \mathbf{u}_i^{\boldsymbol{\lambda}_i,\mathbf{0}}$ and $p_i^{\boldsymbol{\lambda}_i} = p_i^{\boldsymbol{\lambda}_i,\mathbf{0}}$

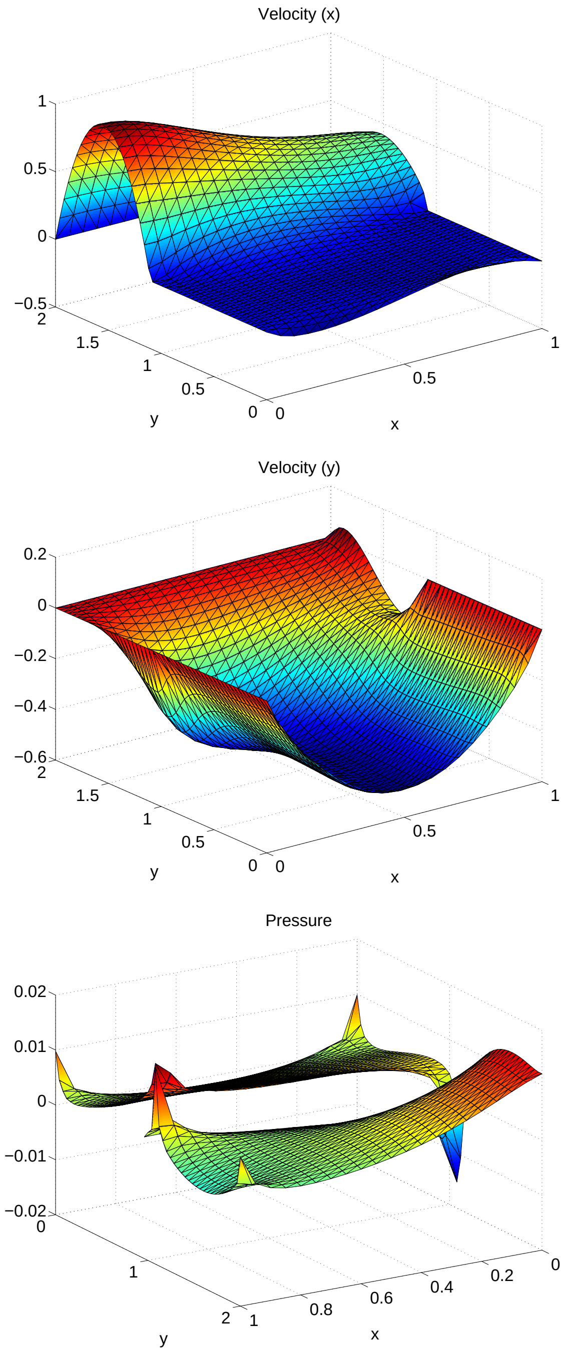

FIG. 4: Test case without analytic solution. Dirichlet boundary condition on l_6. Solution computed by minimizing the functional J_t using Taylor-Hood finite elements.

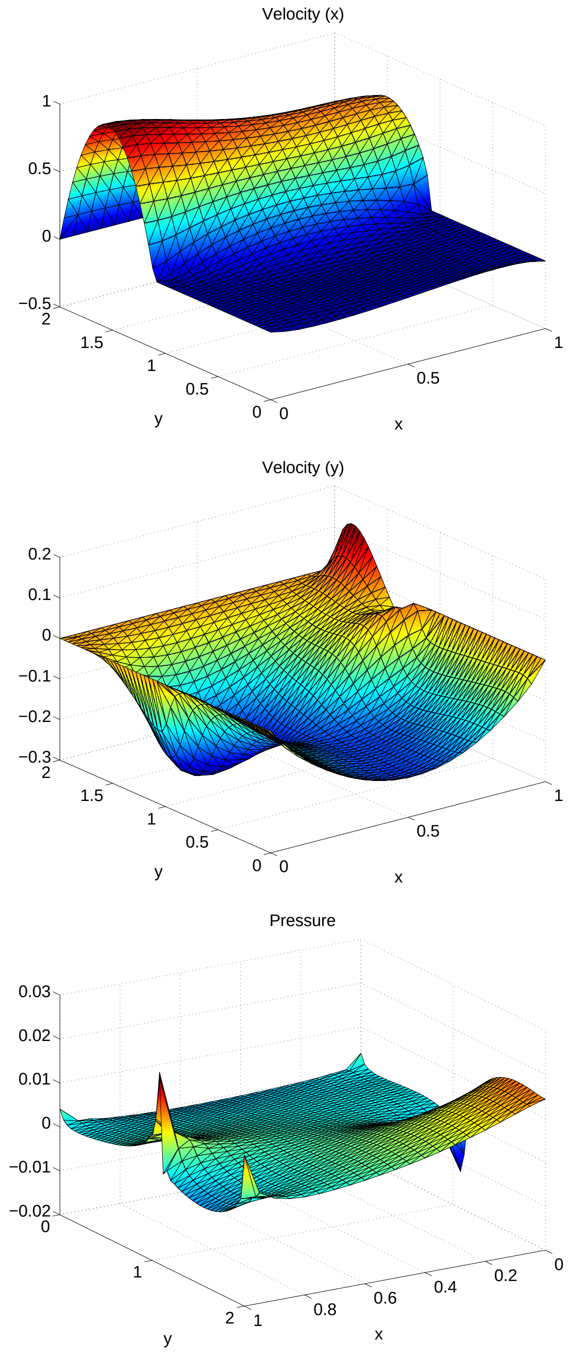

FIG. 5: Test case without analytic solution. Dirichlet boundary condition on l_6. Solution computed by minimizing the functional J_{tf} using Taylor-Hood finite elements.

and $\underline{\mathbf{u}}^{\boldsymbol{\lambda}} = (\mathbf{u}_1^{\boldsymbol{\lambda}_1}, \mathbf{u}_2^{\boldsymbol{\lambda}_2})$, $\underline{p}^{\boldsymbol{\lambda}} = (p_1^{\boldsymbol{\lambda}_1}, p_2^{\boldsymbol{\lambda}_2})$.

TABLE XVIII: Test case without analytic solution. Neumann boundary condition on l_6. Results for the functionals J_t (top), J_f (mid) and J_{tf} (bottom) with Taylor-Hood elements with fixed $h = 0.04$ and $\delta \to 0$.

δ	#iter	inf J_t	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	8	3.771e-21	1.164e-02	2.069e-04	7.870e-04	2.954e-04
$4h$	9	1.942e-20	2.075e-02	3.106e-04	8.418e-04	4.235e-04
$3h$	12	2.629e-23	4.843e-02	6.650e-04	9.458e-04	8.056e-04
$2h$	17	1.470e-25	1.385e-03	2.994e-05	7.864e-05	3.913e-05
h	34	1.077e-24	5.109e-02	5.826e-04	2.688e-04	4.050e-04

δ	#iter	inf J_f	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	9	1.419e-24	7.686e-03	1.113e-04	7.862e-03	7.401e-05
$4h$	11	8.495e-27	1.126e-02	1.562e-04	1.045e-02	7.669e-05
$3h$	14	4.060e-27	1.903e-02	2.647e-04	1.547e-02	7.519e-05
$2h$	19	1.990e-27	2.787e-03	3.860e-05	1.839e-03	1.283e-05
h	38	1.353e-25	2.812e-03	4.766e-05	1.326e-03	3.166e-05

δ	#iter	inf J_{tf}	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	5	1.173e-19	7.973e-04	5.532e-05	3.452e-04	7.837e-05
$4h$	6	1.723e-23	8.329e-04	5.761e-05	3.464e-04	8.162e-05
$3h$	6	2.645e-21	8.210e-04	5.716e-05	3.127e-04	8.108e-05
$2h$	6	3.243e-21	1.868e-04	1.233e-05	5.739e-05	1.728e-05
h	7	2.232e-20	5.438e-04	2.774e-05	1.060e-04	3.649e-05

Then, we can equivalently express the cost functional as

$$J_t(\underline{\boldsymbol{\lambda}}) = \sum_{i=1}^{2} \Big[\frac{1}{2} \|\mathbf{u}_1^{\boldsymbol{\lambda}_1} - \mathbf{u}_2^{\boldsymbol{\lambda}_2}\|^2_{L^2(\Gamma_i)} + (\mathbf{u}_1^{\boldsymbol{\lambda}_1} - \mathbf{u}_2^{\boldsymbol{\lambda}_2}, \mathbf{u}_1^{0,\mathbf{f}} - \mathbf{u}_2^{0,\mathbf{f}})_{L^2(\Gamma_i)} + \frac{1}{2} \|\mathbf{u}_1^{0,\mathbf{f}} - \mathbf{u}_2^{0,\mathbf{f}}\|^2_{L^2(\Gamma_i)} \Big]. \quad (51)$$

In this section we will denote $|||\underline{\boldsymbol{\lambda}}|||_D = \sum_{i=1}^{2} \|\mathbf{u}_1^{\boldsymbol{\lambda}_1} - \mathbf{u}_2^{\boldsymbol{\lambda}_2}\|_{L^2(\Gamma_i)}$.

Lemma 5.1 *If the boundary conditions imposed on the Stokes problem* (1) *satisfy Assumption 2.1, then* $|||\underline{\boldsymbol{\lambda}}|||_D$ *defines a norm on the space* $\boldsymbol{\Lambda}^D$.

Proof. Since $|||\underline{\boldsymbol{\lambda}}|||_D$ is obviously a semi-norm on $\boldsymbol{\Lambda}^D$, we only have to prove that, if $|||\underline{\boldsymbol{\lambda}}|||_D = 0$, then $\underline{\boldsymbol{\lambda}} = \underline{\mathbf{0}}$. Obviously, $|||\underline{\boldsymbol{\lambda}}|||_D = 0$ implies that $\mathbf{u}_1^{\boldsymbol{\lambda}_1} = \mathbf{u}_2^{\boldsymbol{\lambda}_2}$ a.e. on $\Gamma_1 \cup \Gamma_2$. As $\mathbf{u}_i^{\boldsymbol{\lambda}_i}$, $p_i^{\boldsymbol{\lambda}_i}$ is the solution of (49) with $\mathbf{f} = \mathbf{0}$, $(\mathbf{w}, q) = (\mathbf{u}^{\boldsymbol{\lambda}_1}_{1|\Omega_{12}} - \mathbf{u}^{\boldsymbol{\lambda}_2}_{2|\Omega_{12}}, p^{\boldsymbol{\lambda}_1}_{1|\Omega_{12}} - p^{\boldsymbol{\lambda}_2}_{2|\Omega_{12}})$ satisfies

$$\begin{aligned} -\mathbf{div}\, \mathbf{T}(\mathbf{w}, q) &= \mathbf{0} \ \text{ in } \Omega_{12} \\ \operatorname{div} \mathbf{w} &= 0 \ \text{ in } \Omega_{12} \\ \mathbf{w} &= \mathbf{0} \ \text{ on } \Gamma_1 \cup \Gamma_2 \end{aligned} \quad (52)$$

with suitable homogeneous boundary conditions on $\partial\Omega_{12} \cap \partial\Omega$. Since $\mathbf{u}_i^{\boldsymbol{\lambda}_i}$ belongs to $H^1(\Omega_i)$, condition $(52)_3$ has to be interpreted in the sense of traces of zeroth order of H^1 functions on $\Gamma_1 \cup \Gamma_2$.

TABLE XIX: Test case without analytic solution. Neumann boundary condition on l_6. Results for the functionals J_t (top), J_f (mid) and J_{tf} (bottom) with stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with fixed $p = 6$ and $\delta \to 0$. By * we denote that the method did not converge within 250 iterations.

δ	#iter	inf J_t	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	8	7.054e-24	1.398e-02	8.173e-05	1.539e-04	6.044e-05
0.1	14	1.005e-25	3.046e-02	1.121e-04	1.233e-04	8.945e-05
0.05	25	1.388e-23	6.101e-02	1.964e-04	8.335e-05	1.249e-04
0.02	65	3.579e-23	9.699e-01	3.047e-03	6.782e-05	1.272e-03
0.01	211	8.323e-21	6.444e+00	1.995e-02	8.936e-05	5.879e-03

δ	#iter	inf J_f	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	8	3.724e-25	1.240e-02	5.379e-05	1.271e-04	3.804e-05
0.1	14	2.542e-25	1.619e-02	6.113e-05	1.056e-03	3.920e-05
0.05	26	1.076e-24	1.899e-02	7.107e-05	1.229e-03	3.609e-05
0.02	107	2.023e-23	1.876e-02	7.074e-05	9.042e-04	2.798e-05
0.01	250*	4.232e-17	5.735e-01	2.461e-03	2.657e-02	7.751e-06

δ	#iter	inf J_{tf}	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	5	1.208e-22	1.288e-02	7.894e-05	3.015e-04	5.537e-05
0.1	6	2.256e-21	1.660e-02	7.874e-05	2.277e-04	5.645e-05
0.05	7	2.385e-22	1.690e-02	7.153e-05	1.438e-04	5.033e-05
0.02	7	1.144e-20	1.373e-02	4.956e-05	6.421e-05	3.249e-05
0.01	7	6.640e-21	1.048e-02	3.657e-05	2.670e-05	2.421e-05

TABLE XX: Test case without analytic solution. Piecewise linear interfaces. Neumann boundary condition on l_6. Results for the functionals J_{tf} with stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with fixed $p = 6$ and $\delta \to 0$.

δ	#iter	inf J_{tf}	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	6	2.908e-24	1.614e-02	7.436e-05	4.385e-04	5.197e-05
0.1	6	2.593e-20	1.709e-02	6.813e-05	2.951e-04	4.782e-05
0.05	7	8.702e-22	1.637e-02	5.600e-05	1.589e-04	4.031e-05
0.02	9	5.131e-19	1.439e-02	4.559e-05	7.481e-05	3.995e-05
0.01	9	1.323e-18	1.114e-02	4.391e-05	3.330e-05	3.991e-05

Following the same arguments used in the proof of Proposition 2.1, it can be shown that problem (52) is well-posed and its solution is $\mathbf{w} = \mathbf{0}$ and $q = const$. Thus, $\mathbf{u}_1^{\boldsymbol{\lambda}_1} = \mathbf{u}_2^{\boldsymbol{\lambda}_2}$ and $p_1^{\boldsymbol{\lambda}_1} + C_1 = p_2^{\boldsymbol{\lambda}_2} + C_2$ a.e. in Ω_{12} with $C_1, C_2 \in \mathbb{R}$, $q = C_2 - C_1$, and we can define

$$\overline{\mathbf{u}} = \begin{cases} \mathbf{u}_1^{\boldsymbol{\lambda}_1} & \text{in } \Omega_1 \setminus \Omega_{12} \\ \mathbf{u}_1^{\boldsymbol{\lambda}_1} = \mathbf{u}_2^{\boldsymbol{\lambda}_2} & \text{in } \Omega_{12} \\ \mathbf{u}_2^{\boldsymbol{\lambda}_2} & \text{in } \Omega_2 \setminus \Omega_{12} \end{cases} \quad (53)$$

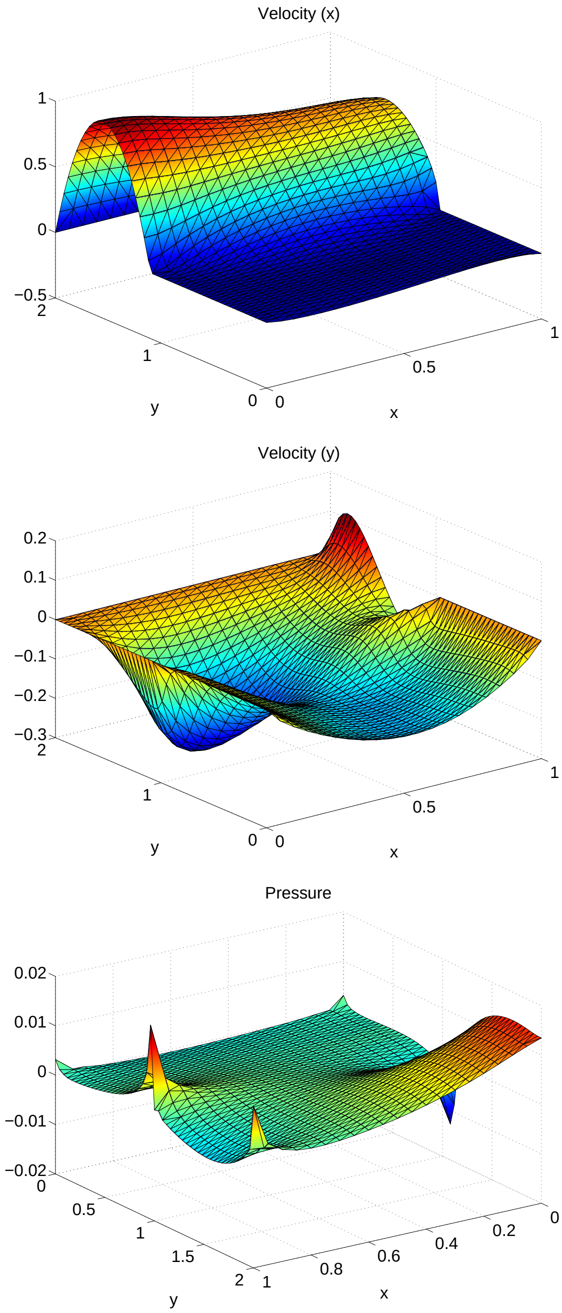

FIG. 6: Test case without analytic solution. Neumann boundary condition on l_6. Reference monodomain solution computed using Taylor-Hood finite elements.

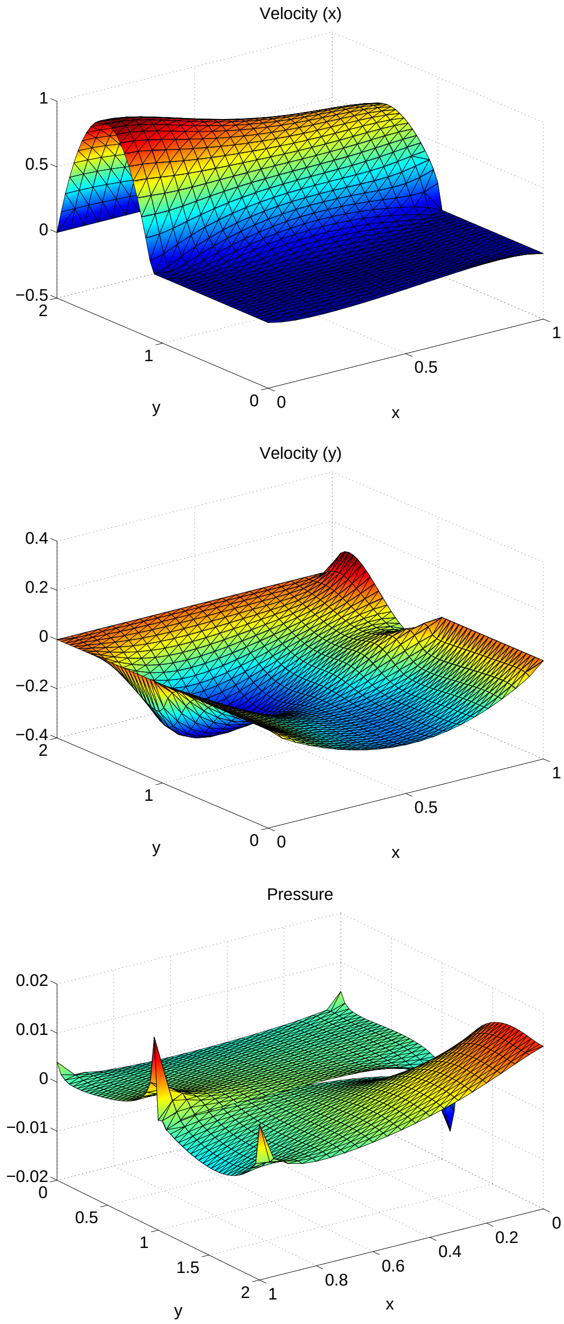

FIG. 7: Test case without analytic solution. Neumann boundary condition on l_6. Solution computed by minimizing the functional J_t using Taylor-Hood finite elements.

and

$$\overline{p} = \begin{cases} p_1^{\boldsymbol{\lambda}_1} + C_1 & \text{in } \Omega_1 \setminus \Omega_{12} \\ p_1^{\boldsymbol{\lambda}_1} + C_1 = p_2^{\boldsymbol{\lambda}_2} + C_2 & \text{in } \Omega_{12} \\ p_2^{\boldsymbol{\lambda}_2} + C_2 & \text{in } \Omega_2 \setminus \Omega_{12}. \end{cases} \qquad (54)$$

By construction, the pair $(\overline{\mathbf{u}}, \overline{p})$ satisfies a Stokes problem in Ω with null force and homogeneous boundary conditions with $\Gamma_N \neq \emptyset$. This problem is well-posed and, in particular, $\overline{\mathbf{u}} = \mathbf{0}$ a.e. in Ω. This implies that $\overline{\mathbf{u}} = \mathbf{0}$ on $\Gamma_1 \cup \Gamma_2$ and, for $i = 1, 2$, $\boldsymbol{\lambda}_i = \mathbf{0}$ in $\boldsymbol{\Lambda}_i$. □

Although we cannot guarantee that $\boldsymbol{\Lambda}^D$ is a complete space with respect to the norm $|||\underline{\boldsymbol{\lambda}}|||_D$, we can construct its completion, say $\hat{\boldsymbol{\Lambda}}^D$, with respect to such norm. In practice, we will always consider a finite dimensional space

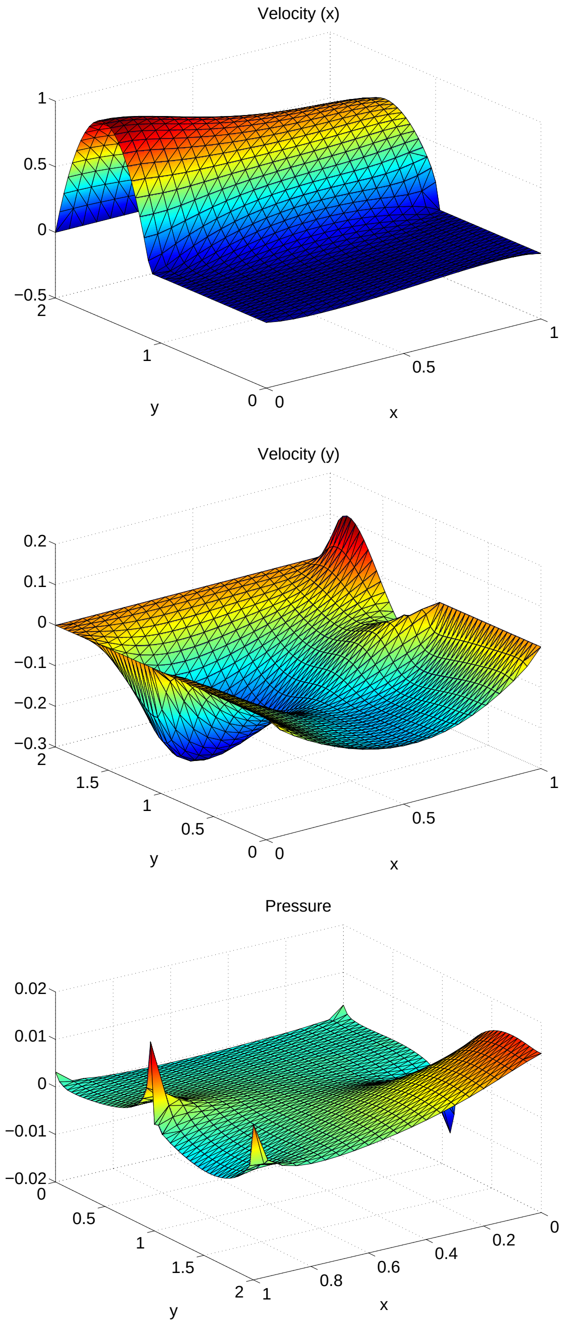

FIG. 8: Test case without analytic solution. Neumann boundary condition on l_6. Solution computed by minimizing the functional J_{tf} using Taylor-Hood finite elements.

$\mathbf{\Lambda}_h^D \subset \mathbf{\Lambda}^D \subseteq \hat{\mathbf{\Lambda}}^D$ and, at the discrete level, all norms are equivalent. Thus, this would not be a problem for the application that we have in mind. For the sake of notation, in the following we will still denote the completion of $\mathbf{\Lambda}^D$ by the same symbol.

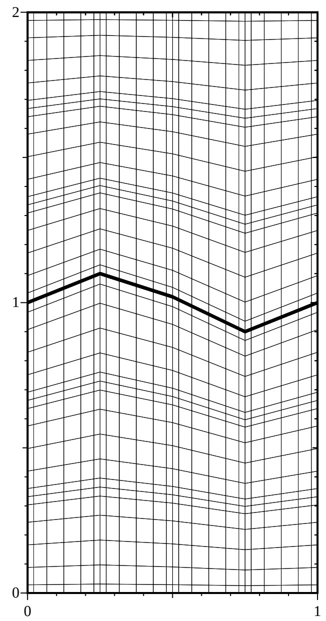

FIG. 9: Computational mesh for stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements in the case of piecewise linear interfaces. In the figure $\delta = 0.01$.

Theorem 5.1 *Consider the minimization problem*

$$\inf_{\underline{\boldsymbol{\lambda}} \in \mathbf{\Lambda}^D} J_t(\underline{\boldsymbol{\lambda}}). \tag{55}$$

If Assumption 2.1 holds, problem (55) *has a unique solution satisfying*

$$_{(\mathbf{\Lambda}^D)'}\langle J_t'(\underline{\boldsymbol{\lambda}}), \underline{\boldsymbol{\mu}} \rangle_{\mathbf{\Lambda}^D} = \sum_{i=1}^{2} (\mathbf{u}_1^{\boldsymbol{\lambda}_1,\mathbf{f}} - \mathbf{u}_2^{\boldsymbol{\lambda}_2,\mathbf{f}}, \mathbf{u}_1^{\boldsymbol{\mu}_1} - \mathbf{u}_2^{\boldsymbol{\mu}_2})_{L^2(\Gamma_i)} = 0 \tag{56}$$

for all $\underline{\boldsymbol{\mu}} \in \mathbf{\Lambda}^D$.

Proof. For any $\underline{\boldsymbol{\lambda}} \in \mathbf{\Lambda}^D$, let us define

$$\pi(\underline{\boldsymbol{\lambda}}, \underline{\boldsymbol{\mu}}) = \frac{1}{2} \sum_{i=1}^{2} (\mathbf{u}_1^{\boldsymbol{\lambda}_1} - \mathbf{u}_2^{\boldsymbol{\lambda}_2}, \mathbf{u}_1^{\boldsymbol{\mu}_1} - \mathbf{u}_2^{\boldsymbol{\mu}_2})_{L^2(\Gamma_i)},$$

$$L(\underline{\boldsymbol{\mu}}) = -\frac{1}{2} \sum_{i=1}^{2} (\mathbf{u}_1^{\mathbf{0},\mathbf{f}} - \mathbf{u}_2^{\mathbf{0},\mathbf{f}}, \mathbf{u}_1^{\boldsymbol{\mu}_1} - \mathbf{u}_2^{\boldsymbol{\mu}_2})_{L^2(\Gamma_i)}$$

so that

$$J_t(\underline{\boldsymbol{\lambda}}) = \pi(\underline{\boldsymbol{\lambda}}, \underline{\boldsymbol{\lambda}}) - 2L(\underline{\boldsymbol{\lambda}}) + \frac{1}{2} \sum_{i=1}^{2} \|\mathbf{u}_1^{\mathbf{0},\mathbf{f}} - \mathbf{u}_2^{\mathbf{0},\mathbf{f}}\|^2_{L^2(\Gamma_i)}.$$

The bilinear form $\pi : \mathbf{\Lambda}^D \times \mathbf{\Lambda}^D \to \mathbb{R}$ is symmetric by definition and, thanks to Lemma 5.1, is continuous and coercive with respect to the norm $|||\underline{\boldsymbol{\lambda}}|||_D$. Moreover,

TABLE XXI: Test case without analytic solution. Neumann boundary condition on l_6. Results obtained for the functional J_{tf} with Taylor-Hood elements with fixed $h = 0.04$ and $\delta \to 0$. The viscosity is $\nu = 10^{-2}$ (top), $\nu = 10^{-4}$ (mid), $\nu = 10^{-6}$ (bottom).

$\nu = 10^{-2}$

δ	#iter	$\inf J_{tf}$	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	5	2.688e-18	1.668e-03	2.766e-04	3.452e-04	3.918e-04
$4h$	6	4.242e-22	1.740e-03	2.880e-04	3.464e-04	4.081e-04
$3h$	6	6.553e-20	1.726e-03	2.858e-04	3.127e-04	4.054e-04
$2h$	6	8.063e-20	4.077e-04	6.166e-05	5.739e-05	8.638e-05
h	7	5.523e-19	1.150e-03	1.387e-04	1.060e-04	1.824e-04

$\nu = 10^{-4}$

δ	#iter	$\inf J_{tf}$	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	5	8.632e-22	3.544e-04	2.766e-06	3.452e-04	3.918e-06
$4h$	6	4.311e-26	3.731e-04	2.880e-06	3.464e-04	4.081e-06
$3h$	6	6.614e-24	3.558e-04	2.858e-06	3.127e-04	4.054e-06
$2h$	6	8.109e-24	6.072e-05	6.166e-07	5.739e-05	8.638e-07
h	7	5.583e-23	2.281e-04	1.387e-06	1.060e-04	1.824e-06

$\nu = 10^{-6}$

δ	#iter	$\inf J_{tf}$	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
$5h$	5	5.704e-22	3.147e-04	2.766e-08	3.452e-04	3.918e-08
$4h$	6	8.725e-30	3.321e-04	2.880e-08	3.464e-04	4.081e-08
$3h$	6	6.694e-28	3.131e-04	2.858e-08	3.127e-04	4.054e-08
$2h$	6	8.158e-28	4.540e-05	6.166e-09	5.739e-05	8.638e-09
h	7	5.584e-27	1.983e-04	1.387e-08	1.060e-04	1.824e-08

TABLE XXII: Test case without analytic solution. Neumann boundary condition on l_6. Results obtained for the functional J_{tf} with stabilized $\mathbb{Q}_p - \mathbb{Q}_p$ elements with fixed $p = 6$ and $\delta \to 0$. The viscosity is $\nu = 10^{-2}$ (top), $\nu = 10^{-4}$ (mid), $\nu = 10^{-6}$ (bottom).

$\nu = 10^{-2}$

δ	#iter	$\inf J_{tf}$	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	5	2.202e-21	1.288e-02	3.947e-04	3.015e-04	2.769e-04
0.1	6	5.438e-20	1.660e-02	3.937e-04	2.277e-04	2.823e-04
0.05	7	5.672e-21	1.690e-02	3.576e-04	1.438e-04	2.517e-04
0.02	7	2.684e-19	1.373e-02	2.478e-04	6.421e-05	1.624e-04
0.01	7	1.594e-19	1.048e-02	1.829e-04	2.670e-05	1.211e-04

$\nu = 10^{-4}$

δ	#iter	$\inf J_{tf}$	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	5	3.063e-25	1.288e-02	3.947e-06	3.015e-04	2.769e-06
0.1	6	5.648e-24	1.660e-02	3.937e-06	2.277e-04	2.823e-06
0.05	7	5.994e-25	1.690e-02	3.576e-06	1.438e-04	2.517e-06
0.02	7	3.142e-23	1.373e-02	2.478e-06	6.421e-05	1.624e-06
0.01	7	8.512e-23	1.048e-02	1.829e-06	2.670e-05	1.211e-06

$\nu = 10^{-6}$

δ	#iter	$\inf J_{tf}$	e_1^u	e_0^p	$e_{12,0}^u$	$e_{12,0}^p$
0.2	5	1.017e-26	1.288e-02	3.947e-08	3.015e-04	2.769e-08
0.1	6	2.382e-25	6.667e-02	4.800e-08	2.277e-04	2.823e-08
0.05	7	4.591e-25	1.690e-02	3.576e-08	1.438e-04	2.517e-08
0.02	7	6.312e-24	1.373e-02	2.478e-08	6.421e-05	1.624e-08
0.01	7	3.803e-21	1.048e-02	1.829e-08	2.670e-05	1.211e-08

$L : \mathbf{\Lambda}^D \to \mathbb{R}$ is a linear continuous functional. Then, being $(\mathbf{\Lambda}^D, |||\cdot|||_D)$ a Hilbert space (recall that now $\mathbf{\Lambda}^D$ denotes its completion with respect to the norm $|||\cdot|||_D$), applying classical results of calculus of variations (see, e.g., [12, Theorem 1.1]), the existence and uniqueness of the solution is guaranteed.

The Euler-Lagrange equation (56) follows by observing that, for all $\underline{\boldsymbol{\lambda}}, \underline{\boldsymbol{\mu}} \in \mathbf{\Lambda}^D$, ${}_{(\mathbf{\Lambda}^D)'}\langle J_t'(\underline{\boldsymbol{\lambda}}), \underline{\boldsymbol{\mu}}\rangle_{\mathbf{\Lambda}^D} = 2\pi(\underline{\boldsymbol{\lambda}}, \underline{\boldsymbol{\mu}}) - 2L(\underline{\boldsymbol{\mu}})$.
□

Remark 5.1 *Notice that, although the definition of the functional J_t involves the difference between the traces of the velocity on $\Gamma_1 \cup \Gamma_2$ only, the requirement that $\partial\Omega_{12} \cap \Gamma_N \neq \emptyset$ guarantees that the local pressures p_1 and p_2 will match in the overlapping region, i.e., $p_1 = p_2$ a.e. in Ω_{12}.*

5.1.1. The optimality system for Dirichlet controls

After Theorem 5.1, we assume that Assumption 2.1 is satisfied. More in particular, we consider the case $\partial\Omega_{12} \cap \Gamma_N \neq \emptyset$ and $\Gamma_D \neq \emptyset$ so that the constants C_1 and C_2 of Lemma 5.1 are both null. In the other cases, we would require that $p_i, q_i \in Q_{i,0}$, and the non-null constants C_1, C_2 are those identified in the proof of Proposition 2.1.

The Euler-Lagrange equation (56) becomes:

$$ {}_{(\mathbf{\Lambda}^D)'}\langle J_t'(\underline{\boldsymbol{\lambda}}), \underline{\boldsymbol{\mu}}\rangle_{\mathbf{\Lambda}^D} = \int_{\Gamma_1\cup\Gamma_2} (\mathbf{u}_1^{\boldsymbol{\lambda}_1,\mathbf{f}} - \mathbf{u}_2^{\boldsymbol{\lambda}_2,\mathbf{f}}) \cdot (\mathbf{u}_1^{\boldsymbol{\mu}_1} - \mathbf{u}_2^{\boldsymbol{\mu}_2}) = 0 \tag{57} $$

for all $\underline{\boldsymbol{\mu}} \in \mathbf{\Lambda}^D$.

Solving equation (57) is equivalent to solving the following optimality system: find $\underline{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) \in \mathbf{\Lambda}^D$ and, for $i = 1, 2$, $(\mathbf{u}_i, p_i) \in \boldsymbol{V}_{i,0} \times Q_i$, $(\mathbf{w}_i, q_i) \in \boldsymbol{V}_{i,0} \times Q_i$ such that

$$ \begin{aligned} -\mathbf{div}\,\mathbf{T}(\mathbf{u}_i, p_i) &= \mathbf{f} && \text{in } \Omega_i \\ \operatorname{div}\mathbf{u}_i &= 0 && \text{in } \Omega_i \\ \mathbf{u}_i &= \boldsymbol{\lambda}_i && \text{on } \Gamma_i \\ \mathbf{T}(\mathbf{u}_i, p_i)\cdot\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N^i \end{aligned} \tag{58} $$

$$ \begin{aligned} -\mathbf{div}\,\mathbf{T}(\mathbf{w}_i, q_i) &= \mathbf{0} && \text{in } \Omega_i \\ \operatorname{div}\mathbf{w}_i &= 0 && \text{in } \Omega_i \\ \mathbf{w}_i &= (-1)^{i+1}(\mathbf{u}_1 - \mathbf{u}_2) && \text{on } \Gamma_i \\ \mathbf{T}(\mathbf{w}_i, q_i)\cdot\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N^i \end{aligned} \tag{59} $$

and, for all $(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in \boldsymbol{\Lambda}^D$,

$$\int_{\Gamma_1} ((\mathbf{u}_1 - \mathbf{u}_2) + \mathbf{w}_2)\boldsymbol{\mu}_1 \, d\Gamma + \int_{\Gamma_2} (-(\mathbf{u}_1 - \mathbf{u}_2) + \mathbf{w}_1)\boldsymbol{\mu}_2 \, d\Gamma = 0. \tag{60}$$

Proposition 5.1 *The optimality system* (58)-(60) *has a unique solution whose control component* $\underline{\boldsymbol{\lambda}} \in \boldsymbol{\Lambda}^D$ *is the solution of the Euler-Lagrange equation* (57).

Proof. Let $\underline{\boldsymbol{\lambda}}$ be the solution of (50). Theorem 5.1 guarantees that such solution exists and is unique. Then, it is also a solution of (58)-(60). Indeed, the solution satisfies (57) which implies that $\mathbf{u}_1^{\boldsymbol{\lambda}_1,\mathbf{f}} = \mathbf{u}_2^{\boldsymbol{\lambda}_2,\mathbf{f}}$ on $\Gamma_1 \cup \Gamma_2$. As a consequence the solutions $(\mathbf{w}_i, q_i)$ of (59) are identically null and (60) is satisfied.

We prove now that this solution is unique. Consider first the case $\mathbf{f} = \mathbf{0}$. We define the operator $\chi : \boldsymbol{\Lambda}^D \to (\boldsymbol{\Lambda}^D)'$,

$$\begin{aligned} {}_{(\boldsymbol{\Lambda}^D)'}\langle \chi(\underline{\boldsymbol{\lambda}}), \underline{\boldsymbol{\mu}} \rangle_{\boldsymbol{\Lambda}^D} &= \int_{\Gamma_1} ((\mathbf{u}_1^{\boldsymbol{\lambda}_1} - \mathbf{u}_2^{\boldsymbol{\lambda}_2}) + \mathbf{w}_2^{\underline{\boldsymbol{\lambda}}})\boldsymbol{\mu}_1 \, d\Gamma \\ &+ \int_{\Gamma_2} (-(\mathbf{u}_1^{\boldsymbol{\lambda}_1} - \mathbf{u}_2^{\boldsymbol{\lambda}_2}) + \mathbf{w}_1^{\underline{\boldsymbol{\lambda}}})\boldsymbol{\mu}_2 \, d\Gamma \end{aligned} \tag{61}$$

where, for $i = 1, 2$, $\mathbf{u}_i^{\boldsymbol{\lambda}_i}$, $\mathbf{w}_i^{\underline{\boldsymbol{\lambda}}}$ are solutions of (58) and (59), respectively, with $\mathbf{f} = \mathbf{0}$. The operator χ is linear and continuous, and $ker(\chi) = \{\mathbf{0}\}$. Indeed, thanks to (59), $\mathbf{w}_i^{\underline{\boldsymbol{\lambda}}} \in [H^{1/2}(\Gamma_i)]^d$ and, if $\underline{\boldsymbol{\lambda}} \in ker(\chi)$, due to (61), $\mathbf{w}_2^{\underline{\boldsymbol{\lambda}}} = -(\mathbf{u}_1^{\boldsymbol{\lambda}_1} - \mathbf{u}_2^{\boldsymbol{\lambda}_2})$ on Γ_1 and $\mathbf{w}_1^{\underline{\boldsymbol{\lambda}}} = (\mathbf{u}_1^{\boldsymbol{\lambda}_1} - \mathbf{u}_2^{\boldsymbol{\lambda}_2})$ on Γ_2. Thus, for $i = 1, 2$, $j = 3 - i$, $\mathbf{w}_i^{\underline{\boldsymbol{\lambda}}}$ satisfies the system

$$\begin{aligned} -\mathbf{div}\,\mathbf{T}(\mathbf{w}_i^{\underline{\boldsymbol{\lambda}}}, q_i^{\underline{\boldsymbol{\lambda}}}) &= \mathbf{0} \quad \text{in } \Omega_i \\ \operatorname{div} \mathbf{w}_i^{\underline{\boldsymbol{\lambda}}} &= 0 \quad \text{in } \Omega_i \\ \mathbf{w}_i^{\underline{\boldsymbol{\lambda}}} &= -\mathbf{w}_j^{\underline{\boldsymbol{\lambda}}} \quad \text{on } \Gamma_i \\ \mathbf{w}_i^{\underline{\boldsymbol{\lambda}}} &= \mathbf{0} \quad \text{on } \Gamma_D^i \\ \mathbf{T}(\mathbf{w}_i^{\underline{\boldsymbol{\lambda}}}, q_i^{\underline{\boldsymbol{\lambda}}}) \cdot \mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma_N^i \end{aligned}$$

We define $\tilde{\mathbf{w}} = \mathbf{w}_{1|\Omega_{12}}^{\underline{\boldsymbol{\lambda}}} + \mathbf{w}_{2|\Omega_{12}}^{\underline{\boldsymbol{\lambda}}}$ and $\tilde{q} = q_{1|\Omega_{12}}^{\underline{\boldsymbol{\lambda}}} + q_{2|\Omega_{12}}^{\underline{\boldsymbol{\lambda}}}$ in Ω_{12}. By construction $(\tilde{\mathbf{w}}, \tilde{q}) \in \boldsymbol{V}_{12} \times L^2(\Omega_{12})$ and they satisfy the problem

$$\begin{aligned} -\mathbf{div}\,\mathbf{T}(\tilde{\mathbf{w}}, \tilde{q}) &= \mathbf{0} \quad \text{in } \Omega_{12} \\ \operatorname{div} \tilde{\mathbf{w}} &= 0 \quad \text{in } \Omega_{12} \\ \tilde{\mathbf{w}} &= \mathbf{0} \quad \text{on } \Gamma_1 \cup \Gamma_2 \\ \tilde{\mathbf{w}} &= \mathbf{0} \quad \text{on } \Gamma_D \cap \partial\Omega_{12} \\ \mathbf{T}(\tilde{\mathbf{w}}, \tilde{q}) \cdot \mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma_N \cap \partial\Omega_{12} \end{aligned}$$

whose solution is identically null. Thus, $\mathbf{w}_1^{\underline{\boldsymbol{\lambda}}} = -\mathbf{w}_2^{\underline{\boldsymbol{\lambda}}}$ and $q_1^{\underline{\boldsymbol{\lambda}}} = -q_2^{\underline{\boldsymbol{\lambda}}}$ in Ω_{12} and we can define

$$\mathbf{w} = \begin{cases} \mathbf{w}_1^{\underline{\boldsymbol{\lambda}}} & \text{in } \Omega_1 \setminus \Omega_{12} \\ \mathbf{w}_1^{\underline{\boldsymbol{\lambda}}} = -\mathbf{w}_2^{\underline{\boldsymbol{\lambda}}} & \text{in } \Omega_{12} \\ \mathbf{w}_2^{\underline{\boldsymbol{\lambda}}} & \text{in } \Omega_2 \setminus \Omega_{12} \end{cases}$$

and

$$q = \begin{cases} q_1^{\underline{\boldsymbol{\lambda}}} & \text{in } \Omega_1 \setminus \Omega_{12} \\ q_1^{\underline{\boldsymbol{\lambda}}} = -q_2^{\underline{\boldsymbol{\lambda}}} & \text{in } \Omega_{12} \\ q_2^{\underline{\boldsymbol{\lambda}}} & \text{in } \Omega_2 \setminus \Omega_{12} \end{cases}$$

which satisfy the Stokes problem

$$\begin{aligned} -\mathbf{div}\,\mathbf{T}(\mathbf{w}, q) &= \mathbf{0} \quad \text{in } \Omega \\ \operatorname{div} \mathbf{w} &= 0 \quad \text{in } \Omega \\ \mathbf{w} &= \mathbf{0} \quad \text{on } \Gamma_D \\ \mathbf{T}(\mathbf{w}, q) \cdot \mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma_N \end{aligned}$$

whose unique solution is $\mathbf{w} = \mathbf{0}$ and $q = 0$. Thus, we can conclude that $\mathbf{w}_i^{\underline{\boldsymbol{\lambda}}} = \mathbf{0}$ in Ω_i $(i = 1, 2)$ and $\mathbf{u}_1^{\boldsymbol{\lambda}_1} = \mathbf{u}_2^{\boldsymbol{\lambda}_2}$ on $\Gamma_1 \cup \Gamma_2$.

Applying a similar argument to the state equations (58) with $\mathbf{f} = \mathbf{0}$ and defining $\tilde{\mathbf{w}} = \mathbf{w}_{1|\Omega_{12}}^{\underline{\boldsymbol{\lambda}}} - \mathbf{w}_{2|\Omega_{12}}^{\underline{\boldsymbol{\lambda}}}$ and $\tilde{q} = q_{1|\Omega_{12}}^{\underline{\boldsymbol{\lambda}}} - q_{2|\Omega_{12}}^{\underline{\boldsymbol{\lambda}}}$ in Ω_{12}, we can prove that both these functions are null and we can conclude that $\boldsymbol{\lambda}_i = \mathbf{0}$, $i = 1, 2$.

If $\mathbf{f} \neq \mathbf{0}$, for $i = 1, 2$ and $j = 3 - i$, let $\mathbf{w}_i^{\mathbf{f}}, q_i^{\mathbf{f}}$ be the solution of the problem

$$\begin{aligned} -\mathbf{div}\,\mathbf{T}(\mathbf{w}_i^{\mathbf{f}}, q_i^{\mathbf{f}}) &= \mathbf{0} \quad \text{in } \Omega_i \\ \operatorname{div} \mathbf{w}_i^{\mathbf{f}} &= 0 \quad \text{in } \Omega_i \\ \mathbf{w}_i^{\mathbf{f}} &= \mathbf{u}_i^{\mathbf{0},\mathbf{f}} - \mathbf{u}_j^{\mathbf{0},\mathbf{f}} \quad \text{on } \Gamma_i \\ \mathbf{w}_i^{\mathbf{f}} &= \mathbf{0} \quad \text{on } \Gamma_D^i \\ \mathbf{T}(\mathbf{w}_i^{\mathbf{f}}, q_i^{\mathbf{f}}) \cdot \mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma_N^i, \end{aligned}$$

$\mathbf{u}_i^{\mathbf{0},\mathbf{f}}$ being the solutions of (58) with $\boldsymbol{\lambda}_i = \mathbf{0}$. Then, we can write (60) as

$${}_{(\boldsymbol{\Lambda}^D)'}\langle \chi(\underline{\boldsymbol{\lambda}}), \underline{\boldsymbol{\mu}} \rangle_{\boldsymbol{\Lambda}^D} = -{}_{(\boldsymbol{\Lambda}^D)'}\langle \mathcal{A}_f, \underline{\boldsymbol{\mu}} \rangle_{\boldsymbol{\Lambda}^D} \qquad \forall \underline{\boldsymbol{\mu}} \in \boldsymbol{\Lambda}^D,$$

where

$$\begin{aligned} &\mathcal{A}_f : \boldsymbol{\Lambda}^D \to (\boldsymbol{\Lambda}^D)' \\ &{}_{(\boldsymbol{\Lambda}^D)'}\langle \mathcal{A}_f, \underline{\boldsymbol{\mu}} \rangle_{\boldsymbol{\Lambda}^D} = \int_{\Gamma_1} ((\mathbf{u}_1^{\mathbf{0},\mathbf{f}} - \mathbf{u}_2^{\mathbf{0},\mathbf{f}}) + \mathbf{w}_2^{\mathbf{f}})\boldsymbol{\mu}_1 \, d\Gamma \\ &\qquad + \int_{\Gamma_2} (-(\mathbf{u}_1^{\mathbf{0},\mathbf{f}} - \mathbf{u}_2^{\mathbf{0},\mathbf{f}}) + \mathbf{w}_1^{\mathbf{f}})\boldsymbol{\mu}_2 \, d\Gamma. \end{aligned}$$

The thesis follows from the same arguments used before. □

Since the space $\boldsymbol{\Lambda}_h^D$ of discrete Dirichlet controls is a subset of $\boldsymbol{\Lambda}^D$, Lemma 5.1, Theorem 5.1 and Proposition 5.1 hold in the discrete case too and we can conclude that the minimization problem (16)–(17), or, equivalently, the optimality system (18)–(20), has a unique solution.

The minimum of the cost functional J_t is zero thanks to Proposition 2.1.

The real value $\inf_{\boldsymbol{\lambda}_h \in \boldsymbol{\Lambda}_h^D} J_t(\boldsymbol{\lambda}_h)$ attained at convergence, and reported in the second column of the Tables I and II, is about ϵ^2, $\epsilon = 10^{-9}$ being the tolerance in the stopping criterium of Bi-CGStab iterations. We notice that reducing the tolerance ϵ, $\inf_{\boldsymbol{\lambda}_h \in \boldsymbol{\Lambda}_h^D} J_t(\boldsymbol{\lambda}_h)$ reduces too. The errors between the discrete states $(\mathbf{u}_{i,h}, p_{i,h})$ and the exact ones $(\mathbf{u}_i, p_i)$ vanish for $h \to 0$ and increasing p, according to the theoretical convergence rate of $hp-$finite element approximation.

5.2. Analysis of the optimal control problem with Neumann controls

For $i = 1, 2$, let

$$\mathbf{\Lambda}_i^N = [H^{-1/2}(\Gamma_i)]^d \tag{62}$$

denote the spaces of *admissible Neumann controls* and we set

$$\mathbf{\Lambda}^N = \mathbf{\Lambda}_1^N \times \mathbf{\Lambda}_2^N. \tag{63}$$

For $i = 1, 2$, we consider two unknown control functions $\boldsymbol{\lambda}_i \in \mathbf{\Lambda}_i^N$ and the state problems

$$\begin{array}{rcll} -\mathbf{div}\,\mathbf{T}(\mathbf{u}_i^{\boldsymbol{\lambda}_i,\mathbf{f}}, p_i^{\boldsymbol{\lambda}_i,\mathbf{f}}) &=& \mathbf{f} & \text{in } \Omega_i \\ \operatorname{div} \mathbf{u}_i^{\boldsymbol{\lambda}_i,\mathbf{f}} &=& 0 & \text{in } \Omega_i \\ \mathbf{T}(\mathbf{u}_i^{\boldsymbol{\lambda}_i,\mathbf{f}}, p_i^{\boldsymbol{\lambda}_i,\mathbf{f}}) \cdot \mathbf{n} &=& \boldsymbol{\lambda}_i & \text{on } \Gamma_i \end{array} \tag{64}$$

with suitable homogeneous boundary conditions on $\partial\Omega_i \setminus \Gamma_i$. The unknown controls on the interface are obtained by solving the minimization problem

$$\inf_{\substack{\underline{\boldsymbol{\lambda}} \in \mathbf{\Lambda}^N \\ \underline{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)}} \left[\tilde{J}_f(\underline{\boldsymbol{\lambda}}) = \frac{1}{2} \sum_{i=1}^{2} \| \mathbf{T}(\mathbf{u}_1^{\boldsymbol{\lambda}_1,\mathbf{f}}, p_1^{\boldsymbol{\lambda}_1,\mathbf{f}}) \cdot \mathbf{n} - \mathbf{T}(\mathbf{u}_2^{\boldsymbol{\lambda}_2,\mathbf{f}}, p_2^{\boldsymbol{\lambda}_2,\mathbf{f}}) \cdot \mathbf{n} \|^2_{H^{-1/2}(\Gamma_i)} \right]. \tag{65}$$

Denoting by $-\Delta_{\Gamma_i}$ the Laplace-Beltrami operator on Γ_i, for any $\psi, \phi \in H^{-1/2}(\Gamma_i)$ we define the following inner product (see, e.g., [12]):

$$(\psi, \phi)_{H^{-1/2}(\Gamma_i)} = \int_{\Gamma_i} (-\Delta_{\Gamma_i})^{-1/2} \psi \; \phi \, d\Gamma \tag{66}$$

and the related norm $\|\psi\|_{H^{-1/2}(\Gamma_i)} = (\psi, \psi)^{1/2}_{H^{-1/2}(\Gamma_i)}$.

The fractional Laplace-Beltrami operator $(-\Delta_{\Gamma_i})^{-1/2}$ can be defined through a Neumann to Dirichlet map defined from $H^{-1/2}(\Gamma_i)$ to $H^{1/2}(\Gamma_i)$ (see, e.g., [3]). Precisely, for any $\phi \in H^{-1/2}(\Gamma_i)$ we solve the problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega_i \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega_i \setminus \Gamma_i \\ \dfrac{\partial u}{\partial n_\Gamma} = \phi & \text{on } \Gamma_i \end{cases} \tag{67}$$

and we set $(-\Delta_{\Gamma_i})^{-1/2}\phi = u|_{\Gamma_i}$.

From now on, let $(\cdot,\cdot)_{*,i}$ and $\|\cdot\|_{*,i}$ replace $(\cdot,\cdot)_{H^{-1/2}(\Gamma_i)}$ and $\|\cdot\|_{H^{-1/2}(\Gamma_i)}$, respectively.

Equations (65), (64) define an optimal control problem where both the control functions and the observations are of boundary (interface) type.

As for Dirichlet case, thanks to the linearity of the problem, we can equivalently express the cost functional as

$$\begin{aligned} \tilde{J}_f(\underline{\boldsymbol{\lambda}}) = \frac{1}{2} \sum_{i=1}^{2} \Big[& \| \mathbf{T}(\mathbf{u}_1^{\boldsymbol{\lambda}_1}, p_1^{\boldsymbol{\lambda}_1}) \cdot \mathbf{n} - \mathbf{T}(\mathbf{u}_2^{\boldsymbol{\lambda}_2}, p_2^{\boldsymbol{\lambda}_2}) \cdot \mathbf{n} \|_{*,i} \\ & + (\mathbf{T}(\mathbf{u}_1^{\boldsymbol{\lambda}_1}, p_1^{\boldsymbol{\lambda}_1}) \cdot \mathbf{n} - \mathbf{T}(\mathbf{u}_2^{\boldsymbol{\lambda}_2}, p_2^{\boldsymbol{\lambda}_2}) \cdot \mathbf{n}, \\ & \quad \mathbf{T}(\mathbf{u}_1^{\mathbf{0},\mathbf{f}}, p_1^{\mathbf{0},\mathbf{f}}) \cdot \mathbf{n} - \mathbf{T}(\mathbf{u}_2^{\mathbf{0},\mathbf{f}}, p_2^{\mathbf{0},\mathbf{f}}) \cdot \mathbf{n})_{*,i} \\ & + \frac{1}{2} \| \mathbf{T}(\mathbf{u}_1^{\mathbf{0},\mathbf{f}}, p_1^{\mathbf{0},\mathbf{f}}) \cdot \mathbf{n} - \mathbf{T}(\mathbf{u}_2^{\mathbf{0},\mathbf{f}}, p_2^{\mathbf{0},\mathbf{f}}) \cdot \mathbf{n} \|_{*,i} \Big]. \end{aligned} \tag{68}$$

Let us denote

$$|||\underline{\boldsymbol{\lambda}}|||_N = \sum_{i=1}^{2} \left[\| \mathbf{T}(\mathbf{u}_1^{\boldsymbol{\lambda}_1}, p_1^{\boldsymbol{\lambda}_1}) \cdot \mathbf{n} - \mathbf{T}(\mathbf{u}_2^{\boldsymbol{\lambda}_2}, p_2^{\boldsymbol{\lambda}_2}) \cdot \mathbf{n} \|_{*,i} \right.$$

Lemma 5.2 *If $\partial\Omega_{12} \cap \Gamma_D \neq \emptyset$, then $|||\underline{\boldsymbol{\lambda}}|||_N$ defines a norm on the space $\mathbf{\Lambda}^N$.*

Proof. We proceed as done for Dirichlet controls: $|||\underline{\boldsymbol{\lambda}}|||_N$ is always a semi-norm on $\mathbf{\Lambda}^N$, we only have to prove that, if $|||\underline{\boldsymbol{\lambda}}|||_N = 0$, then $\underline{\boldsymbol{\lambda}} = \underline{\mathbf{0}}$. Obviously, $|||\underline{\boldsymbol{\lambda}}|||_N = 0$ implies that $\mathbf{T}(\mathbf{u}_1^{\boldsymbol{\lambda}_1}, p_1^{\boldsymbol{\lambda}_1}) \cdot \mathbf{n} = \mathbf{T}(\mathbf{u}_2^{\boldsymbol{\lambda}_2}, p_2^{\boldsymbol{\lambda}_2}) \cdot \mathbf{n}$ a.e. on $\Gamma_1 \cup \Gamma_2$. In view of Proposition 2.2, starting from $(\mathbf{u}_i^{\boldsymbol{\lambda}_i}, p_i^{\boldsymbol{\lambda}_i})$ we define the pair $(\overline{\mathbf{u}}, \overline{p})$ as in (53), (54), that satisfies a Stokes problem in Ω with null force and homogeneous boundary conditions. This problem is well-posed and, in particular, $\overline{\mathbf{u}} = \mathbf{0}$ and $\overline{p} = 0$ a.e. in Ω. This implies that $\mathbf{T}(\overline{\mathbf{u}}, \overline{p}) \cdot \mathbf{n} = \mathbf{0}$ on $\Gamma_1 \cup \Gamma_2$ and, for $i = 1, 2$, $\boldsymbol{\lambda}_i = \mathbf{0}$ in $\mathbf{\Lambda}_i^N$. □

We cannot guarantee that $\mathbf{\Lambda}^N$ is complete with respect to the norm $|||\underline{\boldsymbol{\lambda}}|||_N$, but we can construct its completion, say $\hat{\mathbf{\Lambda}}^N$, with respect to such norm. For the sake of notation, in the following we will still denote the completion of $\mathbf{\Lambda}^N$ by the same symbol.

Theorem 5.2 *Consider the minimization problem*

$$\inf_{\underline{\boldsymbol{\lambda}} \in \mathbf{\Lambda}^N} \tilde{J}_f(\underline{\boldsymbol{\lambda}}). \tag{69}$$

If $\partial\Omega_{12} \cap \Gamma_D \neq \emptyset$, problem (69) *has a unique solution satisfying*

$$\begin{aligned} & {}_{(\mathbf{\Lambda}^N)'}\langle \tilde{J}'_f(\underline{\boldsymbol{\lambda}}), \underline{\boldsymbol{\mu}} \rangle_{\mathbf{\Lambda}^N} = \\ & \textstyle\sum_{i=1}^{2} (\mathbf{T}(\mathbf{u}_1^{\boldsymbol{\lambda}_1,\mathbf{f}}, p_1^{\boldsymbol{\lambda}_1,\mathbf{f}}) \cdot \mathbf{n} - \mathbf{T}(\mathbf{u}_2^{\boldsymbol{\lambda}_2,\mathbf{f}}, p_2^{\boldsymbol{\lambda}_2,\mathbf{f}}) \cdot \mathbf{n}, \\ & \mathbf{T}(\mathbf{u}_1^{\boldsymbol{\mu}_1}, p_1^{\boldsymbol{\mu}_1}) \cdot \mathbf{n} - \mathbf{T}(\mathbf{u}_2^{\boldsymbol{\mu}_2}, p_2^{\boldsymbol{\mu}_2}) \cdot \mathbf{n})_{*,i} = 0 \end{aligned} \tag{70}$$

for all $\underline{\boldsymbol{\mu}} \in \mathbf{\Lambda}^N$.

Proof. The proof follows the same guidelines of the proof of Theorem 5.1. □

In view of (66), the Euler-Lagrange equation (70) becomes:

$$\sum_{i=1}^{2}\int_{\Gamma_i}(-\Delta_{\Gamma_i})^{-1/2}\Big(\mathbf{T}(\mathbf{u}_i^{\boldsymbol{\lambda}_i,\mathbf{f}},p_i^{\boldsymbol{\lambda}_i,\mathbf{f}})\cdot\mathbf{n} -\mathbf{T}(\mathbf{u}_j^{\boldsymbol{\lambda}_j,\mathbf{f}},p_j^{\boldsymbol{\lambda}_j,\mathbf{f}})\cdot\mathbf{n}\Big)\cdot \Big(\mathbf{T}(\mathbf{u}_i^{\boldsymbol{\mu}_i},p_i^{\boldsymbol{\mu}_i})\cdot\mathbf{n}-\mathbf{T}(\mathbf{u}_j^{\boldsymbol{\mu}_j},p_j^{\boldsymbol{\mu}_j})\cdot\mathbf{n}\Big)d\Gamma=0 \quad (71)$$

for all $\boldsymbol{\mu}\in\boldsymbol{\Lambda}^N$ and $j=3-i$.

Solving equation (71) is equivalent to solving the following optimality system: find $\underline{\boldsymbol{\lambda}}=(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)\in\boldsymbol{\Lambda}^N$ and, for $i=1,2$, $(\mathbf{u}_i,p_i)\in\boldsymbol{V}_{i,0}\times Q_i$, $(\mathbf{w}_i,q_i)\in\boldsymbol{V}_{i,0}\times Q_i$ such that

$$\begin{array}{rcll} -\mathbf{div}\,\mathbf{T}(\mathbf{u}_i,p_i) &=& \mathbf{f} & \text{in } \Omega_i\\ \operatorname{div}\mathbf{u}_i &=& 0 & \text{in } \Omega_i\\ \mathbf{T}(\mathbf{u}_i,p_i)\cdot\mathbf{n} &=& \boldsymbol{\lambda}_i & \text{on } \Gamma_i\\ \mathbf{T}(\mathbf{u}_i,p_i)\cdot\mathbf{n} &=& \mathbf{0} & \text{on } \Gamma_N^i \end{array} \quad (72)$$

$$\begin{array}{rcll} -\mathbf{div}\,\mathbf{T}(\mathbf{w}_i,q_i) &=& \mathbf{0} & \text{in } \Omega_i\\ \operatorname{div}\mathbf{w}_i &=& 0 & \text{in } \Omega_i\\ \mathbf{T}(\mathbf{w}_i,q_i)\cdot\mathbf{n} &=& (-1)^{i+1}\Big(\mathbf{T}(\mathbf{u}_i,p_i)\cdot\mathbf{n} -\mathbf{T}(\mathbf{u}_j,p_j)\cdot\mathbf{n}\Big) & \text{on } \Gamma_i\\ \mathbf{T}(\mathbf{w}_i,q_i)\cdot\mathbf{n} &=& \mathbf{0} & \text{on } \Gamma_N^i \end{array} \quad (73)$$

and, for all $(\boldsymbol{\mu}_1,\boldsymbol{\mu}_2)\in\boldsymbol{\Lambda}^N$,

$$\sum_{i=1}^{2}\int_{\Gamma_i}(-\Delta_{\Gamma_i})^{-1/2}\Big(\mathbf{T}(\mathbf{u}_i,p_i)\cdot\mathbf{n}-\mathbf{T}(\mathbf{u}_j,p_j)\cdot\mathbf{n} +\mathbf{T}(\mathbf{w}_j,q_j)\cdot\mathbf{n}\Big)\boldsymbol{\mu}_i d\Gamma=0 \quad (74)$$

for $j=3-i$.

Proposition 5.2 *The optimality system* (72)-(74) *has a unique solution whose control component* $\underline{\boldsymbol{\lambda}}\in\boldsymbol{\Lambda}^N$ *is the solution of the Euler-Lagrange equation* (71).

Proof. Let $\underline{\boldsymbol{\lambda}}$ be the solution of (65). Theorem 5.2 guarantees that such solution exists and is unique. Then, it is also a solution of (72)-(74). Indeed, the solution satisfies (71) which implies that $\mathbf{T}(\mathbf{u}_1^{\boldsymbol{\lambda}_1,\mathbf{f}},p_1^{\boldsymbol{\lambda}_1,\mathbf{f}})\cdot\mathbf{n}=\mathbf{T}(\mathbf{u}_2^{\boldsymbol{\lambda}_2,\mathbf{f}},p_2^{\boldsymbol{\lambda}_2,\mathbf{f}})\cdot\mathbf{n}$ on $\Gamma_1\cup\Gamma_2$. As a consequence the solutions $(\mathbf{w}_i,q_i)$ of (73) are identically null and (74) is satisfied.

To prove that this solution is unique, we proceed as in the proof of Proposition 5.1, by exploiting linearity, continuity and coercivity of the Laplace-Beltrami operator (see (67)). □

In view of Proposition 2.2, the infimum of $\tilde{J}_f$ is zero.

The cost functional $\tilde{J}_f$ differs from J_f defined in (24) in the choice of the norm. As a matter of fact, at the continuous level we cannot guarantee that the fluxes $\mathbf{T}(\mathbf{u}_i^{\boldsymbol{\lambda}_i,\mathbf{f}},p_i^{\boldsymbol{\lambda}_i,\mathbf{f}})\cdot\mathbf{n}$ are L^2 functions, being $[H^{-1/2}(\Gamma_i)]^d$ their natural space, while the discrete fluxes are more regular and belong to $[L^2(\Gamma_i)]^d$, as we have shown in Section 3 3.4.

Since the space $\boldsymbol{\Lambda}_h^N$ of discrete Neumann controls is a subset of $\boldsymbol{\Lambda}^N$, we can conclude that also the minimization problem

$$\inf_{\substack{\underline{\boldsymbol{\lambda}}_h\in\boldsymbol{\Lambda}_h^N\\ \underline{\boldsymbol{\lambda}}_h=(\boldsymbol{\lambda}_{1,h},\boldsymbol{\lambda}_{2,h})}} \tilde{J}_f(\underline{\boldsymbol{\lambda}}_h) \quad (75)$$

has a unique solution (thanks to Lemma 5.2 and Theorem 5.2) and it can be computed by solving the optimality system (72)–(74) (by Proposition 5.2).

Following the same guidelines of Lemma 5.2 and Theorem 5.2 we can prove that also the minimization problem (24) has a unique solution.

At the discrete level the solutions computed by solving (24) and (75) could not coincide. Nevertheless, the results of Tables VI and VII show that $\inf_{\boldsymbol{\lambda}_h\in\boldsymbol{\Lambda}_h^N} J_f(\boldsymbol{\lambda}_h)\simeq\epsilon^2$, where $\epsilon=10^{-9}$ is the tolerance used in the stopping test of Bi-CGStab iterations. Moreover, as for Dirichlet controls, the errors between the discrete states $(\mathbf{u}_{i,h},p_{i,h})$ and the exact ones $(\mathbf{u}_i,p_i)$ vanish for $h\to 0$ and increasing p, according to the theoretical convergence rate of $hp-$finite element approximation.

Solving (24) instead of (75) is obviously more attractive since no additional Laplace-Beltrami problems like (67) have to be solved at each Bi-CGStab iteration to update the numerical solution.

For what concerns the minimization problems (30) and (31) with mixed controls, we can apply the analysis developed for both Dirichlet and Neumann controls and draw the same conclusions reached above for the cost functional J_f.

6. CONCLUSIONS

We have studied the ICDD method for the mathematical formulation and the numerical solution of the Stokes problem. This method rests on the reformulation of the original boundary value problem as an optimal control problem involving control variables that represent the trace of the velocity or the normal stress across the subdomain interfaces. We have shown that choosing control variables of mixed type allows to set up a robust numerical method with convergence rate independent of the discretization parameters as well as of the size of the overlapping region. Possible extensions of this work could consider the case of decomposition with more than two subdomains and heterogeneous couplings like, e.g., the Stokes/Darcy problem (see [6]).

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[1] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang. *Spectral Methods. Fundamentals in Single Domains*. Scientific Computation. Springer-Verlag, Berlin, 2006.

[2] C. Canuto, P. Gervasio, and A. Quarteroni. Finite-element preconditioning of G-NI spectral methods. *SIAM J. Sci. Comput.*, 31(6):4422–4451, 2009/10.

[3] M. Discacciati, P. Gervasio, and A. Quarteroni. Heterogeneous mathematical models in fluid dynamics ans associated solution algorithms. In *Multiscale and Adaptivity: Modeling, Numerics and Applications*, volume 2040 of *Lecture Notes in Mathematics*, chapter 2, pages 57–123. Springer, Berlin and Heidelberg, 2012. Lectures given at the C.I.M.E. Summer School held in Cetraro, July 2009.

[4] M. Discacciati, P. Gervasio, and A. Quarteroni. The Interface Control Domain Decomposition (ICDD) method for elliptic problems. *SIAM J. Control Optim.*, 2013. Accepted.

[5] M. Discacciati, P. Gervasio, and A. Quarteroni. Interface Control Domain Decomposition (ICDD) methods for coupled diffusion and advection-diffusion problems. Technical Report 18, MOX, Politecnico di Milano, 2013. Submitted to Int. J. Numer. Meth. Fluids.

[6] M. Discacciati, P. Gervasio, and A. Quarteroni. Interface Control Domain Decomposition (ICDD) methods for the Stokes/Darcy problem. Technical report, 2013. In preparation.

[7] L.P. Franca and S.L. Frey. Stabilized finite element methods: II. the incompressible Navier-Stokes equations. *Comput. Meth. Appl. Mech. Engrg.*, 99:209–233, 1992.

[8] P. Gervasio and F. Saleri. Stabilized spectral element approximation for the Navier-Stokes equations. *Numerical Methods for Partial Differential Equations*, 14:115–141, 1998.

[9] V. Girault and P.A. Raviart. *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*. Springer, Berlin, 1986.

[10] R. Glowinski, Q.V. Dinh, and J. Periaux. Domain decomposition methods for nonlinear problems in fluid dynamics. *Comput. Methods Appl. Mech. Engrg.*, 40(1):27–109, 1983.

[11] T.J.R. Hughes, L.P. Franca, and M. Balestra. A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška-Brezzi condition: a stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations. *Comput. Meth. Appl. Mech. Engrg.*, 59:85–99, 1986.

[12] J.-L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations*. Springer, New York, 1971.

[13] J.-L. Lions and O. Pironneau. Algorithmes parallèles pour la solution de problèmes aux limites. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(11):947–952, 1998.

[14] E. Miglio, A. Quarteroni, and F. Saleri. Coupling of free surface and groundwater flows. *Comput. Fluids*, 32(1):73–83, 2003.

[15] A. Quarteroni. *Numerical Models for Differential Problems*. Springer, Milan, 2009.

[16] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*. Springer, Berlin, 1994.

[17] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations*. The Clarendon Press, Oxford University Press, New York, 1999.

[18] B.F. Smith, P.E. Bjørstad, and W.D. Gropp. *Domain Decomposition*. Cambridge University Press, Cambridge, 1996.

[19] A. Toselli and O. Widlund. *Domain Decomposition Methods – Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

[20] H.A. van der Vorst. Bi-CGSTAB: a fast and smoothly converging variant of Bi-CG for the solution of nonsymmetric linear systems. *SIAM J. Sci. Statist. Comput.*, 13(2):631–644, 1992.