Discretization of the 1d Poisson equation

Given $\Omega = (x_a, x_b)$, $\partial\Omega =$ boundary of Ω, given the functions $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, we look for the approximation of the solution $u : \Omega \to \mathbb{R}$ of the Poisson equation

$$\begin{cases} -u'' = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

by the centered 2nd-order finite difference scheme:

$$\begin{cases} -\dfrac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f(x_i) & i = 2, \dots, N-1 \\ u_1 = g(x_a), \;\; u_N = g(x_b) \end{cases}$$

or equivalently

$$\begin{cases} -u_{i-1} + 2u_i - u_{i+1} = h^2 f(x_i) & i = 2, \dots, N-1 \\ u_1 = g(x_a), \;\; u_N = g(x_b) \end{cases}$$

The mesh grid We fix $N \in \mathbb{N}$ and we define the equispaced points x_i (for $i = 1, \ldots, N$) in $\Omega = (x_a, x_b)$, with $x_1 = x_a$ and $x_b = x_N$. Then we set $h = (x_b - x_a)/(N-1)$ (the space between two consecutive points).

The unknown vector

It is a column vector $\mathtt{u}$ with N entries, it will hold the approximations of the exact solution at the point x_i.

The matrix approximating the Laplace operator by centered 2nd-order finite difference scheme

(We move h^2 on the right hand side)

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix}$$

```
e=ones(N,1);
A=spdiags([-e,2*e,-e],(-1:1),N,N);
A(1,:)=zeros(1,N); A(1,1)=1;
A(N,:)=zeros(1,N); A(N,N)=1;
```

The right hand side with Dirichlet boundary conditions

$\mathbf{b} = [u_a, h^2 f(x_2), \dots, h^2 f(x_{N-1}), u_b]^T$.

```
f=@(x)[...]; ua=...; ub=...;
b=h^2*f(x); b(1)=ua; b(N)=ub;
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The linear system

Solve: $A\mathbf{u} = \mathbf{b}$

The plot

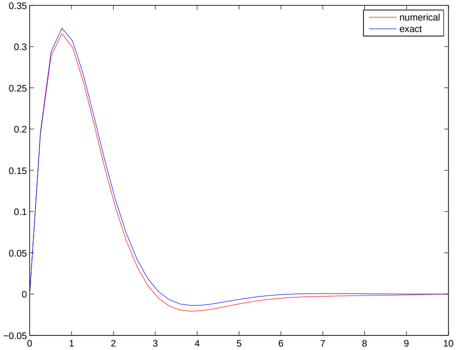
```
plot(x,u)
```

The error (if we know the exact solution and we want to check the code)

```
uex=@(x)[...];
UEX=uex(x);
err=max(abs(UEX-u));
```

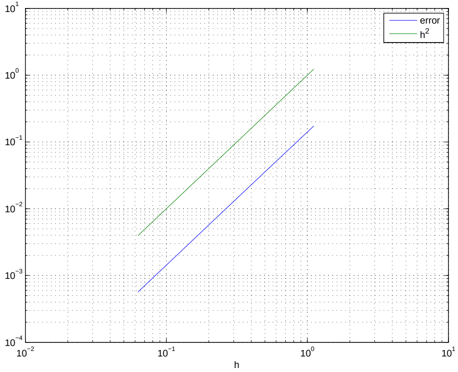
First test case (to check if the code works well)

Exercise 1. (espde1) $\Omega = (0, 10)$, $f(x) = 2\cos(x)/e^x$, $g(x) = \sin(x)/e^x$.
The exact solution is $u_{ex}(x) = \sin(x)/e^x$.
If $N = 40$, the error $err = \max_{x_i} |u_{ex}(x) - u(x)| \simeq 0.0094$



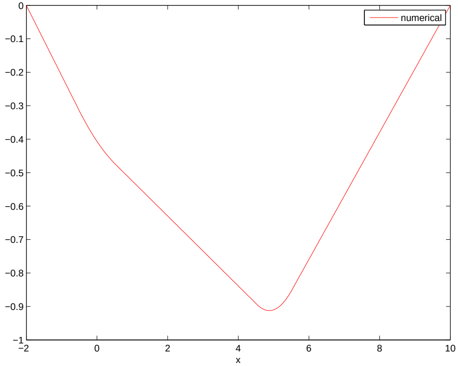
Error analysis

Now take $N = 10, 20, 40, 80, 160,$ solve the Poisson problem and collect the errors in a vector.
Plot the solutions computed with various N and plot the errors vs $h = (x_b - x_a)/(N-1)$, to check that it decays as h^2.



Exercise 2. $\Omega = (-2, 10)$, $f(x) = -0.1\chi_{(-1/2,1/2)} - 0.3\chi_{(4.5,5.5)}$, $u_a = u_b = 0$.

$f(x)$ represents the discontinuous load on an elastic string with tension equal to one, which is fixed at the end-points. The solution $u(x)$ of the Poisson problem is the displacement of the bar.



The heat equation

Given $\Omega = (x_a, x_b)$, $\partial\Omega =$ boundary of Ω, $(t_0, T) \subset \mathbb{R}$, given the functions $f : \Omega \times (t_0, T) \to \mathbb{R}$ and $g : \partial\Omega \times (t_0, T) \to \mathbb{R}$, given the thermal diffusivity μ, we look for the approximation of the solution $u : \Omega \times (t_0, T) \to \mathbb{R}$ of the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} - \mu \frac{\partial^2 u}{\partial x^2} = f & \text{in } \Omega \times (t_0, T) \\ u = g & \text{on } \partial\Omega \times (t_0, T) \\ u = u^0 & \text{in } \Omega \times \{t_0\} \end{cases}$$

by approximating the time derivative by BE and the space derivative by the centered 2nd order finite difference scheme. $\{x_1, x_2, \dots, x_N\}$ are equispaced points in Ω with $x_1 = x_a$ and $x_N = x_b$, h is the step along x, so that $u_j(t) = u(x_j, t)$.

The *semi-discretization* of the heat equation yields a system of ordinary differential equations of the following form

$$\begin{cases} \dfrac{d\mathbf{u}}{dt}(t) = -\dfrac{\mu}{h^2}\mathrm{A}\mathbf{u}(t) + \mathbf{f}(t) & \forall t > 0, \\ \mathbf{u}(0) = \mathbf{u}^0, \end{cases}$$

where $\mathbf{u}(t) = (u_1(t), \ldots, u_N(t))^T$ is the vector of unknowns, $\mathbf{f}(t) = (f_1(t), \ldots, f_N(t))^T$, $\mathbf{u}^0 = (u^0(x_1), \ldots, u^0(x_N))^T$, and A is the tridiagonal matrix introduced for the Poisson problem.
Now we discretize the time interval: $\{t^{(0)}, t^{(1)}, \ldots, t^{(k)}, \ldots\}$ are equispaced points in $[t_0, T]$. The time-step is Δt

The BE method applied to the heat equation reads

$$\begin{cases} \dfrac{\mathbf{u}^{(k+1)} - \mathbf{u}^{(k)}}{\Delta t} = -\dfrac{\mu}{h^2}\mathrm{A}\mathbf{u}^{(k+1)} + \mathbf{f}^{(k+1)}, \qquad k = 0, 1, \ldots \\ \mathbf{u}^0 \text{ given} \end{cases}$$

We can write a matlab function to implement this scheme.

Exercise 3. (espde3) Solve the heat equation in $(x_a, x_b) = (0, 1)$ with $\mu = 1$, $f(x, t) = -\sin(x)\sin(t) + \sin(x)\cos(t)$, initial condition $u(x, 0) = \sin(x)$ and boundary conditions $u(0, t) = 0$ and $u(1, t) = \sin(1)\cos(t)$. In this case the exact solution is $u(x, t) = \sin(x)\cos(t)$. Compute the errors $\max_{i=0,\dots,N} |u(x_i, 1) - u_i^M|$ with respect to the time-step Δt on a uniform space grid with $h = 0.002$.

Exercise 4. (espde4) We consider a homogeneous, three meters long aluminium bar with uniform section. We are interested in simulating the evolution of the temperature in the bar starting from a suitable initial condition, by solving the heat equation. If we impose adiabatic conditions on the lateral surface of the bar (i.e. homogeneous Neumann conditions), and Dirichlet conditions at the end sections of the bar, the temperature only depends on the axial space variable (denoted by x). Thus the problem can be modeled by the one-dimensional heat equation with $f = 0$, completed by the initial condition at $t = t_0$ and by Dirichlet boundary conditions at the end points of the reduced computational domain $\Omega = (0, L)$ ($L = 3\mathrm{m}$). Pure aluminium has thermal conductivity $k = 237$ W/(m K), density $\rho = 2700\mathrm{kg/m}^3$ and specific heat capacity $c = 897$ J/(kg K), then its thermal diffusivity is $\mu = 9.786 \cdot 10^{-5}\mathrm{m}^2/\mathrm{s}$. Finally we consider the initial condition $T(x, 0) = 500$ K if $x \in (1, 2)$, 250 K otherwise and the Dirichlet boundary conditions $T(0, t) = T(3, t) = 250$ K. Compute the numerical solution by BE (in time) + FD2 (in space).