Esercitazione del 2 ottobre 2023

1. Scrivere in forma algebrica i numeri complessi

$$w_1 = \frac{7i}{2-i}, \qquad w_2 = \frac{(3-i)\left(\overline{-\frac{2}{3}i-1}\right)}{1+3i}, \qquad w_3 = i^{2023}(1-i)^4.$$

   Individuare la parte reale e la parte immaginaria, e calcolare il modulo di ciascuno.

2. Scrivere in forma algebrica i numeri complessi

$$w_1 = \sqrt{2}\left(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi\right), \quad w_2 = \frac{3}{4}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right), \quad w_3 = \frac{1}{2}\left(\cos\left(\frac{13}{3}\pi\right) - i\sin\left(\frac{13}{3}\pi\right)\right).$$

3. Scrivere in forma trigonometrica i numeri complessi

$$w_1 = \frac{5}{6i}, \qquad w_2 = 5-5i, \qquad w_3 = 3^{3/2}+3i, \qquad w_4 = -\sqrt{7}+\sqrt{21}i.$$

4. Determinare le soluzioni complesse dell'equazione $\overline{z}^2 + z = 0$.

5. Determinare e rappresentare graficamente (identificando $\mathbb{C}$ con $\mathbb{R}^2$) il luogo geometrico degli $z \in \mathbb{C}$ tali che $\mathrm{Re}(z)\cdot(|z+1-i|-3) \geqslant 0$.

6. Determinare e rappresentare graficamente (identificando $\mathbb{C}$ con $\mathbb{R}^2$) il luogo geometrico degli $z \in \mathbb{C}$ tali che $\mathrm{Im}\left(\frac{2}{z}\right) = 3$.

7. (Tema d'esame 7 settembre 2016) Determinare il luogo geometrico A dei punti $z \in \mathbb{C}$ tali che

$$\begin{cases} |z+2|^2 + \mathrm{Re}((z-i)^2) = 3 \\ \mathrm{Im}(z) \geqslant 0. \end{cases}$$

8. Determinare e rappresentare graficamente (identificando $\mathbb{C}$ con $\mathbb{R}^2$) il luogo geometrico degli $z \in \mathbb{C}$ tali che $\mathrm{Re}\left(\frac{z+1+\overline{z}}{z+i}\right) = -1$.

Scrivere in forma algebrica i numeri complessi

$$w_1 = \frac{7i}{2-i}, \qquad w_2 = \frac{(3-i)\left(\overline{-\frac{2}{3}i-1}\right)}{1+3i}, \qquad w_3 = i^{2023}(1-i)^4.$$

Individuare la parte reale e la parte immaginaria, e calcolare il modulo di ciascuno.

$$w_1 = \frac{7i}{2-i} \cdot \frac{2+i}{2+i} = \frac{14i+7i^2}{2^2-i^2} =$$

$$\boxed{z \cdot \overline{z} = |z|^2 \ \forall z \in \mathbb{C}} \qquad = \frac{-7+14i}{5} = -\frac{7}{5} + \frac{14}{5}i$$

$$\mathrm{Re}(w_1) = -\frac{7}{5} \qquad \mathrm{Im}(w_1) = \frac{14}{5}$$

$$|w_1| = \sqrt{(\mathrm{Re}\, w_1)^2 + (\mathrm{Im}\, w_1)^2} =$$

$$= \sqrt{\left(-\frac{7}{5}\right)^2 + \left(\frac{14}{5}\right)^2} = \sqrt{\left(\frac{7}{5}\right)^2 (1+2^2)} = \frac{7}{5}\sqrt{5}$$

$$w_2 = \frac{(3-i)\left(\overline{-\frac{2}{3}i-1}\right)}{1+3i} = \frac{(3-i)\left(-1+\frac{2}{3}i\right)}{1+3i} =$$

$$= \frac{-3+i+2i-\frac{2}{3}i^2}{1+3i} = \frac{-\frac{7}{3}+3i}{1+3i} \cdot \frac{1-3i}{1-3i} =$$

$$= \frac{\left(-\frac{7}{3}+3i\right)(1-3i)}{1^2+3^2} = \frac{-\frac{7}{3}+7i+3i+9}{10} = \frac{\frac{20}{3}+10i}{10} = \frac{2}{3}+i$$

$$\mathrm{Re}(w_2) = \frac{2}{3} \qquad \mathrm{Im}(w_2) = 1$$

$$|w_2| = \sqrt{\left(\frac{2}{3}\right)^2 + 1^2} = \sqrt{\frac{4}{9}+1} = \frac{\sqrt{13}}{3}$$

$$w_3 = i^{2023}(1-i)^4$$

<u>Oss.</u> $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = i^2 \cdot i = -i$,
$i^4 = i^3 \cdot i = -i \cdot i = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$,
$i^8 = 1, \dots$

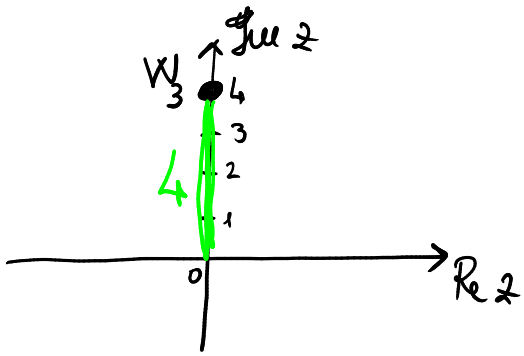
$$i^{2023} = i^{2020} \cdot i^3 = 1 \cdot i^3 = -i$$

$$(1-i)^4 = \left((1-i)^2\right)^2 = \left(1^2 + (-i)^2 + 2 \cdot 1 \cdot (-i)\right)^2 =$$
$$= (1-1-2i)^2 = (-2i)^2 = 4i^2 = -4$$

$$w_3 = i^{2023}(1-i)^4 = -i \cdot (-4) = 4i$$

$\mathrm{Re}(w_3) = 0$ e $\mathrm{Im}(w_3) = 4$

$$|w_3| = \sqrt{0^2 + 4^2} = 4$$



$$z = x + iy =$$
$$= \rho\cos\vartheta + i\rho\sin\vartheta =$$
$$= \rho(\cos\vartheta + i\sin\vartheta)$$

$$x = \rho\cos\vartheta$$
$$y = \rho\sin\vartheta$$

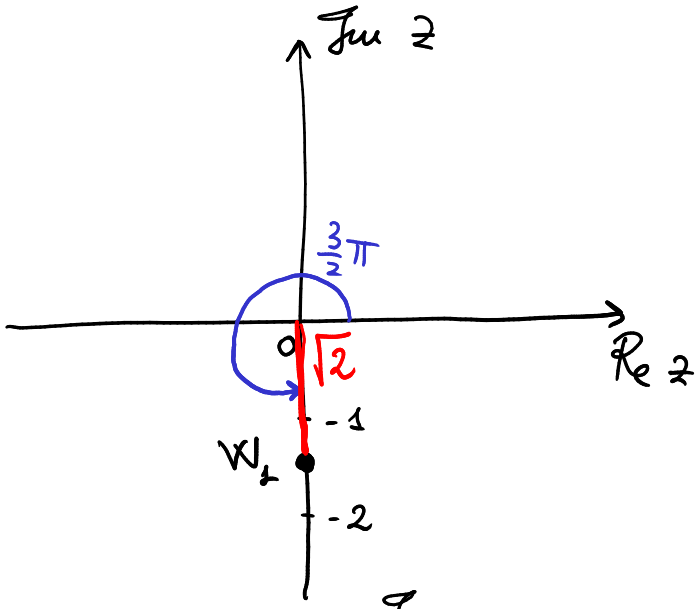
$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases}$$

x parte reale
y parte immaginaria

Scrivere in forma algebrica i seguenti numeri complessi.

$$w_1 = \underbrace{\sqrt{2}}_{\text{modulo}}\left(\cos \underbrace{\boxed{\frac{3}{2}\pi}}_{\text{argomento}} + i \sin \boxed{\frac{3}{2}\pi}\right) =$$

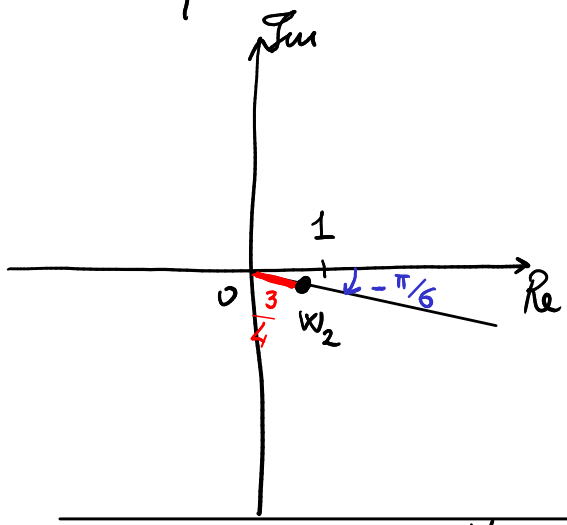
$$= \sqrt{2}\left(0 + i\cdot(-1)\right) = -\sqrt{2}\, i$$



$$w_2 = \frac{3}{4}\left(\cos \boxed{\left(-\frac{\pi}{6}\right)} + i \sin \boxed{\left(-\frac{\pi}{6}\right)}\right) =$$

$$= \frac{3}{4}\left(\cos\frac{\pi}{6} - i \sin\frac{\pi}{6}\right) =$$

$$= \frac{3}{4}\left(\frac{\sqrt{3}}{2} - i\cdot\frac{1}{2}\right) = \frac{3\sqrt{3}}{8} - \frac{3}{8}i$$



$$\boxed{\begin{array}{l} \sin(-\alpha) = -\sin\alpha \quad \forall \alpha \in \mathbb{R} \\ \cos(-\alpha) = \cos\alpha \quad \forall \alpha \in \mathbb{R} \end{array}}$$

$$w_3 = \boxed{\frac{1}{2}}\left(\cos\left(\frac{13}{3}\pi\right) - i \sin\left(\frac{13}{3}\pi\right)\right)$$

$\rho = \frac{1}{2}$ $\qquad$ $\vartheta = -\frac{13}{3}\pi$

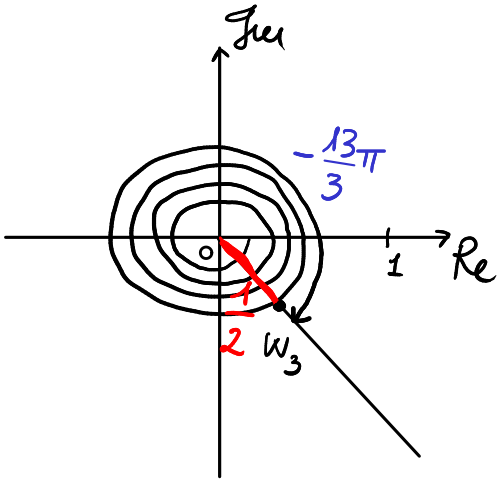
Forma trigonometrica di w_3

$$\frac{1}{2}\left(\cos\left(-\frac{13}{3}\pi\right) + i \sin\left(-\frac{13}{3}\pi\right)\right)$$

$$w_3 = \frac{1}{2}\left(\cos\left(4\pi + \frac{\pi}{3}\right) - i \sin\left(4\pi + \frac{\pi}{3}\right)\right) =$$

$$= \frac{1}{2}\left(\cos\frac{\pi}{3} - i \sin\frac{\pi}{3}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \frac{1}{4} - \frac{\sqrt{3}}{4}i$$



Scrivere in forma trigonometrica i numeri complessi

$$w_1 = \frac{5}{6i}, \qquad w_2 = 5 - 5i, \qquad w_3 = 3^{3/2} + 3i, \qquad w_4 = -\sqrt{7} + \sqrt{21}i.$$

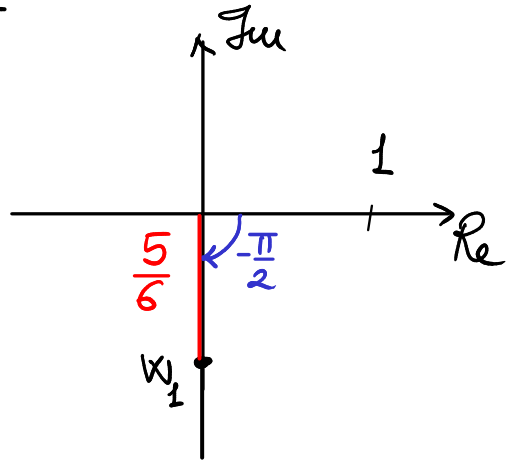
$$w_1 = \frac{5}{6i} \cdot \frac{i}{i} = \frac{5i}{-6} = -\frac{5}{6}i$$

$$\mathrm{Re}(w_1) = 0 \qquad \mathrm{Im}(w_1) = -\frac{5}{6} \qquad \rho_1 = |w_1| = \frac{5}{6}$$

$$\begin{cases} 0 = \frac{5}{6}\cos\vartheta_1 \\ -\frac{5}{6} = \frac{5}{6}\sin\vartheta_1 \end{cases} \iff \begin{cases} \cos\vartheta_1 = 0 \\ \sin\vartheta_1 = -1 \end{cases}$$

$\vartheta_1 = -\frac{\pi}{2}$ va bene

$$w_1 = \frac{5}{6}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$



$$w_2 = 5 - 5i$$

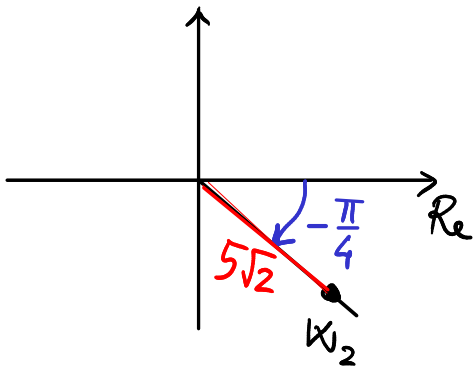
$$\mathrm{Re}(w_2) = 5, \qquad \mathrm{Im}(w_2) = -5$$

$$|w_2| = \sqrt{5^2 + (-5)^2} = 5\sqrt{2} = \rho_2$$

$$\begin{cases} 5 = 5\sqrt{2}\cdot\cos\vartheta_1 \\ -5 = 5\sqrt{2}\cdot\sin\vartheta_1 \end{cases} \iff \begin{cases} \cos\vartheta_1 = \frac{1}{\sqrt{2}} \\ \sin\vartheta_1 = -\frac{1}{\sqrt{2}} \end{cases} \iff \begin{cases} \cos\vartheta_1 = \frac{\sqrt{2}}{2} \\ \sin\vartheta_1 = -\frac{\sqrt{2}}{2} \end{cases}$$

$\vartheta_1 = -\frac{\pi}{4}$ va bene

$$w_2 = 5\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$



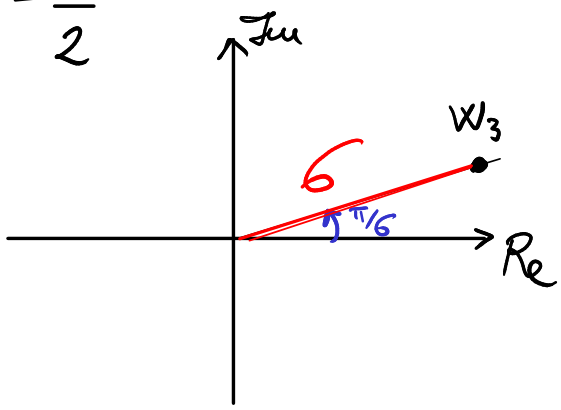
$$w_3 = 3^{3/2} + 3i$$

$$\mathrm{Re}(w_3) = 3^{3/2} = 3\sqrt{3}, \qquad \mathrm{Im}(w_3) = 3$$

$$|w_3| = \sqrt{\left(3^{3/2}\right)^2 + 3^2} = \sqrt{27+9} = 6 = \rho_3$$

$$\begin{cases} 3\sqrt{3} = 6\cos\vartheta_3 \\ 3 = 6\sin\vartheta_3 \end{cases} \iff \begin{cases} \cos\vartheta_3 = \dfrac{\sqrt{3}}{2} \\ \sin\vartheta_3 = \dfrac{1}{2} \end{cases} \qquad \vartheta_3 = \frac{\pi}{6} \text{ va bene}$$

$$w_3 = 6\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

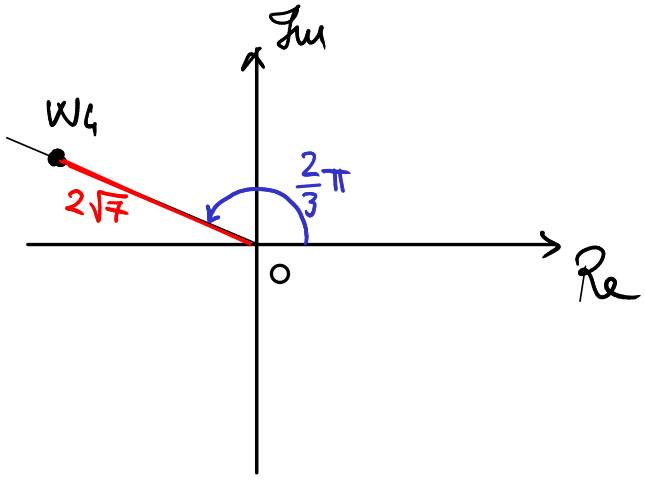


$$w_4 = -\sqrt{7} + \sqrt{21}\,i$$

$$\mathrm{Re}(w_4) = -\sqrt{7}, \qquad \mathrm{Im}(w_4) = \sqrt{21}$$

$$|w_4| = \sqrt{7+21} = \sqrt{28} = 2\sqrt{7} = \rho_4$$

$$\begin{cases} -\sqrt{7} = 2\sqrt{7}\cdot\cos\vartheta_4 \\ \sqrt{21} = 2\sqrt{7}\cdot\sin\vartheta_4 \end{cases} \iff \begin{cases} \cos\vartheta_4 = -\dfrac{1}{2} \\ \sin\vartheta_4 = \dfrac{\sqrt{3}}{2} \end{cases} \qquad \vartheta_4 = \frac{2}{3}\pi \text{ va bene}$$



Determinare le soluzioni complesse dell'equazione $\overline{z}^2 + z = 0$.

$$\overline{z}^2 + z = 0$$

$z = x + iy$, con $x, y \in \mathbb{R}$

$$(x - iy)^2 + (x + iy) = 0$$

$$x^2 + i^2 y^2 - 2ixy + x + iy = 0$$

$$\underbrace{(x^2 - y^2 + x)}_{\text{parte reale}} + i\underbrace{(-2xy + y)}_{\text{parte immaginaria}} = 0$$

$$\begin{cases} x^2 - y^2 + x = 0 \\ -2xy + y = 0 \end{cases} \iff \begin{cases} x^2 - y^2 + x = 0 \\ y(1 - 2x) = 0 \end{cases} \iff$$

$$\begin{cases} x^2 - y^2 + x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x^2 - y^2 + x = 0 \\ 1 - 2x = 0 \end{cases} \iff$$

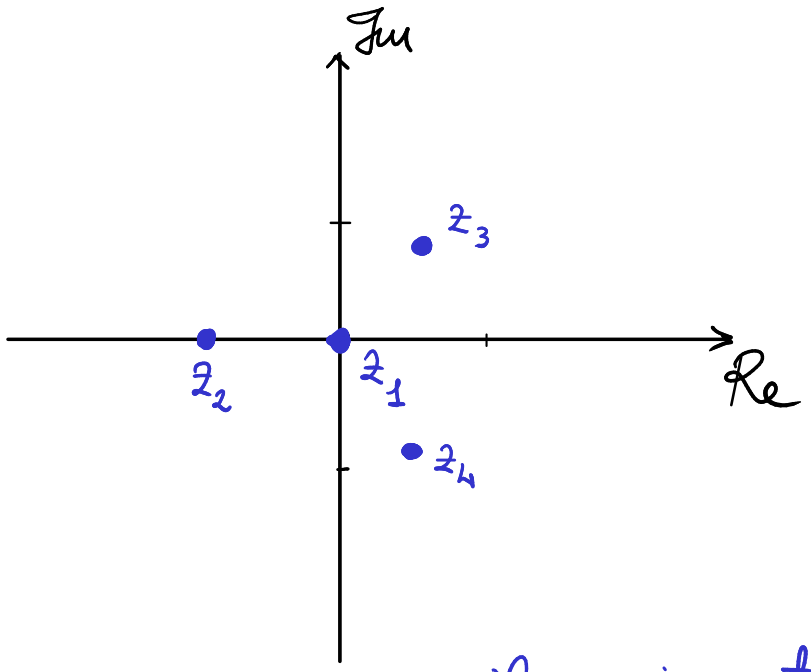
$$\begin{cases} x^2 + x = 0 \\ y = 0 \end{cases} \vee \begin{cases} \frac{1}{4} - y^2 + \frac{1}{2} = 0 \\ x = \frac{1}{2} \end{cases} \iff$$

$$\begin{cases} x(x+1) = 0 \\ y = 0 \end{cases} \vee \begin{cases} y^2 = \frac{3}{4} \\ x = \frac{1}{2} \end{cases}$$

$$\begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=-1 \\ y=0 \end{cases} \vee \begin{cases} x=\frac{1}{2} \\ y=+\frac{\sqrt{3}}{2} \end{cases} \vee \begin{cases} x=\frac{1}{2} \\ y=-\frac{\sqrt{3}}{2} \end{cases}$$

L'equazione data ha in $\mathbb{C}$ come soluzioni

$$z_1=0, \quad z_2=-1, \quad z_3=\frac{1}{2}+\frac{\sqrt{3}}{2}i, \quad z_4=\frac{1}{2}-\frac{\sqrt{3}}{2}i$$



<u>Oss.</u> z_3 e z_4 sono uno il coniugato dell'altro; geometricamente sono simmetrici rispetto all'asse reale

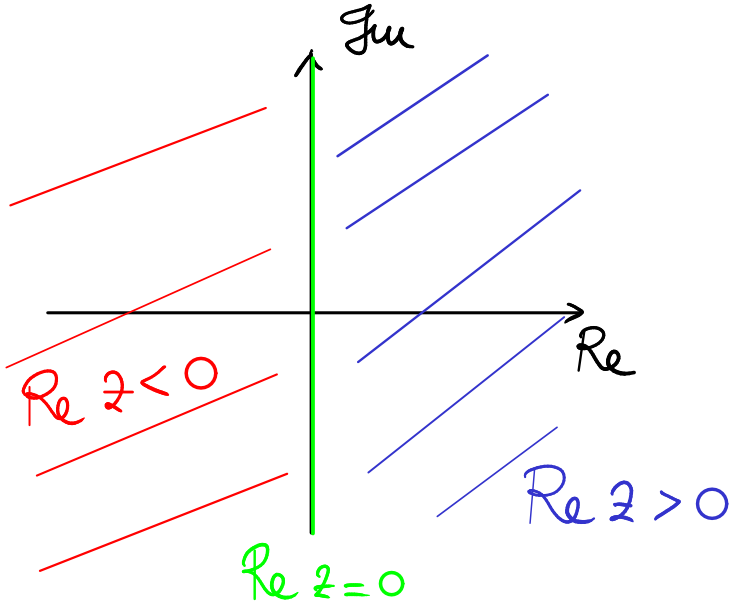
$$z_3 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$z_4 = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$$

Determinare e rappresentare graficamente (identificando $\mathbb{C}$ con $\mathbb{R}^2$) il luogo geometrico degli $z \in \mathbb{C}$ tali che $\mathrm{Re}(z) \cdot (|z+1-i|-3) \geqslant 0$.

$$\mathrm{Re}(z) \cdot \left(|z+1-i| - 3 \right) \geq 0$$

$$\begin{cases} \mathrm{Re}(z) \leq 0 \\ |z+1-i| - 3 \leq 0 \end{cases} \quad \vee \quad \begin{cases} \mathrm{Re}(z) \geq 0 \\ |z+1-i| - 3 \geq 0 \end{cases}$$

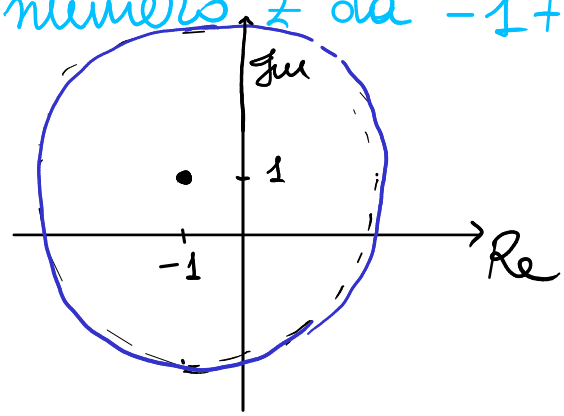


$$|z+1-i| - 3 = 0$$

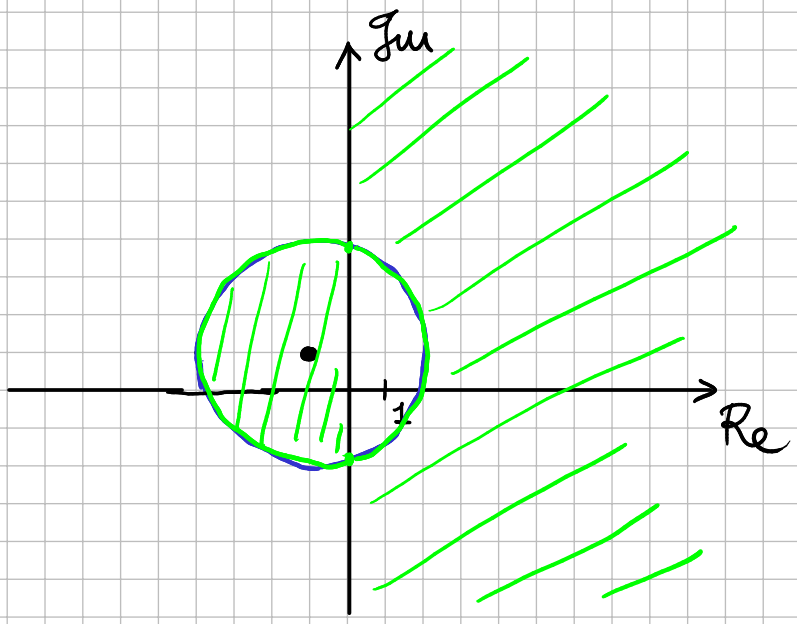
$$|z+1-i| = 3$$

$$\underbrace{|z-(-1+i)|}_{\text{Distanza del numero } z \text{ da } -1+i} = 3$$

Circonferenza con centro $-1+i$ e raggio 3



$$\begin{cases} \mathrm{Re}(z) \leq 0 \\ |z+1-i| \leq 3 \end{cases} \quad \vee \quad \begin{cases} \mathrm{Re}(z) \geq 0 \\ |z+1-i| \geq 3 \end{cases}$$



Determinare e rappresentare graficamente (identificando $\mathbb{C}$ con $\mathbb{R}^2$) il luogo geometrico degli $z \in \mathbb{C}$ tali che $\operatorname{Im}\left(\frac{2}{z}\right) = 3$.

$$\operatorname{Im}\left(\frac{2}{z}\right) = 3$$

C.E.
$z \neq 0$

$z = x + iy$, con $x, y \in \mathbb{R}$

$$\operatorname{Im}\left(\frac{2}{x+iy}\right) = 3$$

$$\operatorname{Im}\left(\frac{2}{x+iy} \cdot \frac{x-iy}{x-iy}\right) = 3$$

$$\operatorname{Im}\left(\frac{2x-2iy}{x^2+y^2}\right) = 3$$

$$\operatorname{Im}\left(\underbrace{\frac{2x}{x^2+y^2}}_{\text{parte reale}} + i \underbrace{\boxed{\frac{-2y}{x^2+y^2}}}_{\text{parte immaginaria}}\right) = 3$$

$$\frac{-2y}{x^2+y^2} = 3$$

$$-2y = 3(x^2+y^2)$$

$$x^2+y^2 = -\frac{2}{3}y$$

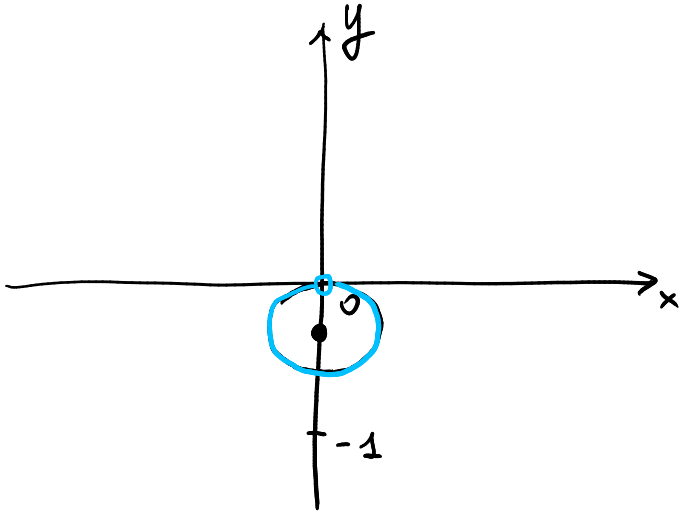
$$x^2+y^2+\frac{2}{3}y = 0$$

Circonferenza in $\mathbb{R}^2$

$$x^2+y^2+ax+by+c=0, \quad \text{con } \left(-\frac{a}{2}\right)^2+\left(-\frac{b}{2}\right)^2-c\geq 0$$

Centro $\left(-\frac{a}{2},-\frac{b}{2}\right)$ Raggio $r=\sqrt{\left(-\frac{a}{2}\right)^2+\left(-\frac{b}{2}\right)^2-c}$

$x^2+y^2+\frac{2}{3}y=0$ è l'equazione della circonferenza di centro $\left(0,-\frac{1}{3}\right)$ e raggio $r=\sqrt{0^2+\left(-\frac{1}{3}\right)^2-0}=\frac{1}{3}$



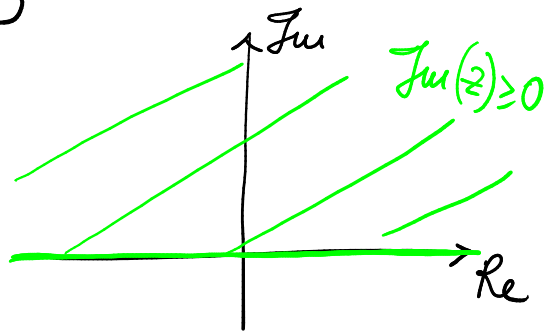
Il luogo geometrico cercato è la circonferenza di centro $\left(0,-\frac{1}{3}\right)$ e raggio $\frac{1}{3}$, privata del punto $(0,0)$

vedi C.E.

(Tema d'esame 7 settembre 2016) Determinare il luogo geometrico A dei punti $z \in \mathbb{C}$ tali che

$$\begin{cases} |z+2|^2 + \mathrm{Re}((z-i)^2) = 3 \\ \mathrm{Im}(z) \geqslant 0. \end{cases}$$

$$\begin{cases} |z+2|^2 + \mathrm{Re}\left((z-i)^2\right) = 3 \\ \mathrm{Im}(z) \geq 0 \end{cases}$$



$z = x + iy$, con $x, y \in \mathbb{R}$

$$|x+iy+2|^2 + \mathrm{Re}\left((x+iy-i)^2\right) = 3$$

$$|(x+2)+iy|^2 + \mathrm{Re}\left((x+i(y-1))^2\right) = 3$$

$$(x+2)^2 + y^2 + \mathrm{Re}\left(x^2 - (y-1)^2 + 2ix(y-1)\right) = 3$$

$$\boxed{\begin{array}{l} z = x+iy, \text{ con } x, y \in \mathbb{R} \\ |z| = \sqrt{x^2+y^2} \end{array}}$$

$$(x+2)^2 + y^2 + x^2 - (y-1)^2 = 3$$

$$x^2 + 4x + \cancel{4} + \cancel{y^2} + x^2 - \cancel{y^2} + 2y - \cancel{1} = \cancel{3}$$

$$2x^2 + 4x + 2y = 0$$

$$y = -x^2 - 2x$$

Parabola in $\mathbb{R}^2$ con asse di simmetria parallelo all'asse y

$$y = ax^2 + bx + c \quad \text{con } a, b, c \in \mathbb{R} \text{ e } a \neq 0$$

Vertice ha ascissa $-\dfrac{b}{2a}$

$$y = -x^2 - 2x$$

$$x^2 + 2x = 0$$

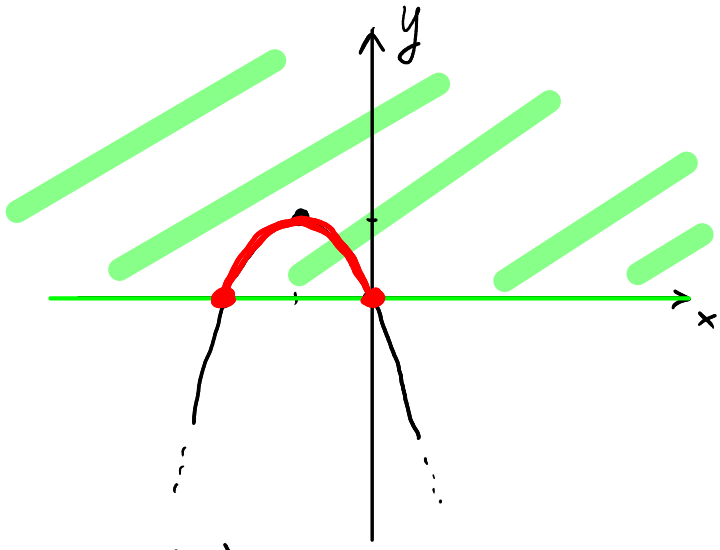
$$x(x+2) = 0$$

$$x = 0 \vee x = -2$$

Ordinata del vertice

$$-(-1)^2 - 2(-1) = 1$$

Vertice $V(-1, 1)$



$$\mathrm{Im}(z) \geq 0 \iff y \geq 0$$

Il luogo geometrico cercato è l'arco della parabola di equazione $y = -x^2 - 2x$ contenuto nel semipiano $y \geq 0$, estremi dell'arco inclusi.

Determinare e rappresentare graficamente (identificando $\mathbb{C}$ con $\mathbb{R}^2$) il luogo geometrico degli $z \in \mathbb{C}$ tali che $\mathrm{Re}\left(\frac{z+1+\overline{z}}{z+i}\right) = -1$.

C.E.
$z \neq -i$

$$\mathrm{Re}\left(\frac{z+1+\overline{z}}{z+i}\right) = -1$$

$z = x + iy$ con $x, y \in \mathbb{R}$

$z + \overline{z} = x + \cancel{iy} + x - \cancel{iy} = 2x$

$z - \overline{z} = \cancel{x} + iy - \cancel{x} + iy = 2iy$

$$\boxed{\begin{aligned} z + \overline{z} &= 2\,\mathrm{Re}(z) \ \forall z \in \mathbb{C} \\ z - \overline{z} &= 2i\,\mathrm{Im}(z) \ \forall z \in \mathbb{C} \end{aligned}}$$

$$\frac{z+1+\overline{z}}{z+i} = \frac{2x+1}{x+i(y+1)} \cdot \frac{x-i(y+1)}{x-i(y+1)} =$$

$$= \frac{2x^2 - 2ix(y+1) + x - i(y+1)}{x^2 + (y+1)^2} =$$

$$= \frac{2x^2 + x}{x^2 + (y+1)^2} + i\,\frac{-2x(y+1) - (y+1)}{x^2 + (y+1)^2}$$

$$\mathrm{Re}\left(\frac{z+1+\overline{z}}{z+i}\right) = -1$$

$$\frac{2x^2 + x}{x^2 + (y+1)^2} = -1$$

$$2x^2 + x = -x^2 - (y+1)^2$$

$$3x^2 + x + (y+1)^2 = 0$$

$$3\left(x^2+\frac{1}{3}x\right)+(y+1)^2=0$$

$$3\left(x+\frac{1}{6}\right)^2+(y+1)^2=\frac{1}{12}$$

$$36\left(x+\frac{1}{6}\right)^2+12(y+1)^2=1$$

$$\frac{\left(x+\frac{1}{6}\right)^2}{\frac{1}{36}}+\frac{(y+1)^2}{\frac{1}{12}}=1$$

Ellisse in $\mathbb{R}^2$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

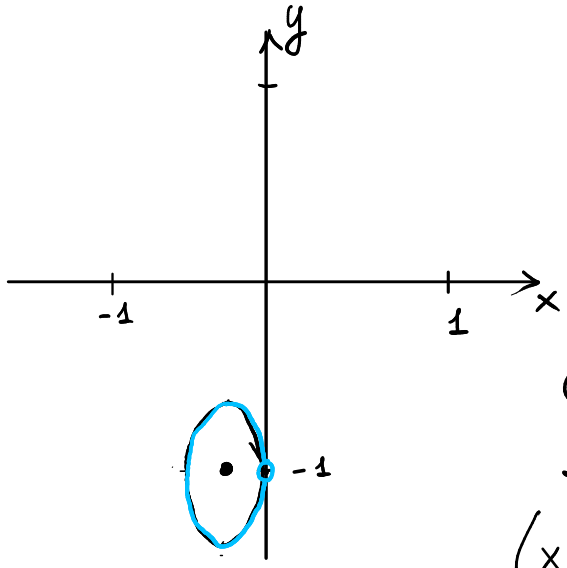
$(a,b>0)$

Centro in $(0,0)$

$$\frac{x^2}{\frac{1}{36}}+\frac{y^2}{\frac{1}{12}}=1$$

$$a=\sqrt{\frac{1}{36}}=\frac{1}{6}$$

$$b=\sqrt{\frac{1}{12}}=\frac{1}{2\sqrt{3}}=\frac{\sqrt{3}}{6}$$



C.E.

$z\neq -i$

$(x,y)\neq(0,-1)$

Il luogo geometrico cercato è l'ellisse di equazione $\frac{\left(x+\frac{1}{6}\right)^2}{1/36}+\frac{(y+1)^2}{1/12}=1$, privata del punto $(0,-1)$.