

Esercitazione del 12 ottobre 2023

1. Calcolare i limiti

$$\lim_{x \rightarrow -\infty} [(3 - 2x^5)(x^{10} + 4)], \quad \lim_{x \rightarrow -\infty} \frac{(3 - 2x)(3x + 5)(4x^8 - 6)}{x - 3x^{10} + 2}, \quad \lim_{x \rightarrow +\infty} \frac{(2x^5 + 3x + 1)(5x^4 + 3x^6 - 2)}{4x^{12} - 5x + 1}.$$

2. Calcolare il limite $\lim_{x \rightarrow -\infty} \left(\frac{2}{3}\right)^{3x}$.

3. Calcolare il limite $\lim_{x \rightarrow +\infty} \log_{\frac{1}{2}}(x^2)$.

4. Calcolare i limiti $\lim_{x \rightarrow 0^+} [x]$ e $\lim_{x \rightarrow 0^-} [x]$.

5. Calcolare i limiti $\lim_{x \rightarrow -\infty} \frac{\sinh x}{\cosh x}$ e $\lim_{x \rightarrow +\infty} \frac{\sinh x}{\cosh x}$.

6. Calcolare il limite $\lim_{x \rightarrow 0^+} \frac{\log 3x}{\log 7x}$.

7. Calcolare il limite $\lim_{x \rightarrow -\infty} \frac{\log x^4}{\log x^2}$.

8. Calcolare i limiti $\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\cos x}{1 - \sin x}$ e $\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\cos x}{1 - \sin x}$.

9. Calcolare il limite $\lim_{x \rightarrow 0} \frac{\tan x}{x}$.

10. Calcolare i limiti $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$ e $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$.

11. Calcolare il limite $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$.

12. Calcolare il limite $\lim_{x \rightarrow -\infty} \frac{x}{x + \sin x}$.

13. Calcolare i limiti $\lim_{x \rightarrow +\infty} \frac{\arctan x}{x}$ e $\lim_{x \rightarrow -\infty} \frac{\arctan x}{x}$.

14. Calcolare il limite $\lim_{x \rightarrow 0} x \sin \frac{2}{x}$.

15. Calcolare il limite $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \cos x}{x}$.

Calcolare i limiti

$$\lim_{x \rightarrow -\infty} [(3-2x^5)(x^{10}+4)], \quad \lim_{x \rightarrow -\infty} \frac{(3-2x)(3x+5)(4x^8-6)}{x-3x^{10}+2}, \quad \lim_{x \rightarrow +\infty} \frac{(2x^5+3x+1)(5x^4+3x^6-2)}{4x^{12}-5x+1}.$$

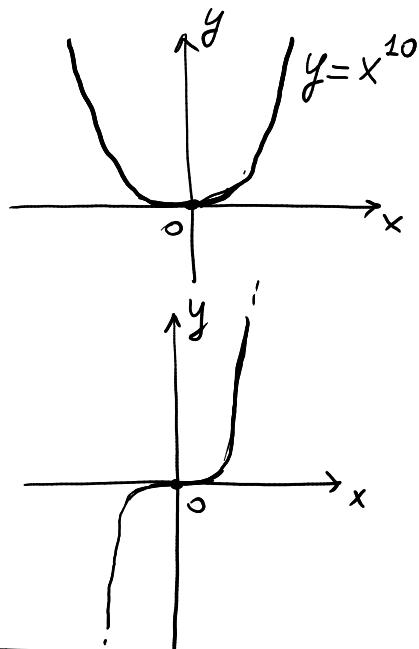
$$\lim_{x \rightarrow -\infty} [(3-2x^5)(x^{10}+4)] = \begin{matrix} \text{algebra} \\ \text{dei} \\ \text{limiti} \end{matrix}$$

(3-2x^5) *(x^{10}+4)*

\downarrow \downarrow

$-\infty$ $+\infty$

$$= (+\infty) \cdot (+\infty) = +\infty$$

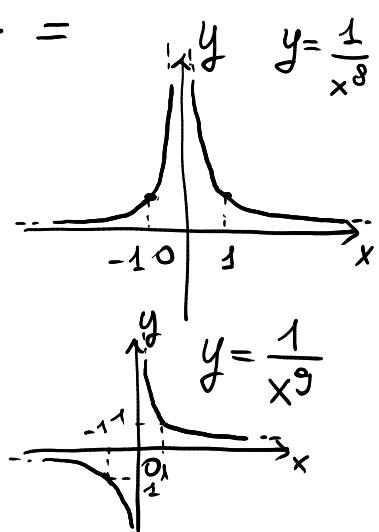


$$\lim_{x \rightarrow -\infty} \frac{(3-2x)(3x+5)(4x^8-6)}{x-3x^{10}+2} =$$

$$= \lim_{x \rightarrow -\infty} \frac{x \left(\frac{3}{x} - 2 \right) \cdot x \left(3 + \frac{5}{x} \right) \cdot x^8 \left(4 - \frac{6}{x^8} \right)}{x^{10} \left(\frac{1}{x^9} - 3 + \frac{2}{x^{10}} \right)} =$$

$$= \lim_{x \rightarrow -\infty} \frac{x^{10} \left(\frac{3}{x} - 2 \right) \left(3 + \frac{5}{x} \right) \left(4 - \frac{6}{x^8} \right)}{x^{10} \left(\frac{1}{x^9} - 3 + \frac{2}{x^{10}} \right)} = \text{A.L.} =$$

$$= \frac{-2 \cdot 3 \cdot 4}{-3} = 8$$



$$\lim_{x \rightarrow +\infty} \frac{(2x^5+3x+1)(5x^4+3x^6-2)}{4x^{12}-5x+1} = \begin{cases} \text{forme indeterminate} \\ \frac{\infty}{\infty} \end{cases}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow +\infty} \frac{\cancel{x^5} \left(2 + \cancel{\frac{3}{x^4}} + \cancel{\frac{1}{x^5}} \right) \cdot \cancel{x^6} \left(\cancel{\frac{5}{x^2}} + 3 - \cancel{\frac{2}{x^6}} \right)}{\cancel{x^{12}} \left(4 - \cancel{\frac{5}{x^{11}}} + \cancel{\frac{1}{x^{12}}} \right)} = \\
 &= \lim_{x \rightarrow +\infty} \frac{\left(2 + \frac{3}{x^4} + \frac{1}{x^5} \right) \left(\frac{5}{x^2} + 3 - \frac{2}{x^6} \right)}{x \left(4 - \frac{5}{x^{11}} + \frac{1}{x^{12}} \right)} \stackrel{AL}{=} \frac{2 \cdot 3}{4 \cdot (+\infty)} = 0
 \end{aligned}$$

Definizione. (infiniti dello stesso ordine)

Siano $x_0 \in \mathbb{R}$ oppure $x_0 = +\infty$ o $x_0 = -\infty$ e f, g funzioni reali definite in un intorno di x_0 escluso al più x_0 .

I limiti $\lim_{x \rightarrow x_0} f(x)$ e $\lim_{x \rightarrow x_0} g(x)$ esistano infiniti

(cioè $\lim_{x \rightarrow x_0} f(x) = +\infty$ o $\lim_{x \rightarrow x_0} f(x) = -\infty$, $\lim_{x \rightarrow x_0} g(x) = +\infty$ o $\lim_{x \rightarrow x_0} g(x) = -\infty$).

- Se $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = l \in \mathbb{R} \setminus \{0\}$, si dice che f e g hanno lo stesso ordine di infinito per $x \rightarrow x_0$, e si scrive $f \sim l \cdot g$ per $x \rightarrow x_0$.

- Se $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0$, si dice che f ha ordine di infinito inferiore rispetto a g per $x \rightarrow x_0$.

- Se $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = +\infty$ oppure $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = -\infty$, si dice

che f ha ordine di infinito superiore rispetto a g per $x \rightarrow x_0$.

$$\lim_{x \rightarrow +\infty} \frac{x^2}{x} = \lim_{x \rightarrow +\infty} x = +\infty$$

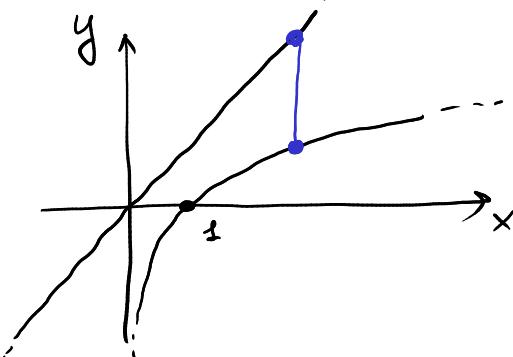
x^2 ha ordine di infinito superiore a x per $x \rightarrow +\infty$

$$\lim_{x \rightarrow +\infty} \frac{8x^3}{3x^3} = \frac{8}{3}$$

$8x^3$ e $3x^3$ hanno lo stesso ordine di infinito per $x \rightarrow +\infty$

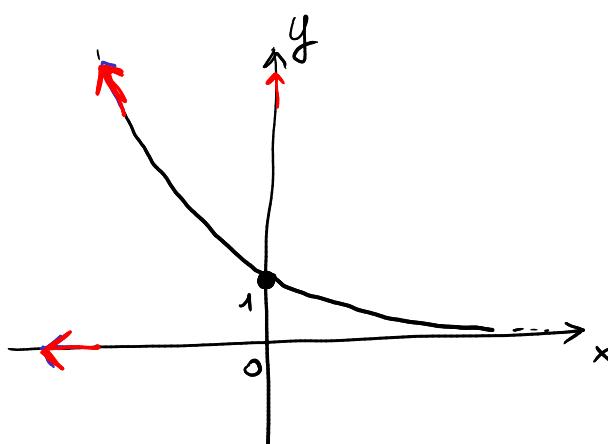
$$\lim_{x \rightarrow +\infty} \frac{\log x}{x} = 0$$

$\log x$ ha ordine di infinito inferiore a x per $x \rightarrow +\infty$



$$\lim_{x \rightarrow +\infty} (2x - 3\log x) = \lim_{x \rightarrow +\infty} \cancel{x} \left(2 - 3 \frac{\log x}{x} \right) \stackrel{AL}{=} +\infty$$

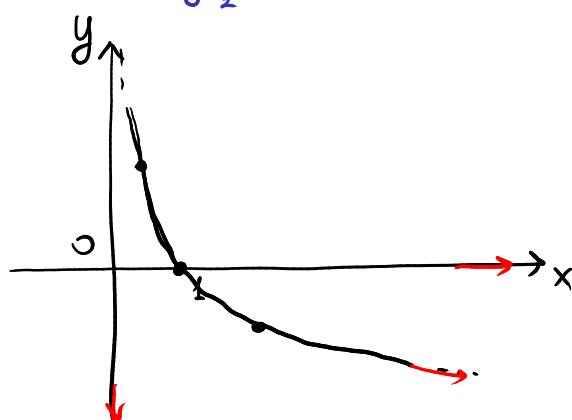
$$\lim_{x \rightarrow -\infty} \left(\frac{2}{3}\right)^{3x} = \lim_{x \rightarrow -\infty} \left[\left(\frac{2}{3}\right)^3\right]^x = \lim_{x \rightarrow -\infty} \left(\frac{8}{27}\right)^x = +\infty$$



$$\lim_{x \rightarrow +\infty} \log_{1/2}(x^2) = \lim_{\substack{x \rightarrow +\infty \\ \{}} 2 \cdot \log_{1/2} x = 2 \cdot (-\infty) = -\infty$$

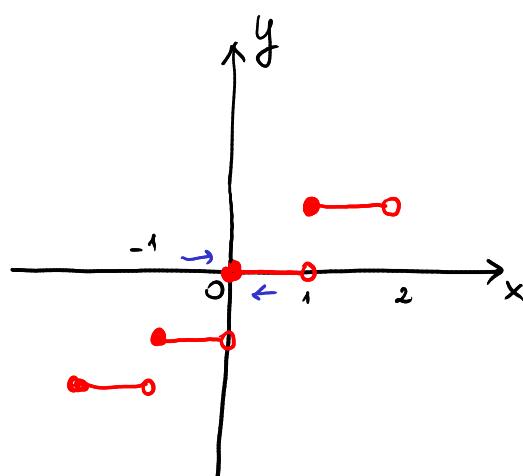
se $A > 0, B \in \mathbb{R}$ allora

$$\log_{1/2} A^B = B \cdot \log_{1/2} A$$



$$\lim_{x \rightarrow 0^+} [x] = 0$$

$$\lim_{x \rightarrow 0^-} [x] = -1$$



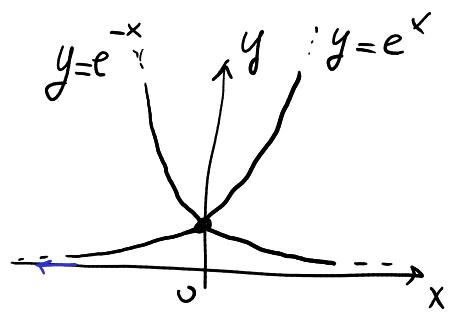
$\lim_{x \rightarrow 0} [x]$ non esiste

SENO IPERBOLICO

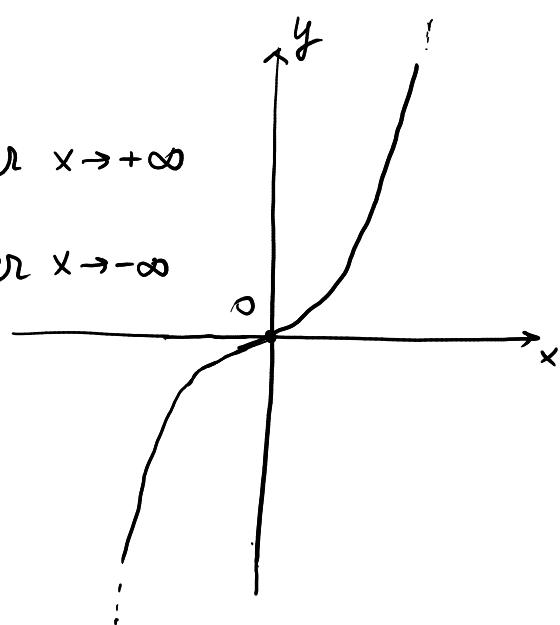
$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \forall x \in \mathbb{R}$$

$$\lim_{x \rightarrow +\infty} \sinh x = \lim_{x \rightarrow +\infty} \frac{e^x - e^{-x}}{2} = +\infty \quad \text{AL}$$

$$\lim_{x \rightarrow -\infty} \sinh x = \lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{2} = -\infty \quad \text{AL}$$



$$\frac{e^x - e^{-x}}{2}$$



$$\sinh 0 = \frac{e^0 - e^0}{2} = \frac{1-1}{2} = 0$$

Seno iperbolico
Dominio \mathbb{R}
Insieme immagine \mathbb{R}

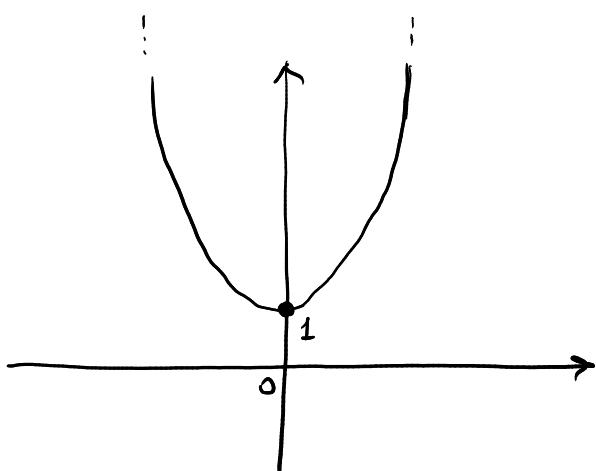
Esercizio: il seno iperbolico è una funzione dispari

COSENO IPERBOLICO

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \forall x \in \mathbb{R}$$

$$\lim_{x \rightarrow +\infty} \cosh x = \lim_{x \rightarrow +\infty} \frac{e^x + e^{-x}}{2} = +\infty$$

$$\lim_{x \rightarrow -\infty} \cosh x = \lim_{x \rightarrow -\infty} \frac{e^x + e^{-x}}{2} = +\infty$$



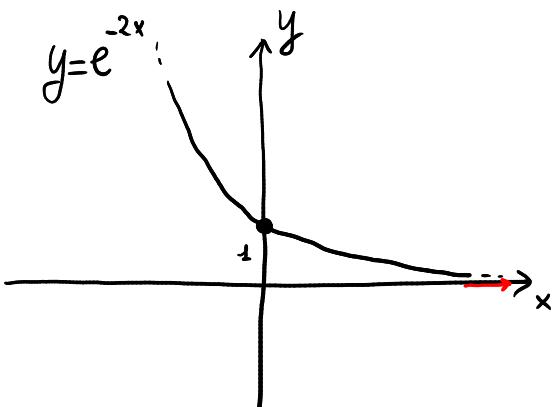
$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

$$\cosh x \sim \frac{1}{2} e^x \text{ per } x \rightarrow +\infty$$

$$\cosh x \sim \frac{1}{2} e^{-x} \text{ per } x \rightarrow -\infty$$

$$\lim_{x \rightarrow +\infty} \frac{\sinh x}{\cosh x} = \lim_{x \rightarrow +\infty} \frac{e^x - e^{-x}}{2} \cdot \frac{2}{e^x + e^{-x}} =$$

$$= \lim_{x \rightarrow +\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow +\infty} \frac{e^x(1 - e^{-2x})}{e^x(1 + e^{-2x})} \stackrel{\text{AL}}{=} \frac{1}{1} = 1$$

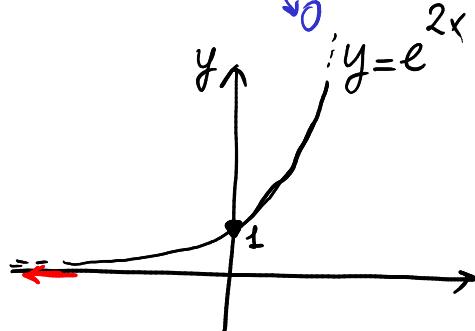


Esercizio:

Il coseno iperbolico è una funzione pari

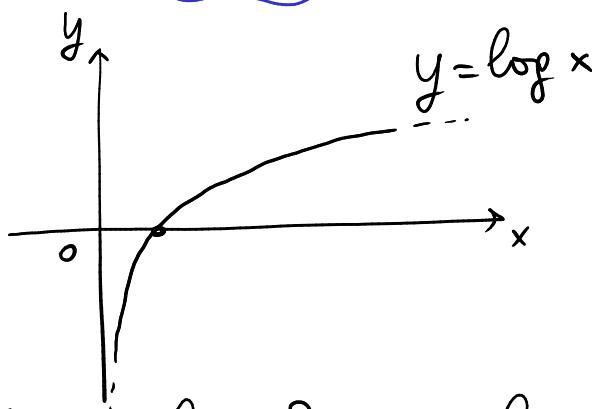
$$\lim_{x \rightarrow -\infty} \frac{\sinh x}{\cosh x} = \lim_{x \rightarrow -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} =$$

$$= \lim_{x \rightarrow -\infty} \frac{e^x (e^{2x} - 1)}{e^x (e^{2x} + 1)} \stackrel{AL}{=} \frac{-1}{1} = -1$$



$$\lim_{x \rightarrow 0^+} \frac{\log 3x}{\log 7x}$$

è una forma indeterminata



$$\lim_{x \rightarrow 0^+} \frac{\log 3x}{\log 7x} = \lim_{x \rightarrow 0^+} \frac{\log 3 + \log x}{\log 7 + \log x} =$$

$$= \lim_{x \rightarrow 0^+} \frac{\log x \left(\frac{\log 3}{\log x} + 1 \right)}{\log x \left(\frac{\log 7}{\log x} + 1 \right)} = \lim_{x \rightarrow 0^+} \frac{\frac{\log 3}{\log x} + 1}{\frac{\log 7}{\log x} + 1} \stackrel{AL}{=} 1$$

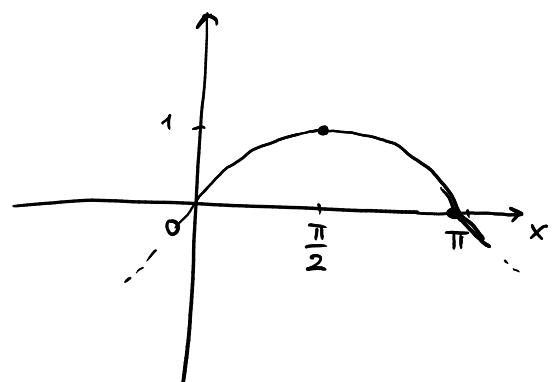
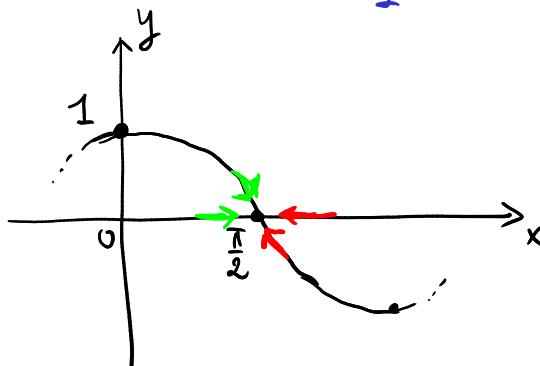
$\log 3x \sim \log 7x$ per $x \rightarrow 0^+$

$$\lim_{x \rightarrow -\infty} \frac{\log x^4}{\log x^2} = \lim_{x \rightarrow -\infty} \frac{4 \cancel{\log |x|}}{2 \cancel{\log |x|}} = 2$$

perché $x^4 = |x|^4$
 $e x^2 = |x|^2$

$$\lim_{\substack{x \rightarrow \frac{\pi}{2}^+}} \frac{\cos x}{1 - \sin x}$$

è una forma indeterminata



$$\lim_{\substack{x \rightarrow \frac{\pi}{2}^+}} \frac{\cos x}{1 - \sin x} = \lim_{\substack{x \rightarrow \frac{\pi}{2}^+}} \frac{\cos x}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} =$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\cos x (1 + \sin x)}{1 - \sin^2 x} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\cos x (1 + \sin x)}{\cos^2 x} =$$

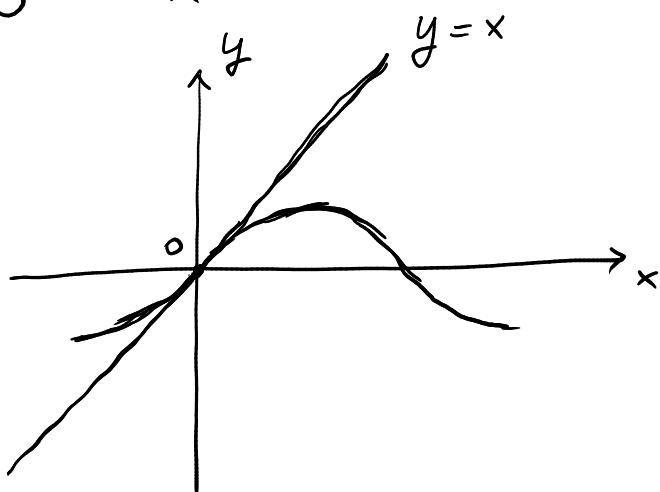
$$= \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{1 + \sin x}{\cos x} = -\infty$$

2
1
o-

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\cos x}{1 - \sin x} = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1 + \sin x}{\cos x} = +\infty$$

Non esiste $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{1 - \sin x}$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$



Definizione (infinitesimi dello stesso ordine)

Siano $x_0 \in \mathbb{R}$ oppure $x_0 = +\infty$ o $x_0 = -\infty$,
 f e g funzioni reali definite in un intorno di x_0 ,
 eccetto al più x_0 .

Siano f e g infinitesime per $x \rightarrow x_0$
 (cioè $\lim_{x \rightarrow x_0} f(x) = 0$ e $\lim_{x \rightarrow x_0} g(x) = 0$)

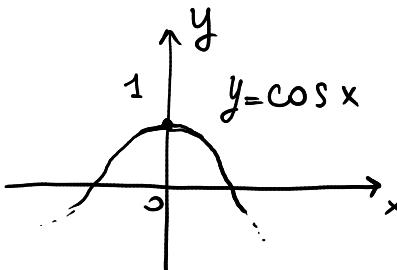
Se $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = l \in \mathbb{R} \setminus \{0\}$, si dice f e g
 sono infinitesime con lo stesso ordine,
 e si scrive $f \sim l \cdot g$ per $x \rightarrow x_0$.

Esempio $\sin x \sim x$ per $x \rightarrow 0$

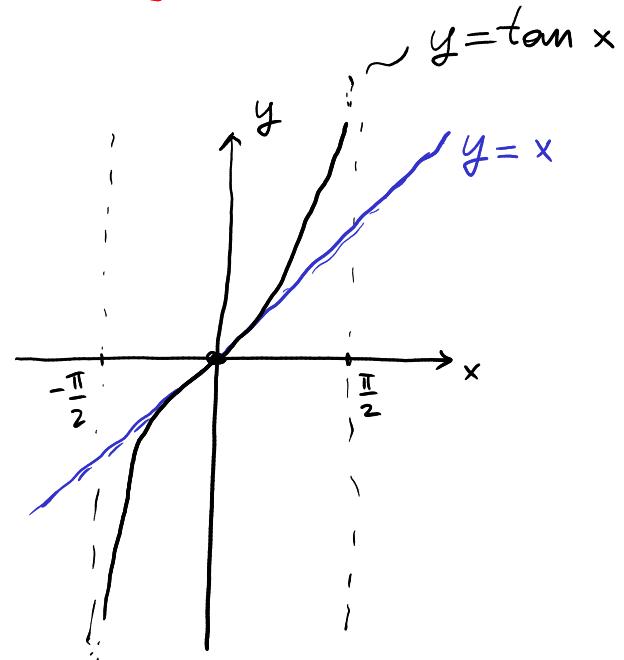
$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) \stackrel{AL}{=} 1$$

$\frac{\sin x}{x} \rightarrow 1$

$\frac{1}{\cos x} \rightarrow 1$



$\tan x \sim x$ per $x \rightarrow 0$



$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x} =$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x^2} \cdot \frac{1}{1 + \cos x} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1 + \cos x} =$$

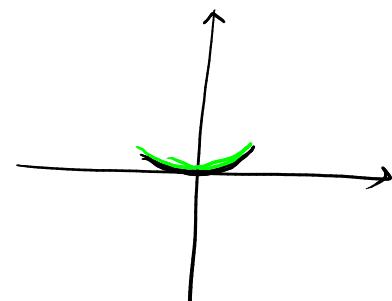
$$= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1}{1 + \cos x} \right) = \frac{1}{2}$$

$\frac{\sin x}{x} \rightarrow 1$

$\frac{\sin x}{x} \rightarrow 1$

$\frac{1}{1 + \cos x} \rightarrow 2$

$$(1 - \cos x) \sim \frac{1}{2} x^2 \text{ per } x \rightarrow 0$$



$$\lim_{x \rightarrow 0} \frac{1-\cos x}{x} = \lim_{x \rightarrow 0} x \cdot \frac{\frac{1-\cos x}{x^2}}{\frac{x^2}{x^2}} = 0$$

\downarrow
0
 \downarrow
 $\frac{1}{2}$

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} =$$

\downarrow
 \downarrow
 \downarrow

$$= \lim_{x \rightarrow 0} \frac{1}{x^3} \left(\frac{\sin x}{\cos x} - \sin x \right) = \lim_{x \rightarrow 0} \frac{\sin x}{x^3} \cdot \left(\frac{1}{\cos x} - 1 \right) =$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{x^3} \cdot \frac{1 - \cos x}{\cos x} =$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x^2} \cdot \frac{1}{\cos x} = \frac{1}{2}$$

\downarrow
1
 \downarrow
 $\frac{1}{2}$
 \downarrow
1

$$(\tan x - \sin x) \sim \frac{1}{2} x^3 \text{ per } x \rightarrow 0$$

$$\lim_{x \rightarrow -\infty} \frac{x}{x + \sin x} = \lim_{x \rightarrow -\infty} \frac{x}{x \left(1 + \frac{\sin x}{x}\right)} =$$

$$= \lim_{x \rightarrow -\infty} \frac{1}{1 + \frac{\sin x}{x}}$$

Basta calcolare $\lim_{x \rightarrow -\infty} \frac{\sin x}{x}$

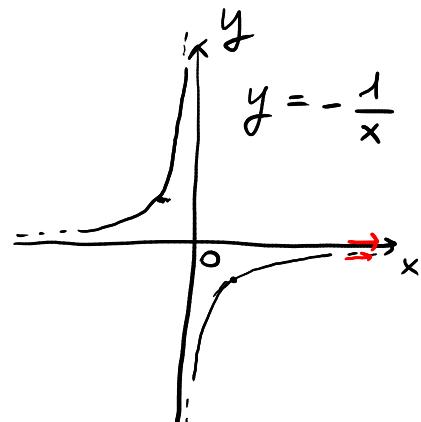
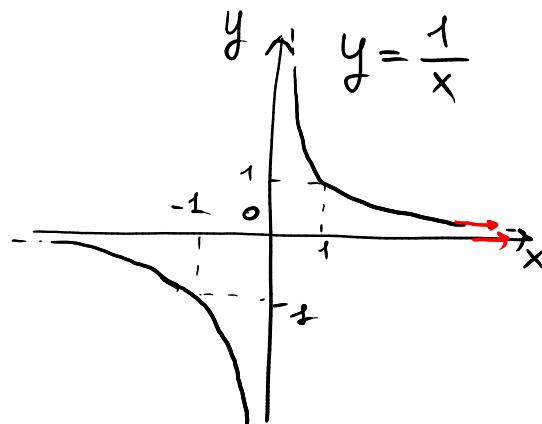
Osservazione. $f(x) = \frac{\sin x}{x}$ è una funzione pari,

$$\text{quindi } \lim_{x \rightarrow -\infty} \frac{\sin x}{x} = \lim_{x \rightarrow +\infty} \frac{\sin x}{x}$$

Applichiamo il II teorema del confronto

$$-1 \leq \sin x \leq 1 \quad \forall x \in \mathbb{R}$$

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \quad \forall x \in (0, +\infty)$$



$$\lim_{x \rightarrow +\infty} \left(-\frac{1}{x}\right) = 0 \quad \text{e} \quad \lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

Per il II teorema del confronto, si conclude che

$$\lim_{x \rightarrow +\infty} \frac{\sin x}{x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{x}{x + \sin x} = \lim_{x \rightarrow -\infty} \frac{1}{1 + \frac{\sin x}{x}} \stackrel{\Delta L}{=} 1$$

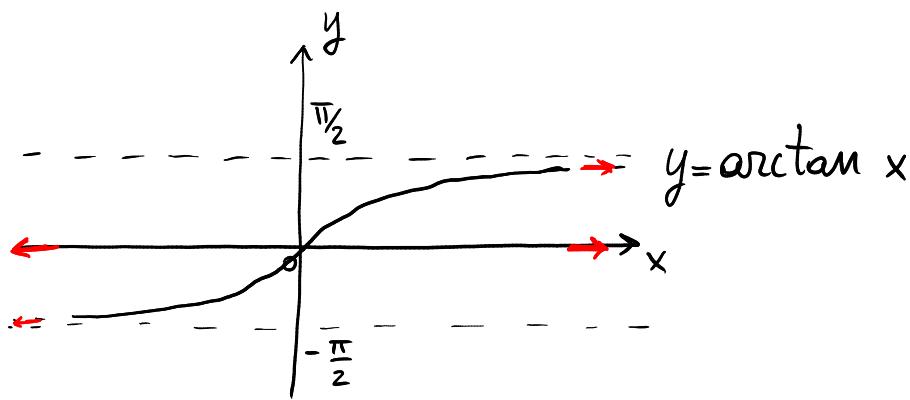
$\frac{\sin x}{x}$

$$\lim_{x \rightarrow +\infty} \frac{\arctan x}{x} \stackrel{\Delta L}{=} 0$$

$x \rightarrow +\infty$

$$\lim_{x \rightarrow -\infty} \frac{\arctan x}{x} \stackrel{\Delta L}{=} 0$$

$x \rightarrow -\infty$



$$\lim_{x \rightarrow 0} x \cdot \sin \frac{2}{x}$$

Osservazione $f(x) = x \cdot \sin \frac{2}{x}$ è pari

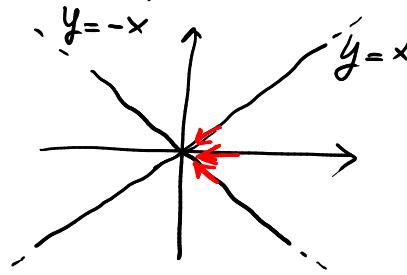
$$\text{Quindi } \lim_{x \rightarrow 0^+} x \cdot \sin \frac{2}{x} = \lim_{x \rightarrow 0^-} x \cdot \sin \frac{2}{x}$$

$$\text{Calcoliamo } \lim_{x \rightarrow 0^+} x \cdot \sin \frac{2}{x}$$

$$-1 \leq \sin \frac{2}{x} \leq 1 \quad \forall x \in (0, +\infty)$$

$$-x \leq x \cdot \sin \frac{2}{x} \leq x \quad \forall x \in (0, +\infty)$$

$$\lim_{x \rightarrow 0^+} (-x) = 0 \quad \text{e} \quad \lim_{x \rightarrow 0^+} x = 0$$



Per il II teorema del confronto, si ha

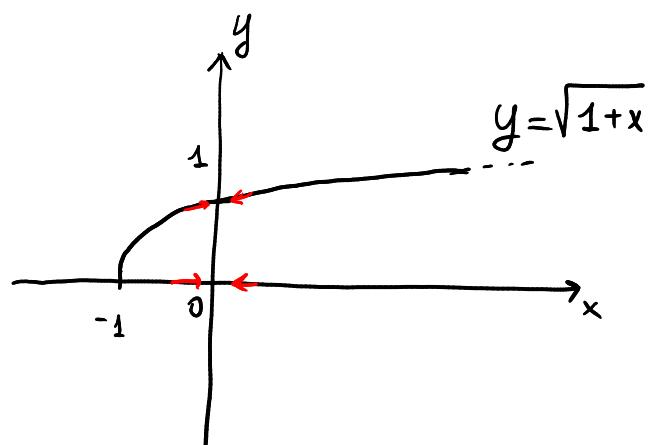
$$\lim_{x \rightarrow 0^+} x \cdot \sin \frac{2}{x} = 0.$$

Quindi anche $\lim_{x \rightarrow 0^-} x \cdot \sin \frac{2}{x} = 0$ (perché la funzione è pari)

e $\lim_{x \rightarrow 0} x \cdot \sin \frac{2}{x} = 0$ (perché i limiti destro e sinistro sono entrambi uguali a 0)

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \cos x}{x}$$

è una forma indeterminata $\frac{0}{0}$



$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \cos x}{x} \cdot \frac{\sqrt{1+x} + \cos x}{\sqrt{1+x} + \cos x} = \\
 &= \lim_{x \rightarrow 0} \frac{1+x - \cos^2 x}{x(\sqrt{1+x} + \cos x)} = \\
 &= \lim_{x \rightarrow 0} \frac{x + \sin^2 x}{x(\sqrt{1+x} + \cos x)} = \\
 &= \lim_{x \rightarrow 0} \left(\frac{x}{x(\sqrt{1+x} + \cos x)} + \frac{\sin^2 x}{x(\sqrt{1+x} + \cos x)} \right) \stackrel{\text{AL}}{=} \frac{1}{2}
 \end{aligned}$$