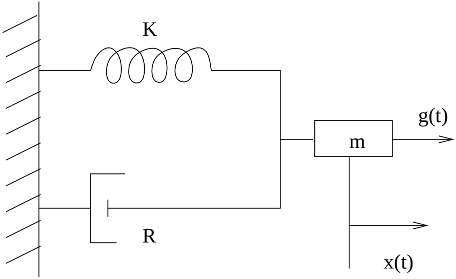
ODE for the driven oscillator with damping



Simple oscillator composed of a mass m, a spring with spring constant K.

The external force $g(t)$ drives the system, where t is the time, while R is the internal damping coefficient (friction).

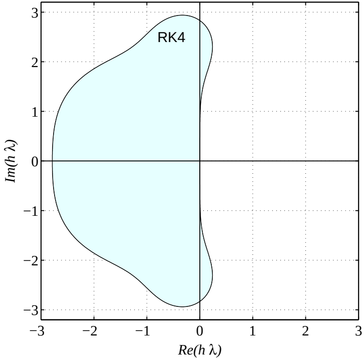
The displacement $x(t)$ of the mass m at time t is the solution of the ode

$$mx''(t) + Rx'(t) + K(x(t) - L) = g(t) \qquad t \geq t_0$$

where L is the rest length of the spring.
It is a 2nd-order ode. Two initial conditions are needed:
$x(t_0) = x_0$ initial position
$x'(t_0) = 0$ initial velocity
Case 1.
Set $g(t) \equiv 0$ (no driven force), $R = 0$ (no friction), $K = 1$, $m = 1$, $L = 1$ $x_0 = 2$, $T = 100$.
Compute the numerical solution with the explicit Runge-Kutta scheme of order 4 and with $h = 0.1$, $h = 1$, $h = 2$, $h = 4$.
In a second time, solve the problem by FE with $h = 0.1$, $h = 0.01$ and $h = 0.001$. Comment on the results.

Solution. The explicit Runge-Kutta scheme of order 4 is:

- explicit,
- one-step,
- fourth-order accurate,
- absolutely stable under conditions on h. Its stability region is:



Download:

`http://dm.ing.unibs.it/gervasio/Nummeth/matlab/rk4.m`

Input and output are as in the previous functions:

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[tn,un]=rk4(odefun,tspan,y0,Nh)
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Let us reduce the 2nd-order ode

$$mx''(t) + Rx'(t) + K(x(t) - L) = g(t)$$

in a system of 1st-order equations:
$y_1(t) = x(t)$,
$y_2(t) = x'(t) = y_1'(t)$
then explicit $x''(t)$, so that
$x''(t) = -(R/m)x'(t) - (K/m)(x(t) - L) + g(t)/m$:

$$\begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = -(R/m)y_2(t) - (K/m)(y_1(t) - L) + g(t)/m \\ y_1(t_0) = x_0 \\ y_2(t_0) = v_0 \end{cases}$$

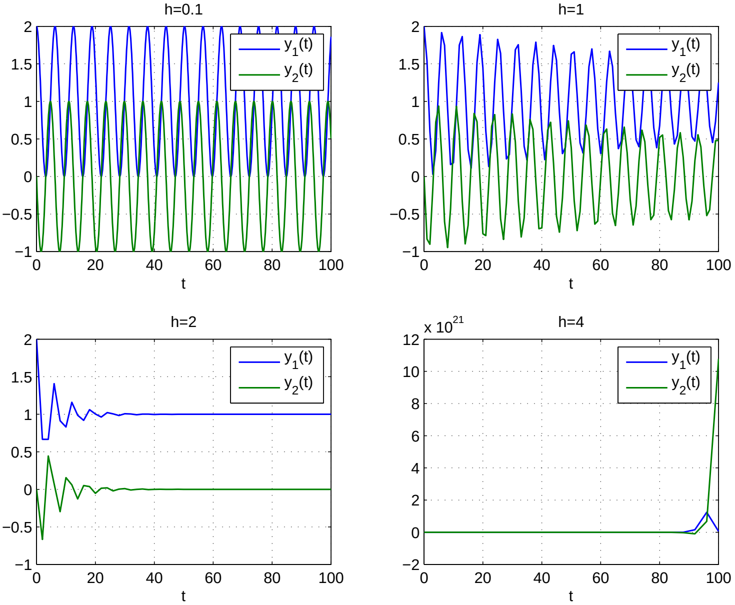
By setting: $\mathbf{y}(t) = [y_1(t), y_2(t)]^t$,
$\mathbf{y}'(t) = [y_1'(t), y_2'(t)]^t$,
$\mathbf{y}(t_0) = [y_1(t_0), y_2(t_0)]^t$,

$$\mathbf{F}(t, \mathbf{y}(t)) = \left[\begin{array}{l} y_2(t) \\ -(R/m)y_2(t) - (K/m)(y_1(t) - L) + g(t)/m, \end{array} \right]$$

the vector form of the ode reads:

$$\begin{cases} \mathbf{y}'(t) = \mathbf{F}(t, \mathbf{y}(t)) & t \geq t_0 \\ \mathbf{y}(t_0) = \mathbf{y}_0 \end{cases}$$

Numerical results for RK4



Comments on the numerical results

The solution is good only when $h = 0.1$, it is periodic without damping.
When $h = 1$, $h = 2$ the solutions vanishes when t grows, but it is not a physical solution, we have not damping in our system. The solution dies since there is a numerical damping that acts as the friction does. The unphysical behavior is due to large h (not accurate results).
When $h = 4$ the numerical solution blows up, in this case the absolute stability is missed out.

Absolute stability for periodic solutions

The absolute stability concept can be extended to periodic solutions.
We say that the scheme is absolutely stable if the numerical solution reflects the behavior of the exact solution when $t \to \infty$.
When $h = 0.1, 1, 2$ the absolute stability is satisfied and the solution remains bounded.
When $h = 4$ the absolute stability is missed out: the solution shows oscillations with increasing amplitude.

Validation of the results

The given problem is linear, since

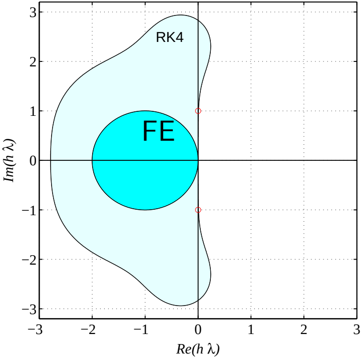
$$\mathbf{F}(t, \mathbf{y}(t)) = A\mathbf{y}(t) + \mathbf{g}(t)$$

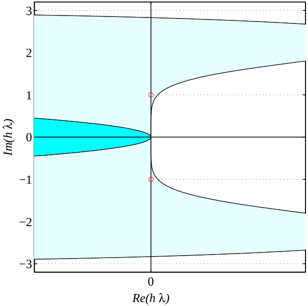
where

$$A = \begin{bmatrix} 0 & 1 \\ -K/m & -R/m \end{bmatrix}$$

The eigenvalues of A are $\lambda_1 = i$, $\lambda_2 = -i$.
In using RK4, we have not a formula that gives explicit bounds on h as for FE, but we have to compare the position of λ_i with the absolute stability region.

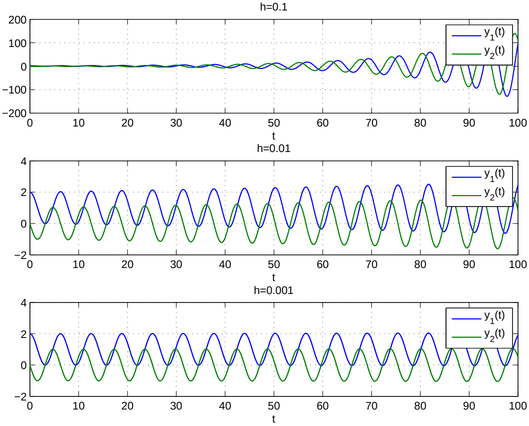
Absolute stability region of RK4 and FE





The eigenvalues $\lambda_{12} = \pm i \in \mathcal{A}_{RK4}$ and $h\lambda_i \in \mathcal{A}_{RK4}$ for any $0 < h \lesssim 2.8$, coherently with our numerical results: oscillations arised only when $h = 4$.

Numerical results for FE



Even if we use a very small h, the numerical solution shows oscillations with increasing amplitude. This is explained by the fact that $\mathcal{A}_{FE}$ does not intersect the imaginary axis and there is no h that leads λ_i inside $\mathcal{A}_{FE}$. Recall that $\mathcal{A}_{FE}$ is open.

When the ode system has pure imaginary eigenvalues, FE is not a candidate to solve such a system.

Case 2. Set $R \neq 0$ and/or $g(t) \neq 0$ and try to solve the same problem as before, analyzing all properties.