Esercitazione del 6 novembre 2023

Prima parte – Limiti di successioni

1. (Prova in itinere 13 novembre 2009) Calcolare il limite

$$\lim_{n\to+\infty} \sqrt[n]{\frac{n^3+1}{n^2-1}}\left(1+\frac{1}{2n}\right)^{3n}.$$

2. (Prova in itinere 14 novembre 2019) Calcolare il limite

$$\lim_{n\to+\infty} \frac{8\cdot n^{7/n} - 7(n+8)^n + \left(\frac{8}{7}\right)^{2n}}{\sqrt{(n+7)^{2n} - n!} - n!}.$$

3. (Prova d'esame 10 aprile 2017) Calcolare il limite

$$\lim_{n\to+\infty} \frac{(n+1)!(n+2)^n \log\left(1+\sin\frac{1}{n}\right)}{n!(n^n+e^2n^2)}.$$

4. (Prova d'esame 9 gennaio 2009) Calcolare al variare di $\alpha \in \mathbb{R}^+$

$$\lim_{n\to+\infty} \frac{\alpha^n + \arctan(n!) + 1 - \cos\frac{1}{n^2}}{7^{\log n} + 2^n + \log(n^3)}.$$

5. (Prova d'esame 13 gennaio 2020) Sia $\alpha \in \mathbb{R}$. Calcolare il limite

$$\lim_{n\to+\infty} [n^\alpha + 4\log(n^n) + \sin(n!)] \sin^2\left(\frac{1}{3\sqrt{n}}\right) \arctan\left(e^{1/n^2} - 1\right).$$

(13/11/2009) Calcolare

$$\lim_{n\to+\infty} \underbrace{\sqrt[n]{\frac{n^3+1}{n^2-1}}\left(1+\frac{1}{2n}\right)^{3n}}_{=a_n} =$$

$$= \lim_{n\to+\infty} \left(\underbrace{\frac{n^3+1}{n^2-1}}_{\to+\infty}\right)^{\overset{\to 0}{1/n}} \cdot \left(\underbrace{1+\frac{1}{2n}}_{\to 1}\right)^{\overset{\to+\infty}{3n}}$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{2n}\right)^{3n} = \lim_{n\to+\infty}\left[\overbrace{\left(1+\frac{1}{2n}\right)^{2n}}^{\to e}\right]^{3/2} = e^{3/2}$$

$$\boxed{\text{NOTAZIONE} \quad \exp(x) = e^x}$$

$$\left(\frac{n^3+1}{n^2-1}\right)^{1/n} = \exp\left(\log\left(\frac{n^3+1}{n^2-1}\right)^{1/n}\right) = \exp\left(\frac{1}{n}\log\overbrace{\left(\frac{n^3+1}{n^2-1}\right)}^{\sim \frac{n^3}{n^2}}\right) \sim$$

$$\sim \exp\left(\frac{1}{n}\log n\right) = e^{\underbrace{\frac{\log n}{n}}_{\to 0}} \longrightarrow 1 \text{ per } n\to+\infty$$

$$\lim_{n\to+\infty} a_n = 1\cdot e^{3/2} = e^{3/2}$$

<u>Osservazione</u>. $e^{3/2} = e\cdot e^{\frac{1}{2}} = e\sqrt{e}$

(14/11/2019) Calcolare

$$\lim_{n\to+\infty} \frac{8\cdot n^{7/n} - 7(n+8)^n + \left(\frac{8}{7}\right)^{2n}}{\sqrt{(n+7)^{2n} - n!} - n!} = a_n$$

$$8\cdot n^{7/n} - 7\underbrace{(n+8)^n}_{\to+\infty} + \underbrace{\left(\frac{8}{7}\right)^{2n}}_{\to+\infty} =$$

$$= 8\cdot \underbrace{e^{\underbrace{\frac{7}{n}\cdot\log n}_{\to 0}}}_{\to 1} - 7(n+8)^n + \left(\frac{8}{7}\right)^{2n} \sim -7(n+8)^n \text{ per } n\to+\infty$$

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{-7(n+8)^n}{\sqrt{(n+7)^{2n}-n!} - n!} \cdot \frac{\sqrt{(n+7)^{2n}-n!} + n!}{\sqrt{(n+7)^{2n}-n!} + n!} =$$

$$= \lim_{n\to+\infty} \frac{-7(n+8)^n\left(\sqrt{(n+7)^{2n}-n!} + n!\right)}{(n+7)^{2n} - n! - (n!)^2}$$

$$(n+7)^{2n} - n! - (n!)^2 = (n+7)^{2n}\left(1 - \underbrace{\frac{n!}{(n+7)^{2n}}}_{\to 0} - \underbrace{\left(\frac{n!}{(n+7)^n}\right)^2}_{\to 0}\right)$$

per confronto di infiniti

$$(n+7)^{2n} - n! - (n!)^2 \sim (n+7)^{2n} \text{ per } n\to+\infty$$

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{-7(n+8)^n\left(\sqrt{(n+7)^{2n}-n!}+n!\right)}{(n+7)^{2n}} =$$

$$= \lim_{n\to+\infty} \frac{-7(n+8)^n\cdot\left(\sqrt{(n+7)^{2n}\left(1-\frac{n!}{(n+7)^{2n}}\right)}+n!\right)}{(n+7)^{2n}} =$$

$$= \lim_{n\to+\infty} \frac{-7(n+8)^n(n+7)^n\left(\sqrt{1-\overset{\to 0}{\frac{n!}{(n+7)^{2n}}}}+\overset{\to 0}{\frac{n!}{(n+7)^n}}\right)}{(n+7)^{2n}} =$$

per confronto di infiniti

$$= \lim_{n\to+\infty} \frac{-7(n+8)^n(n+7)^n}{(n+7)^{2n}} =$$

$$= \lim_{n\to+\infty} \frac{-7(n+8)^n}{(n+7)^n} = \lim_{n\to+\infty} -7\left(\overset{\to 1}{\frac{n+8}{n+7}}\right)^{\overset{\to +\infty}{n}} =$$

$$= \lim_{n\to+\infty} -7\left(\frac{(n+7)+1}{n+7}\right)^n = \lim_{n\to+\infty} -7\left(1+\frac{1}{n+7}\right)^n = -7e$$

<u>OSSERVAZIONE</u>. <u>Non</u> è vero che $(n+8)^n \sim n^n$ e <u>non</u> è vero che $(n+7)^n \sim n^n$ per $n\to+\infty$

$$\lim_{n\to+\infty} \frac{(n+8)^n}{n^n} = \lim_{n\to+\infty}\left(\frac{n+8}{n}\right)^n = \lim_{n\to+\infty}\left(1+\frac{8}{n}\right)^n = e^8$$

È vero che $\boxed{(n+8)^n \sim e^8\cdot n^n \text{ per } n\to+\infty}$

(10/04/2017) Calcolare

$$\lim_{n\to+\infty} \underbrace{\frac{(n+1)!\,(n+2)^n \cdot \log\left(1+\sin\frac{1}{n}\right)}{n!\,(n^n+e^2n^2)}}_{=a_n} =$$

$$= \lim_{n\to+\infty} \frac{(n+1)\cdot \cancel{n!}\,(n+2)^n \cdot \log\left(1+\underbrace{\sin\underbrace{\frac{1}{n}}_{\to 0}}_{\to 0}\right)}{\cancel{n!}\,(n^n+e^2n^2)}$$

$\log(1+t) \sim t$ per $t\to 0$ e $\sin t \sim t$ per $t\to 0$

$$\log\left(1+\sin\frac{1}{n}\right) \sim \sin\frac{1}{n} \sim \frac{1}{n} \quad \text{per } n\to+\infty$$

$$n^n+e^2n^2 = n^n\left(1+\underbrace{\frac{e^2n^2}{n^n}}_{\to 0}\right) \sim n^n \quad \text{per } n\to+\infty$$

(confronto di infiniti)

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{(n+1)(n+2)^n\cdot\frac{1}{n}}{n^n} =$$

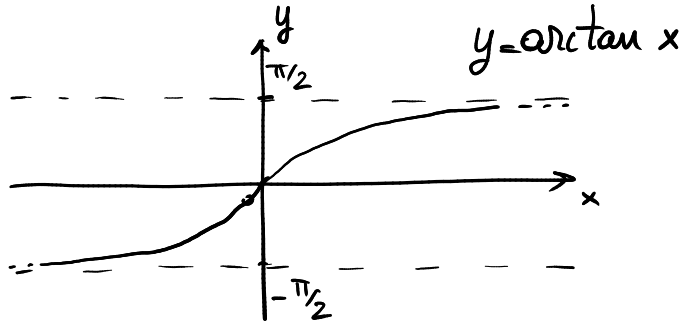
Algebra dei limiti

$$= \lim_{n\to+\infty}\left(\frac{n+2}{n}\right)^n\cdot\frac{n+1}{n} = \lim_{n\to+\infty}\underbrace{\left(1+\frac{2}{n}\right)^n}_{\to e^2}\cdot\underbrace{\frac{n+1}{n}}_{\to 1} = e^2$$

(09/01/2009) Calcolare al variare di $\alpha \in \mathbb{R}^+$

$$\lim_{n\to+\infty} \frac{\alpha^n + \arctan(n!) + 1 - \cos\frac{1}{n^2}}{7^{\log n} + 2^n + \log(n^3)} = a_n$$

$$\underbrace{\alpha^n}_{+\infty} + \underbrace{\arctan(n!)}_{\frac{\pi}{2}} + \underbrace{1-\cos\frac{1}{n^2}}_{0} \sim \alpha^n \quad \text{per } n\to+\infty$$



$$\underbrace{7^{\log n}}_{+\infty} + \underbrace{2^n}_{+\infty} + \underbrace{\log(n^3)}_{+\infty} = e^{(\log 7)(\log n)} + e^{n\cdot\log 2} + 3\log n =$$

$$= e^{n\cdot\log 2}\left(e^{\overbrace{(\log 7)(\log n) - n\log 2}^{-\infty}} + 1 + \underbrace{\frac{3\log n}{e^{n\cdot\log 2}}}_{0}\right) \sim 2^n \quad \text{per } n\to+\infty$$

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{\alpha^n}{2^n} = \lim_{n\to+\infty}\left(\frac{\alpha}{2}\right)^n$$

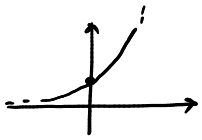
$\frac{\alpha}{2} > 1 \iff \alpha > 2$

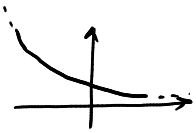
$\alpha > 2 \Rightarrow$ base $\frac{\alpha}{2} > 1$

$\alpha = 2 \Rightarrow$ base $\frac{\alpha}{2} = 1$

$0 < \alpha < 2 \Rightarrow$ base $\frac{\alpha}{2} < 1$ e $\frac{\alpha}{2} > 0$

$$\lim_{n \to +\infty} a_n = \begin{cases} +\infty & \text{se } \alpha > 2 \\ 1 & \text{se } \alpha = 2 \\ 0 & \text{se } 0 < \alpha < 2 \end{cases}$$





(13/01/2020) Calcolare al variare di $\alpha \in \mathbb{R}$

$$\lim_{n\to+\infty} \left[n^{\alpha} + 4\log(n^n) + \sin(n!)\right] \sin^2\left(\frac{1}{3\sqrt{n}}\right) \arctan\left(e^{\frac{1}{n^2}} - 1\right)$$

$$\sin^2\left(\underbrace{\frac{1}{3\sqrt{n}}}_{\to 0}\right) = \left(\sin\left(\frac{1}{3\sqrt{n}}\right)\right)^2 \sim \left(\frac{1}{3\sqrt{n}}\right)^2 = \frac{1}{9n} \text{ per } n\to+\infty$$

$\sin t \sim t$ per $t \to 0$

$$\arctan\left(\underbrace{e^{\overset{\to 0}{1/n^2}} - 1}_{\to 0}\right) \sim e^{\frac{1}{n^2}} - 1 \sim \frac{1}{n^2} \text{ per } n\to+\infty$$

$(e^t - 1) \sim t$ per $t \to 0$ $\qquad$ $\arctan t \sim t$ per $t \to 0$

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \left[n^{\alpha} + 4n\log(n) + \sin(n!)\right] \cdot \frac{1}{9n} \cdot \frac{1}{n^2} =$$

$$= \lim_{n\to+\infty} \frac{1}{9}\left[n^{\alpha} + 4n\log(n) + \sin(n!)\right] \cdot \frac{1}{n^3} =$$

$$= \lim_{n\to+\infty} \frac{1}{9}\left[n^{\alpha-3} + 4\overset{\to 0}{\frac{\log(n)}{n^2}} + \underbrace{\frac{\sin(n!)}{n^3}}_{\to 0}\right]$$

perchè $\sin(n!)$ è una succ. limitata e $\frac{1}{n^3}$ è una succ. infinitesima

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} \frac{1}{9} n^{\alpha-3} = \begin{cases} +\infty & \text{se } \alpha > 3 \\ \frac{1}{9} & \text{se } \alpha = 3 \\ 0 & \text{se } \alpha < 3 \end{cases}$$

$$\lim_{n \to +\infty} n^{\beta} = \begin{cases} +\infty & \text{se } \beta > 0 \\ 1 & \text{se } \beta = 0 \\ 0 & \text{se } \beta < 0 \end{cases}$$

$\alpha - 3 > 0 \iff \alpha > 3$

Esercitazione del 6 novembre 2023

Seconda parte – Calcolo di derivate e punti di non derivabilità

1. (Test 3 febbraio 2020) Sia
$$f(x) = \begin{cases} \dfrac{\cos(3x) - 1}{3x} & \text{se } x < 0 \\ \sqrt{x^2 + 49} - 7 & \text{se } x \geqslant 0. \end{cases}$$
Discutere la derivabilità di f in $x = 0$.

2. (Test 13 gennaio 2020) Sia
$$f(x) = \begin{cases} x^2 \sqrt{\dfrac{49}{x^2} + \log|x|} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0. \end{cases}$$
Discutere la derivabilità di f in $x = 0$.

3. (Prova d'esame 1 settembre 2017) Siano $\alpha \in \,]7, +\infty[$ e $f\colon \mathbb{R} \to \mathbb{R}$ definita da
$$f(x) = \begin{cases} x^{\alpha-7} \log x & \text{se } x > 0 \\ \sqrt{-x} & \text{se } x \leqslant 0. \end{cases}$$
Al variare di $\alpha \in \,]7, +\infty[$ discutere la derivabilità di f nel punto $x = 0$ e classificare il tipo di non derivabilità.

4. Si consideri la funzione $f\colon \operatorname{dom}(f) \subseteq \mathbb{R} \to \mathbb{R}$ definita da $f(x) = \tan x$. Calcolare la funzione derivata prima di f e determinarne il dominio, classificando eventuali punti di non derivabilità.

(03/02/2020)

$$f(x) = \begin{cases} \dfrac{\cos(3x) - 1}{3x} & \text{se } x < 0 \\ \sqrt{x^2 + 49} - 7 & \text{se } x \geq 0 \end{cases}$$

Discutere la derivabilità di f in $x = 0$

- Continuità in 0 ?

$$f(0) = \sqrt{0^2 + 49} - 7 = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\sqrt{x^2 + 49} - 7 \right) = 0$$

per continuità delle funzioni elementari

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\cos(\overset{\to 0}{3x}) - 1}{\underset{\to 0}{3x}} = 0$$

$$\lim_{t \to 0} \frac{1 - \cos t}{t^2} = \frac{1}{2}, \qquad \lim_{t \to 0} \frac{1 - \cos t}{t} = 0$$

Concludiamo che f è continua in $x = 0$

- Calcoliamo la derivata destra e la derivata sinistra in $x = 0$

$$f'_+(0) = \lim_{x \to 0^+} \frac{f(x) - \overset{=0}{f(0)}}{x - 0} = \lim_{x \to 0^+} \frac{f(x)}{x} =$$

$$= \lim_{x \to 0^+} \frac{\sqrt{x^2 + 49} - 7}{x} \cdot \frac{\sqrt{x^2 + 49} + 7}{\sqrt{x^2 + 49} + 7} =$$

$$= \lim_{x\to 0^+} \frac{x^2 + \cancel{49} - \cancel{49}}{x\left(\sqrt{x^2+49} + 7\right)} = \lim_{x\to 0^+} \frac{\overset{\to 0}{x}}{\underset{\downarrow\, 14}{\sqrt{x^2+49} + 7}} = 0$$

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{f(x)}{x} =$$

$$= \lim_{x\to 0^-} \frac{\cos(3x) - 1}{3x^2} = \lim_{x\to 0^-} \frac{-(1-\cos(3x))}{3x^2} =$$

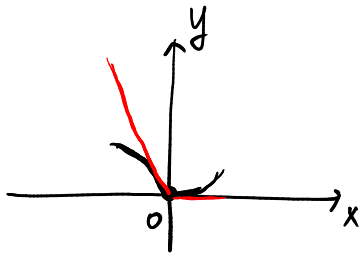
$$\boxed{(1-\cos t) \sim \frac{1}{2} t^2 \text{ per } t \to 0}$$

$$= \lim_{x\to 0^-} \frac{-\frac{1}{2}(3x)^2}{3x^2} = \lim_{x\to 0^-} \frac{-\frac{9}{2}x^2}{3x^2} = -\frac{3}{2}$$

$$f'_-(0) = -\frac{3}{2}, \qquad f'_+(0) = 0$$

f non è derivabile in $x=0$

$x=0$ è un punto angoloso



(13/01/2020)

$$f(x)=\begin{cases} x^2\sqrt{\frac{49}{x^2}+\log|x|} & \text{se } x\neq 0 \\ 0 & \text{se } x=0 \end{cases}$$

Discutere la derivabilità in $x=0$

. Studiamo la continuità in 0

$f(0)=0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2\sqrt{\frac{49}{x^2}+\log|x|}\right) =$$

$$= \lim_{x\to 0^+} x^2\sqrt{\frac{49+x^2\log x}{x^2}} = \lim_{x\to 0^+} \frac{\cancel{x^2}^{\,x}}{\cancel{x}_{\,1}}\sqrt{49+\underbrace{x^2\log x}_{\to 0}} = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2\sqrt{\frac{49}{x^2}+\log|x|} =$$

$$= \lim_{x\to 0^-} x^2\sqrt{\frac{49+x^2\log(-x)}{x^2}} = \lim_{x\to 0^-} \frac{\cancel{x^2}^{\,x}}{\cancel{-x}_{\,-1}}\sqrt{49+\underbrace{x^2\log(-x)}_{\to 0}} = 0$$

f è continua in $x=0$

. Calcoliamo la derivata destra e la derivata sinistra in $x=0$

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - \overbrace{f(0)}^{=0}}{x-0} = \lim_{x\to 0^+} \frac{f(x)}{x} =$$

se $x>0$, allora $|x| = x$

$$= \lim_{x\to 0^+} \frac{\cancel{x^2}^{\,x}}{\cancel{x}_{\,1}} \sqrt{\frac{49}{x^2} + \log|x|} = \lim_{x\to 0^+} x \sqrt{\frac{49 + x^2 \log x}{x^2}} =$$

$$= \lim_{x\to 0^+} \frac{\cancel{x}^{\,1}}{\cancel{x}_{\,1}} \sqrt{49 + \underbrace{x^2 \log x}_{\to 0}} = 7$$

se $x<0$ allora $|x| = -x$

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} x \sqrt{\frac{49}{x^2} + \log|x|} =$$

$$= \lim_{x\to 0^-} x \sqrt{\frac{49 + x^2 \log(-x)}{x^2}} = \lim_{x\to 0^-} \frac{\cancel{x}^{\,1}}{\cancel{-x}_{\,-1}} \sqrt{49 + x^2 \log(-x)} = -7$$

$f'_+(0) = 7, \quad f'_-(0) = -7$

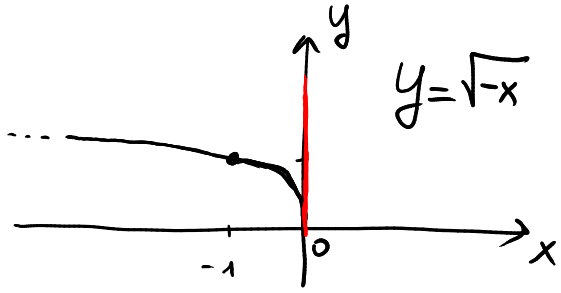
f non è derivabile in $x=0$

$x=0$ è un punto angoloso per f

(01/09/2017) Siano $\alpha \in (7,+\infty)$ e $f:\mathbb{R}\to\mathbb{R}$ funzione definita da

$$f(x) = \begin{cases} x^{\alpha-7}\log x & \text{se } x>0 \\ \sqrt{-x} & \text{se } x\le 0 \end{cases}$$

Discutere la derivabilità di f in $x=0$



- Continuità di f in $x=0$

$f(0) = \sqrt{-0} = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{-x} = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^{\alpha-7}\log x = 0$

0 perchè $\alpha>7$

$$\boxed{\text{Se } \alpha>0 \text{ e } \beta>0 \quad \lim_{t\to+\infty} \frac{(\log x)^{\alpha}}{x^{\beta}} = 0}$$

f è continua in $x=0$

- Calcoliamo $f'_-(0)$ e $f'_+(0)$

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x)-\overbrace{f(0)}^{=0}}{x-0} = \lim_{x\to 0^-} \frac{f(x)}{x} =$$

$$= \lim_{x\to 0^-} \frac{\sqrt{-x}}{x} = \lim_{x\to 0^-} \frac{\sqrt{-x}}{-(-x)} = \lim_{x\to 0^-}\left(-\sqrt{\frac{-x}{(-x)^2}}\right) =$$

$$= \lim_{x\to 0^-}\left(-\sqrt{\frac{\cancel{-x}^{\,-1}}{\cancel{x^2}_{\,x}}}\right) = \lim_{x\to 0^-} \frac{-1}{\underbrace{\sqrt{-x}}_{\to 0^+}} = -\infty$$

Alternativa

$$\lim_{x\to 0^-} \frac{\sqrt{-x}}{x} = \lim_{x\to 0^-} \frac{(-x)^{1/2}}{-(-x)} = \lim_{x\to 0^-}\left[-(-x)^{\frac{1}{2}-1}\right] =$$

$$= -\lim_{x\to 0^-} (-x)^{-1/2} = -\infty$$

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{f(x)}{x} =$$

$$= \lim_{x\to 0^+} \frac{x^{\alpha-7}\log x}{x} = \lim_{x\to 0^+} x^{\alpha-8}\log x$$

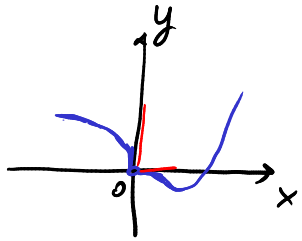
$$\lim_{x\to 0^+} x^{\alpha-8} = \begin{cases} 0 & \text{se } \alpha>8 \\ 1 & \text{se } \alpha=8 \\ +\infty & \text{se } \alpha<8 \end{cases}$$

$$f'_+(0) = \lim_{x\to 0^+} x^{\alpha-8}\underbrace{\log x}_{-\infty} = \begin{cases} 0 & \text{se } \alpha>8 \\ 1\cdot(-\infty) & \text{se } \alpha=8 \\ +\infty\cdot(-\infty) & \text{se } 7<\alpha<8 \end{cases}$$

$$= \begin{cases} 0 & \text{se } \alpha > 8 \\ -\infty & \text{se } 7 < \alpha \leq 8 \end{cases}$$

Se $\alpha > 8$, allora

$f'_-(0) = -\infty$ e $f'_+(0) = 0$

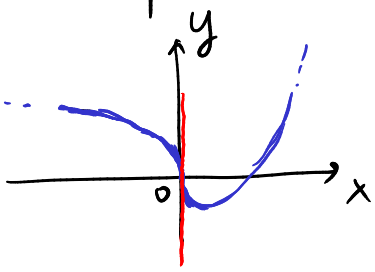


$x = 0$ è un punto angoloso per f

Se $7 < \alpha \leq 8$, allora

$f'_-(0) = -\infty$ e $f'_+(0) = -\infty$

$x = 0$ è un punto a tangente verticale



$$f(x) = \tan x \qquad \mathrm{dom}(f) = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\}$$

$$f(x) = \frac{\overbrace{\sin x}^{=g(x)}}{\underbrace{\cos x}_{=h(x)}} \qquad \left(\frac{g}{h}\right)'(x) = \frac{g'(x)\cdot h(x) - g(x)\cdot h'(x)}{[h(x)]^2}$$

$$f'(x) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

$$\mathrm{dom}(f') = \mathrm{dom}(f) = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\}$$

Non ci sono punti di non derivabilità