Esercitazione del 23 novembre 2023

1. Costruire il polinomio di Taylor di ordine 3 e centro $x_0 = \dfrac{\pi}{6}$ della funzione $f(x) = \cos 2x$.

2. Costruire il polinomio di Maclaurin di ordine 4 della funzione $f(x) = \arcsin x$.

3. Calcolare il limite $\displaystyle\lim_{x\to 0} \frac{\sin x - \cos x + 1}{2x + x^2 + 1 - \frac{e^x - 1}{x}}$.

4. Calcolare il limite $\displaystyle\lim_{x\to 0} \frac{x(e^x - 1 - x\cos x)}{3x - \arctan(3x)}$.

5. Calcolare il limite $\displaystyle\lim_{x\to 0} \frac{e^{x^2/2} - \cos x - x^2}{x^4}$.

6. Calcolare il limite $\displaystyle\lim_{x\to +\infty} x^2 \left(\sqrt[3]{x^3+1} - \sqrt[3]{x^3-1}\right)$.

7. (Prova d'esame 28 giugno 2013) Al variare di $\beta \in \mathbb{R}$, calcolare il limite
$$\lim_{x\to 2^+} \frac{\sqrt[3]{x-2} - 3\log^2(1 + \sqrt[3]{x-2})}{(\sinh(x-2) - \sin(x-2))^\beta}.$$

8. Calcolare il limite $\displaystyle\lim_{x\to +\infty} \frac{3\cos\left(\frac{1}{x}\right)\left(\sin\frac{1}{x} - \sinh\frac{1}{x}\right)}{4\left(e^{1/x} - 1\right)^4}$.

9. Calcolare il limite $\displaystyle\lim_{n\to +\infty} \frac{\cos\left(\frac{1}{n}\right) - \cosh\left(\frac{1}{n}\right)}{n^\beta}$, al variare di $\beta \in \mathbb{R}$.

10. Calcolare il limite $\displaystyle\lim_{x\to 0^+} \frac{\log\left(1 - \cos(3x^2) - \frac{9}{2}x^4\right)}{\log x}$.

11. (Prova d'esame 10 febbraio 2021) Sia $\alpha \in \mathbb{R}$. Calcolare, se esiste, il limite della successione
$$\lim_{n\to +\infty} \sin\left(\frac{2^n}{n}\right) \cdot \frac{\sqrt{1 + \dfrac{1}{n^2}} - 1}{n^\alpha \left(\sinh\left(\dfrac{1}{n}\right) - \sin\left(\dfrac{1}{n}\right)\right)}$$
al variare di α.

Costruire il polinomio di Taylor di ordine 3 e centro $x_0 = \frac{\pi}{6}$ della funzione $f(x) = \cos 2x$.

$$f(x) = \cos 2x \qquad f\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$f'(x) = -\sin(2x)\cdot 2 \qquad f'\left(\frac{\pi}{6}\right) = -2\cdot\sin\frac{\pi}{3} = -2\cdot\frac{\sqrt{3}}{2} = -\sqrt{3}$$

$$f''(x) = -2\cdot\cos(2x)\cdot 2 = \qquad f''\left(\frac{\pi}{6}\right) = -4\cos\frac{\pi}{3} = -2$$
$$= -4\cos(2x)$$

$$f'''(x) = -4\cdot(-\sin(2x))\cdot 2 = \qquad f'''\left(\frac{\pi}{6}\right) = 8\sin\frac{\pi}{3} = 4\sqrt{3}$$
$$= 8\sin 2x$$

$$p_3(x) = f\left(\frac{\pi}{6}\right) + \frac{f'\left(\frac{\pi}{6}\right)}{1!}\left(x-\frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3 =$$

$$= \frac{1}{2} - \sqrt{3}\left(x-\frac{\pi}{6}\right) - \frac{2}{2}\left(x-\frac{\pi}{6}\right)^2 + \frac{4\sqrt{3}}{6}\left(x-\frac{\pi}{6}\right)^3 =$$

$$= \frac{1}{2} - \sqrt{3}\left(x-\frac{\pi}{6}\right) - \left(x-\frac{\pi}{6}\right)^2 + \frac{2}{3}\sqrt{3}\left(x-\frac{\pi}{6}\right)^3$$

Quindi

$$\cos 2x = \frac{1}{2} - \sqrt{3}\left(x-\frac{\pi}{6}\right) - \left(x-\frac{\pi}{6}\right)^2 + \frac{2}{3}\sqrt{3}\left(x-\frac{\pi}{6}\right)^3 + \underbrace{o\left(\left(x-\frac{\pi}{6}\right)^3\right)}_{\text{Resto di Peano}}$$

per $x \to \frac{\pi}{6}$

$$f'''(x) = 8\sin(2x)$$

$$f^{(4)}(x) = 8\cdot\cos(2x)\cdot 2 = 16\cos(2x)$$

Formule di Taylor con il resto di Lagrange

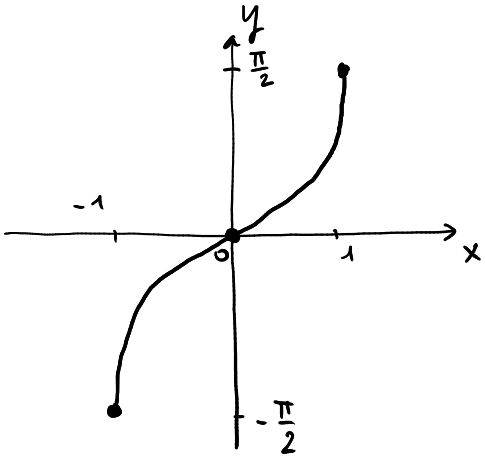
$$\cos(2x) = \frac{1}{2} - \sqrt{3}\left(x - \frac{\pi}{6}\right) - \left(x - \frac{\pi}{6}\right)^2 + \frac{2}{3}\sqrt{3}\left(x - \frac{\pi}{6}\right)^3 +$$

$$+ \frac{16\cos(2\xi)}{4!}\left(x - \frac{\pi}{6}\right)^4 =$$

$$= \frac{1}{2} - \sqrt{3}\left(x - \frac{\pi}{6}\right) - \left(x - \frac{\pi}{6}\right)^2 + \frac{2}{3}\sqrt{3}\left(x - \frac{\pi}{6}\right)^3 + \frac{2}{3}\cos(2\xi)\left(x - \frac{\pi}{6}\right)^4$$

per qualche ξ compreso fra x e $\frac{\pi}{6}$

Costruire il polinomio di Maclaurin di ordine 4 della funzione $f(x) = \arcsin x$.



f è dispari $\qquad f(0)=0$

f' è pari

f'' è dispari $\qquad f''(0)=0$

f''' è pari

$f^{(4)}$ è dispari $\qquad f^{(4)}(0)=0$

$$f(x) = \arcsin x$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \qquad f'(0)=1$$

$$f''(x) = -\frac{1}{\cancel{2}}(1-x^2)^{-\frac{1}{2}-1}\cdot(-\cancel{2}x) = x(1-x^2)^{-\frac{3}{2}}$$

$$f'''(x) = (1-x^2)^{-\frac{3}{2}} + x\cdot\left(-\frac{3}{\cancel{2}}\right)(1-x^2)^{-\frac{3}{2}-1}\cdot(-\cancel{2}x) =$$

$$= (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}} =$$

$$= (1-x^2)^{-\frac{5}{2}}(1-x^2+3x^2) =$$

$$= (1-x^2)^{-5/2}(2x^2+1)$$

$$f'''(0) = (1-0^2)^{-5/2}(2\cdot 0^2+1) = 1\cdot 1 = 1$$

$$P_4(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 =$$

$$= x + \frac{1}{6}x^3$$

Quindi

$$\arcsin x = x + \frac{x^3}{6} + o(x^4) \quad \text{per } x \to 0$$

Calcolare $\lim_{x\to 0} \dfrac{\sin x - \cos x + 1}{2x + x^2 + 1 - \dfrac{e^x - 1}{x}}$

$$\lim_{x\to 0} \left(\sin x - \cos x + 1\right) = 0 - 1 + 1 = 0$$

$$\lim_{x\to 0} \left(2x + x^2 + 1 - \underbrace{\frac{e^x - 1}{x}}_{\to 1}\right) = 0 + 0 + 1 - 1 = 0$$

$$\sin x = x - \frac{x^3}{6} + o(x^4) \quad \text{per } x \to 0$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3) \quad \text{per } x \to 0$$

$$\sin x - \cos x + 1 = x - \frac{x^3}{6} + o(x^4) - \cancel{1} + \frac{x^2}{2} + o(x^2) + \cancel{1}$$

$$= x + o(x) \quad \text{per } x \to 0$$

Quindi il numeratore si comporta come x per $x \to 0$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) \quad \text{per } x \to 0$$

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2} + \frac{x^2}{6} + o(x^2) \quad \text{per } x \to 0$$

$$2x + x^2 + 1 - \frac{e^x - 1}{x} = 2x + \underbrace{x^2}_{=o(x)} + \cancel{1} - \cancel{1} - \frac{x}{2} - \underbrace{\frac{x^2}{6}}_{=o(x)} + o(x^2) =$$

$$= \frac{3}{2}x + o(x) \quad \text{per } x \to 0$$

Dunque il denominatore si comporta come $\frac{3}{2}x$ per $x \to 0$

$$\lim_{x \to 0} \frac{\sin x - \cos x + 1}{2x + x^2 + 1 - \frac{e^x - 1}{x}} = \lim_{x \to 0} \frac{x + o(x)}{\frac{3}{2}x + o(x)} =$$

$$= \lim_{x \to 0} \frac{\cancel{x}}{\frac{3}{2}\cancel{x}} = \frac{2}{3}$$

Calcolare $\lim_{x\to 0} \dfrac{x\left(e^x - 1 - x\cos x\right)}{3x - \arctan(3x)}$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) \quad \text{per } x\to 0$$

$$x\cos x = x\left(1 - \frac{x^2}{2} + o(x^3)\right) =$$

$$= x - \frac{x^3}{2} + o(x^4) \quad \text{per } x\to 0$$

$$e^x - 1 - x\cos x = \cancel{x} + \frac{x^2}{2} + o(x^2) - \cancel{x} + \underbrace{\frac{x^3}{2}}_{=o(x^2)} + o(x^4) =$$

$$= \frac{x^2}{2} + o(x^2) \quad \text{per } x\to 0$$

$$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} + o(t^6) \quad \text{per } t\to 0$$

$$3x - \arctan(3x) = 3x - \left(3x - \frac{(3x)^3}{3} + o(x^4)\right) =$$

$$= \cancel{3x} - \cancel{3x} + 9x^3 + o(x^4) \quad \text{per } x\to 0$$

$$\lim_{x\to 0} \frac{x\left(e^x - 1 - x\cos x\right)}{3x - \arctan(3x)} = \lim_{x\to 0} \frac{x\left(\frac{x^2}{2} + o(x^2)\right)}{9x^3 + o(x^4)} =$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{2} + o(x^3)}{9x^3 + o(x^4)} = \lim_{x\to 0} \frac{\cancel{x^3}/2}{9\cancel{x^3}} = \frac{1}{18}$$

Calcolare $\lim_{x\to 0} \dfrac{e^{x^2/2} - \cos x - x^2}{x^4}$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \quad \text{per } x \to 0$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + o(t^3) \quad \text{per } t \to 0$$

$$e^{x^2/2} = 1 + \frac{x^2}{2} + \frac{1}{2}\cdot\left(\frac{x^2}{2}\right)^2 + \frac{1}{6}\cdot\left(\frac{x^2}{2}\right)^3 + o(x^6)$$

$$e^{x^2/2} - \cos x - x^2 = \cancel{1} + \cancel{\frac{x^2}{2}} + \frac{x^4}{8} + \overbrace{\frac{x^6}{48} + o(x^6)}^{=o(x^5)}$$

$$- \cancel{1} + \cancel{\frac{x^2}{2}} - \frac{x^4}{24} - \cancel{x^2} =$$

$$= \frac{x^4}{12} + o(x^5) \quad \text{per } x \to 0$$

$$\lim_{x\to 0} \frac{e^{x^2/2} - \cos x - x^2}{x^4} = \lim_{x\to 0} \frac{\frac{x^4}{12} + o(x^5)}{x^4} =$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}/12}{\cancel{x^4}} = \frac{1}{12}$$

Calcolare $\lim_{x\to+\infty} x^2\left(\sqrt[3]{x^3+1}-\sqrt[3]{x^3-1}\right)$

Fissato $\alpha\in\mathbb{R}$, vale lo sviluppo

$$(1+t)^\alpha = 1+\alpha t+o(t) \quad \text{per } t\to 0$$

$$\lim_{x\to+\infty} x^2\left(\sqrt[3]{x^3+1}-\sqrt[3]{x^3-1}\right) = \lim_{x\to+\infty} x^2\left(\sqrt[3]{x^3\left(1+\frac{1}{x^3}\right)}-\sqrt[3]{x^3\left(1-\frac{1}{x^3}\right)}\right)=$$

$$= \lim_{x\to+\infty} x^3\left(\sqrt[3]{1+\underbrace{\frac{1}{x^3}}_{\downarrow\, 0}}-\sqrt[3]{1-\underbrace{\frac{1}{x^3}}_{\downarrow\, 0}}\right) = \circledast$$

$$(1+t)^{1/3} = 1+\frac{1}{3}t+o(t) \quad \text{per } t\to 0$$

$$\left(1+\frac{1}{x^3}\right)^{1/3} = 1+\frac{1}{3}\cdot\frac{1}{x^3}+o\left(\frac{1}{x^3}\right) \quad \text{per } x\to+\infty$$

$$\left(1-\frac{1}{x^3}\right)^{1/3} = 1-\frac{1}{3}\cdot\frac{1}{x^3}+o\left(\frac{1}{x^3}\right) \quad \text{per } x\to+\infty$$

$$\circledast = \lim_{x\to+\infty} x^3\left(\cancel{1}+\frac{1}{3x^3}+o\left(\frac{1}{x^3}\right)-\cancel{1}+\frac{1}{3x^3}+o\left(\frac{1}{x^3}\right)\right) =$$

$$= \lim_{x\to+\infty} x^3\left(\frac{2}{3x^3}+o\left(\frac{1}{x^3}\right)\right) =$$

$$= \lim_{x\to+\infty}\left(\frac{2}{3}+o(1)\right) = \frac{2}{3}$$

Fissato $\beta \in \mathbb{R}$, calcolare il limite

$$\lim_{x \to 2^+} \frac{\sqrt[3]{x-2} - 3 \cdot \log^2\left(1+\sqrt[3]{x-2}\right)}{\left(\sinh(x-2) - \sin(x-2)\right)^\beta} =$$

$$\boxed{\begin{array}{l} t = x-2 \\ \text{se } x \to 2^+ \text{ allora} \\ t \to 0^+ \end{array}}$$

$$= \lim_{t \to 0^+} \frac{t^{1/3} - 3 \cdot \log^2(1+t^{1/3})}{(\sinh t - \sin t)^\beta}$$

$$\log(1+y) = y + o(y) \quad \text{per } y \to 0$$

$$\log(1+t^{1/3}) = t^{1/3} + o(t^{1/3}) \quad \text{per } t \to 0$$

$$t^{1/3} - 3 \cdot \log^2(1+t^{1/3}) = t^{1/3} - 3 \cdot \left(t^{1/3} + o(t^{1/3})\right)^2 =$$

$$= t^{1/3} - 3 \cdot \left(t^{2/3} + \underbrace{2t^{1/3} o(t^{1/3}) + o(t^{2/3})}_{= o(t^{2/3})}\right) =$$

$$= t^{1/3} \underbrace{- 3 \cdot t^{2/3} + o(t^{2/3})}_{= o(t^{1/3})} = t^{1/3} + o(t^{1/3})$$

per $t \to 0$

$$\sinh t = t + \frac{t^3}{6} + o(t^4) \quad \text{per } t \to 0$$

$$\sin t = t - \frac{t^3}{6} + o(t^4) \quad \text{per } t \to 0$$

$$\sinh t - \sin t = \cancel{t} + \frac{t^3}{6} + o(t^4) - \cancel{t} + \frac{t^3}{6} + o(t^4) =$$

$$= \frac{t^3}{3} + o(t^4) \quad \text{per } t \to 0$$

$$(\sinh t - \sin t) \sim \frac{t^3}{3} \quad \text{per } t \to 0$$

$$\lim_{t\to 0^+} \frac{t^{1/3} - 3\log^2(1+t^{1/3})}{(\sinh t - \sin t)^\beta} = \lim_{t\to 0^+} \frac{t^{1/3}}{\left(\frac{t^3}{3}\right)^\beta} =$$

$$= \lim_{t\to 0^+} 3^\beta \cdot t^{\frac{1}{3}-3\beta} = \begin{cases} 0 & \text{se } \beta < \frac{1}{9} \quad \text{(esp. positivo)} \\ 3^{1/9} & \text{se } \beta = \frac{1}{9} \quad \text{(esp. nullo)} \\ +\infty & \text{se } \beta > \frac{1}{9} \quad \text{(esp. negativo)} \end{cases}$$

$$\frac{1}{3} - 3\beta > 0 \iff 3\beta < \frac{1}{3} \iff \beta < \frac{1}{9}$$

Calcolare il limite

$$\lim_{x\to+\infty} \frac{3\cos\left(\frac{1}{x}\right)\left(\sin\frac{1}{x} - \sinh\frac{1}{x}\right)}{4\left(e^{1/x}-1\right)^4} =$$

$\frac{1}{x} = y$

se $x \to +\infty$, allora $y \to 0^+$

$$= \lim_{y\to 0^+} \frac{3\overbrace{\cos y}^{\to 1}\overbrace{\left(\sin y - \sinh y\right)}^{\to 0}}{4\underbrace{\left(e^y-1\right)}_{\to 0}{}^4} = \circledast$$

$$\sin y - \sinh y = \cancel{y} - \frac{y^3}{6} + o(y^4) - \cancel{y} - \frac{y^3}{6} + o(y^4) =$$

$$= -\frac{y^3}{3} + o(y^4) \quad \text{per } y \to 0$$

$$(e^y - 1) \sim y \quad \text{per } y \to 0$$

$$(e^y-1)^4 \sim y^4 \quad \text{per } y \to 0$$

$$\circledast = \lim_{y\to 0^+} \frac{3\cos y \overbrace{\left(-\frac{y^3}{3} + o(y^4)\right)}^{\sim -\frac{y^3}{3}}}{4\underbrace{\left(e^y-1\right)^4}_{\sim y^4}} =$$

$$= \lim_{y\to 0^+} \frac{3\cdot\left(-\frac{y^3}{3}\right)}{4y^4} = \lim_{y\to 0^+} \frac{-1}{4y} = -\infty$$

Calcolare, al variare di $\beta \in \mathbb{R}$, il limite

$$\lim_{n \to +\infty} \frac{\cos\left(\frac{1}{n}\right) - \cosh\left(\frac{1}{n}\right)}{n^\beta}$$

$$\cos t = 1 - \frac{t^2}{2} + o(t^3) \quad \text{per } t \to 0$$

$$\cosh t = 1 + \frac{t^2}{2} + o(t^3) \quad \text{per } t \to 0$$

$$\lim_{n \to +\infty} \frac{\cos\left(\frac{1}{n}\right) - \cosh\left(\frac{1}{n}\right)}{n^\beta} =$$

$$= \lim_{n \to +\infty} \frac{\left(\cancel{1} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right) - \left(\cancel{1} + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right)}{n^\beta} =$$

$$= \lim_{n \to +\infty} \frac{-\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)}{n^\beta} = \lim_{n \to +\infty} \frac{-n^{-2}}{n^\beta} =$$

$$= \lim_{n \to +\infty} \left(-n^{-2-\beta}\right) = \begin{cases} -\infty & \text{se } \beta < -2 \\ -1 & \text{se } \beta = -2 \\ 0 & \text{se } \beta > -2 \end{cases}$$

$$-2 - \beta > 0 \iff \beta < -2$$

Calcolare il limite

$$\lim_{x\to 0^+} \frac{\log\left(1-\cos(3x^2)-\frac{9}{2}x^4\right)}{\log x}$$

$\to 1-1-0 \to -\infty$

$\to -\infty$

$$\lim_{x\to 0^+} \frac{\log\left(1-\cos(3x^2)-\frac{9}{2}x^4\right)}{\log x} \overset{(H)}{=}$$

$$\overset{(H)}{=} \lim_{x\to 0^+} \frac{\frac{1}{1-\cos(3x^2)-\frac{9}{2}x^4}\cdot\left(\sin(3x^2)\cdot 6x - \frac{9}{\cancel{2}_{1}}\cdot \cancel{4}^{2}x^3\right)}{\frac{1}{x}} =$$

$$= \lim_{x\to 0^+} \frac{6x^2\sin(3x^2)-18x^4}{1-\cos(3x^2)-\frac{9}{2}x^4} = \lim_{x\to 0^+} \frac{6x^2\left(\sin(3x^2)-3x^2\right)}{1-\cos(3x^2)-\frac{9}{2}x^4} =$$

$$= (\Delta)$$

$$\sin t = t - \frac{t^3}{6} + o(t^4) \quad \text{per } t\to 0$$

$$\sin(3x^2)-3x^2 = \cancel{3x^2} - \frac{(3x^2)^3}{6} + o(x^8) - \cancel{3x^2} =$$

$$= -\frac{9}{2}x^6 + o(x^8) \quad \text{per } x\to 0$$

$$\left(\sin(3x^2)-3x^2\right) \sim -\frac{9}{2}x^6 \quad \text{per } x\to 0$$

$$\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{24} + o(t^5) \quad \text{per } t \to 0$$

$$\cos(3x^2) = 1 - \frac{(3x^2)^2}{2} + \frac{(3x^2)^4}{24} + o(x^{10}) =$$

$$= 1 - \frac{9}{2}x^4 + \frac{27}{8}x^8 + o(x^{10}) \quad \text{per } x \to 0$$

$$1 - \cos(3x^2) - \frac{9}{2}x^4 = \cancel{1} - \cancel{1} + \cancel{\frac{9}{2}x^4} - \frac{27}{8}x^8 + o(x^{10}) - \cancel{\frac{9}{2}x^4} =$$

$$= -\frac{27}{8}x^8 + o(x^{10}) \sim -\frac{27}{8}x^8 \quad \text{per } x \to 0$$

$$(\Delta) = \lim_{x \to 0^+} \frac{\overset{3}{\cancel{6}}\cancel{x^2} \cdot \left(-\frac{9}{\underset{1}{\cancel{2}}}\cancel{x^6}\right)}{-\frac{27}{8}\cancel{x^8}} = \frac{-27}{-\frac{27}{8}} = 8$$

Fissato $\alpha \in \mathbb{R}$, calcolare il limite

$$\lim_{n \to +\infty} \sin\left(\frac{2^n}{n}\right) \cdot \frac{\sqrt{1+\frac{1}{n^2}} - 1}{n^\alpha \left(\sinh\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n}\right)\right)}.$$

$\lim_{n \to +\infty} \frac{2^n}{n} = +\infty$ per confronto di infiniti

Non esiste $\lim_{n \to +\infty} \sin\left(\frac{2^n}{n}\right)$, ma la successione $a_n = \sin\left(\frac{2^n}{n}\right)$ è limitata, perchè $-1 \leq \sin t \leq 1 \ \forall t \in \mathbb{R}$.

$$(1+t)^{1/2} = 1 + \frac{1}{2}t + o(t) \quad \text{per } t \to 0$$

$$\sqrt{1+\frac{1}{n^2}} - 1 = \cancel{1} + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) - \cancel{1} \quad \text{per } n \to +\infty$$

$$\sqrt{1+\frac{1}{n^2}} - 1 \sim \frac{1}{2n^2} \quad \text{per } n \to +\infty$$

$$\sinh\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n}\right) = \cancel{\frac{1}{n}} + \frac{1}{6n^3} + o\left(\frac{1}{n^4}\right) - \cancel{\frac{1}{n}} + \frac{1}{6n^3} + o\left(\frac{1}{n^4}\right) =$$

$$= \frac{1}{3n^3} + o\left(\frac{1}{n^4}\right) \quad \text{per } n \to +\infty$$

$$\sinh\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n}\right) \sim \frac{1}{3n^3} \quad \text{per } n \to +\infty$$

$$\lim_{n\to+\infty} \underbrace{\frac{\sqrt{1+\frac{1}{n^2}}-1}{n^\alpha\left(\sinh\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n}\right)\right)}}_{=b_n} = \lim_{n\to+\infty} \frac{\frac{1}{2n^2}}{n^\alpha \cdot \frac{1}{3n^3}} =$$

$$= \lim_{n\to+\infty} \frac{3n^3}{2n^2\cdot n^\alpha} = \lim_{n\to+\infty} \frac{3}{2} n^{1-\alpha} = \begin{cases} +\infty & \text{se } \alpha<1 \\ \frac{3}{2} & \text{se } \alpha=1 \\ 0 & \text{se } \alpha>1 \end{cases}$$

$$\boxed{1-\alpha>0 \iff \alpha<1}$$

$$\lim_{n\to+\infty} \underbrace{\sin\left(\frac{2^n}{n}\right)}_{\text{succ. limitata}} \cdot \underbrace{b_n}_{\text{succ. infinitesima}} = 0 \quad \text{se } \alpha>1$$

Se $\alpha \leq 1$, allora non esiste $\lim_{n\to+\infty} \underbrace{\sin\left(\frac{2^n}{n}\right)}_{\text{oscilla fra } -1 \text{ e } 1} \cdot \underbrace{b_n}_{\text{tende a } +\infty \text{ o a } \frac{3}{2}}$