Esercitazione del 25 settembre 2023

In ciascuno dei seguenti esercizi è data una funzione $f\colon \operatorname{dom}(f) \subseteq \mathbb{R} \to \mathbb{R}$.

- Tracciare il grafico di f tramite opportune trasformazioni applicate ai grafici delle funzioni elementari.
- Individuare il dominio e l'insieme immagine di f.

① $f(x) = (1-x)^{\sqrt{3}}$

② $f(x) = (|x|+1)^{-\sqrt{2}}$

③ $f(x) = e^{|x|} + 1$

④ $f(x) = \log(x-3)$

⑤ $f(x) = \log|x+1|$

In ciascuno dei seguenti esercizi è data una funzione $f\colon \operatorname{dom}(f) \subseteq \mathbb{R} \to \mathbb{R}$.

- Determinare il dominio di f.
- Determinare l'estremo superiore e l'estremo inferiore di $\operatorname{dom}(f)$. Stabilire anche se tale insieme ammette massimo o minimo.

① $f(x) = \sqrt{e^{x+1} - e^x} + \dfrac{1}{\sqrt{e^{2x+1} - e^x}} - \sqrt{\dfrac{(4x^2-20x+25)(4x^2-x-3)}{[(x^3+1)^4+\sqrt{7}](3x-1-4x^2)}}$

② $f(x) = \begin{cases} \sqrt{7-x^2} & \text{se } 0 \leqslant x < 2 \\ \dfrac{1}{\sqrt{x+3}+\sqrt{5}-2} & \text{se } -3 \leqslant x < 0 \\ \sqrt{\dfrac{|3-2x|+1}{|x^2-3|-2}} & \text{se } x < -3 \end{cases}$

③ $f(x) = \dfrac{x^4}{[x]} - \dfrac{\sqrt[3]{x}}{2}(\sqrt{x-1}+x-3)^{-\frac{1}{4}}$

④ $f(x) = \sqrt{e^{2x} - 4e^x + 3} \cdot (1-\log^2 x)^{-\frac{1}{2}}$

25 settembre 2023

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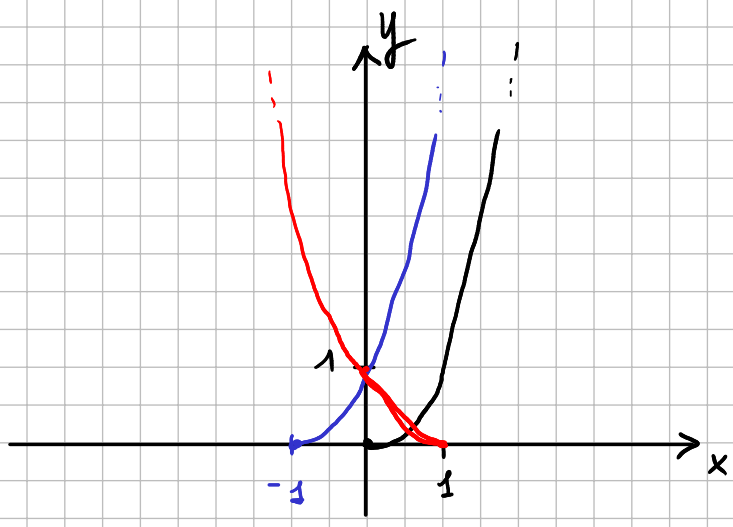
$$f(x) = (1-x)^{\sqrt{3}}$$



$x \longmapsto x^{\sqrt{3}}$

$x \longmapsto (x+1)^{\sqrt{3}}$

traslazione di vettore $(-1,0)$

$x \longmapsto (-x+1)^{\sqrt{3}}$

simmetria rispetto all'asse y

$x \longmapsto x^{\sqrt{3}}$

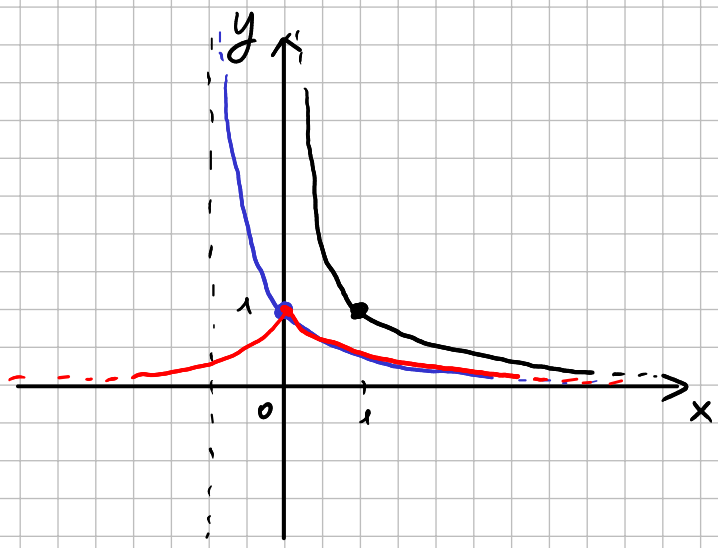
$x \longmapsto (-x)^{\sqrt{3}}$

$x \longmapsto (-(x-1))^{\sqrt{3}}$

$\mathrm{dom}(f) = (-\infty, 1]$

$\mathrm{im}(f) = [0, +\infty)$

$$f(x) = (|x|+1)^{-\sqrt{2}}$$



$x \longmapsto x^{-\sqrt{2}}$

$x \longmapsto (x+1)^{-\sqrt{2}}$

trasl. di vettore $(-1,0)$

$x \longmapsto (|x|+1)^{-\sqrt{2}}$

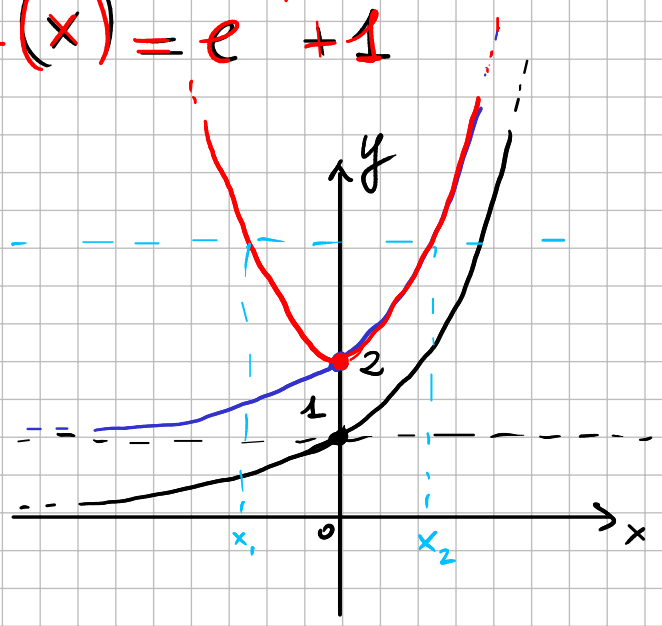
Data una funzione g, si ha

$$g(|x|) = \begin{cases} g(x) & \text{se } x \geq 0 \\ g(-x) & \text{se } x < 0 \end{cases}$$

$\mathrm{dom}(f) = \mathbb{R}$

$\mathrm{im}(f) = (0, 1]$

$$f(x) = e^{|x|} + 1$$



$x \longmapsto e^x$

$x \longmapsto e^x + 1$

traslazione di vettore $(0,1)$

$x \longmapsto e^{|x|} + 1$

$\mathrm{dom}(f) = \mathbb{R}$ $\qquad$ $\mathrm{im}(f) = [2, +\infty)$

Ogni elemento di $(2,+\infty)$ ha esattamente 2 controimmagini, mentre 2 ha un'unica controimmagine

La funzione $f : \mathbb{R} \longrightarrow [2,+\infty)$ non è iniettiva

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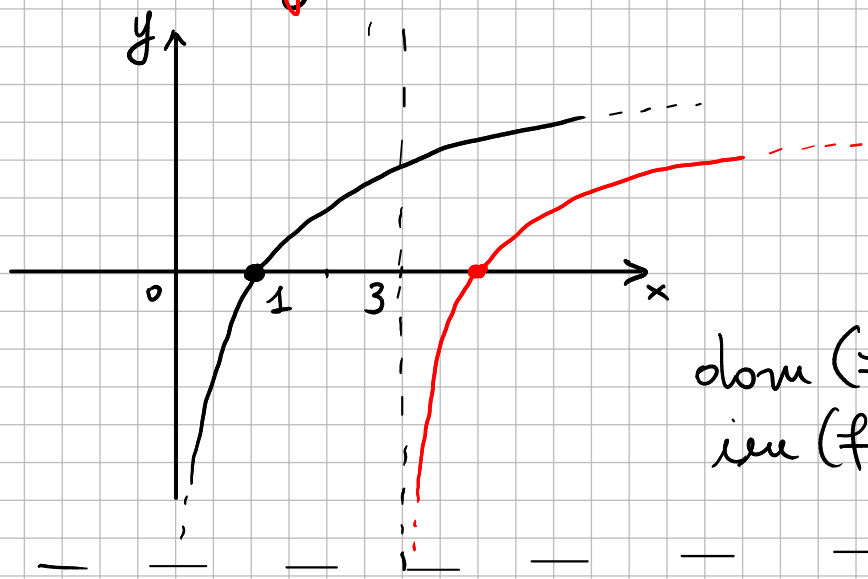
NOTAZIONE

$\log(\cdot)$ LOGARITMO NATURALE

$$f(x) = \log(x-3)$$



$x \longmapsto \log x$

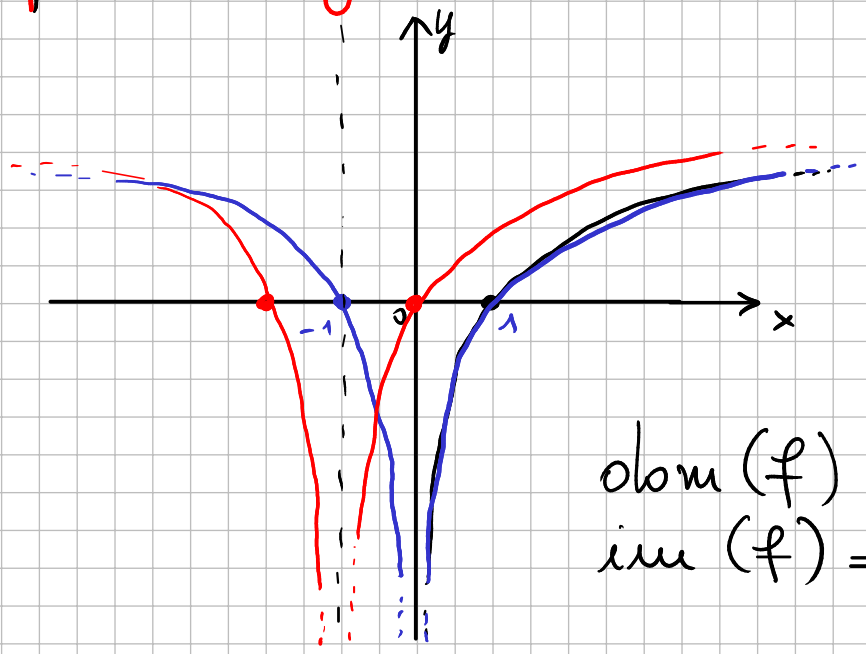
$x \longmapsto \log(x-3)$

trasl. di vettore $(3,0)$

$\mathrm{dom}(f) = (3, +\infty)$

$\mathrm{im}(f) = \mathbb{R}$

$f(x) = \log|x+1|$



$x \longmapsto \log x$

$x \longmapsto \log|x|$

$x \longmapsto \log|x+1|$

traslazione di vett. $(-1,0)$

$\mathrm{dom}(f) = \mathbb{R} \setminus \{-1\}$

$\mathrm{im}(f) = \mathbb{R}$

$$f(x) = \sqrt{e^{x+1} - e^x} + \frac{1}{\sqrt{e^{2x+1} - e^x}} - \sqrt{\frac{(4x^2 - 20x + 25)(4x^2 - x - 3)}{[(x^3+1)^4 + \sqrt{7}](3x - 1 - 4x^2)}}$$

Determinare il dominio di f

$$\begin{cases} e^{x+1} - e^x \geq 0 \\ e^{2x+1} - e^x > 0 \\ \dfrac{(4x^2-20x+25)(4x^2-x-3)}{[(x^3+1)^4+\sqrt{7}](3x-1-4x^2)} \geq 0 \\ [(x^3+1)^4+\sqrt{7}](3x-1-4x^2) \neq 0 \end{cases} \iff \begin{cases} x > -1 \\ -\frac{3}{4} \leq x \leq 1 \ \vee \ x = \frac{5}{2} \end{cases}$$

$$e^{x+1} - e^x \geq 0$$

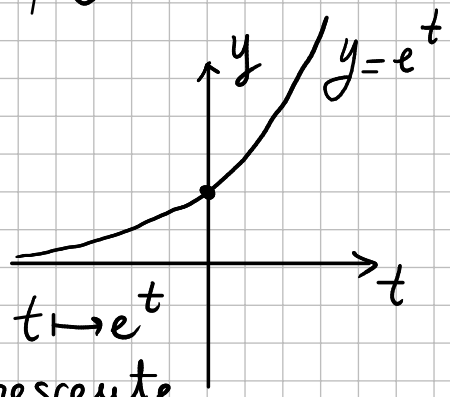
$$e^{x+1} \geq e^x$$

$$x + 1 \geq x$$

$$1 \geq 0$$

$$\forall x \in \mathbb{R}$$

perchè $t \mapsto e^t$ è str. crescente



$$e^{2x+1} - e^x > 0$$

$$e^{2x+1} > e^x$$

$$2x + 1 > x$$

$$x > -1$$

$$\frac{(4x^2-20x+25)(4x^2-x-3)}{[(x^3+1)^4+\sqrt{7}](3x-1-4x^2)} \geq 0$$

$$\frac{(2x-5)^2(4x^2-x-3)}{[(x^3+1)^4+\sqrt{7}](3x-1-4x^2)} \geq 0$$

$(2x-5)^2 \geq 0 \quad \forall x \in \mathbb{R}$ $\qquad$ $(2x-5)^2 = 0 \iff x = \frac{5}{2}$

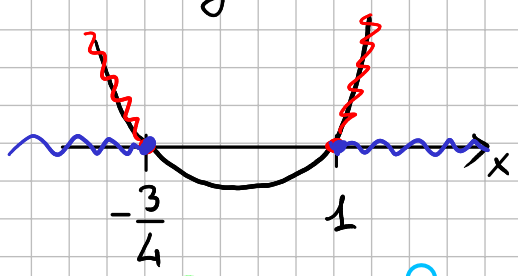
$4x^2 - x - 3 \geq 0$

$4x^2 - x - 3 = 0$

$(x-1)(4x+3) = 0$

$x = 1 \vee x = -\frac{3}{4}$

$y = 4x^2 - x - 3$



$x \leq -\frac{3}{4} \vee x \geq 1$

$\underbrace{(x^3+1)^4}_{\geq 0 \ \forall x \in \mathbb{R}} + \underbrace{\sqrt{7}}_{>0} > 0 \quad \forall x \in \mathbb{R}$

$3x - 1 - 4x^2 > 0$

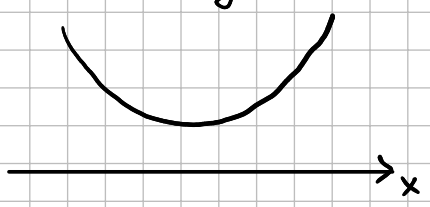
$4x^2 - 3x + 1 < 0$

$4x^2 - 3x + 1 = 0$

$\Delta = (-3)^2 - 4 \cdot 4 \cdot 1 < 0$

L'equaz. non ha soluzioni reali

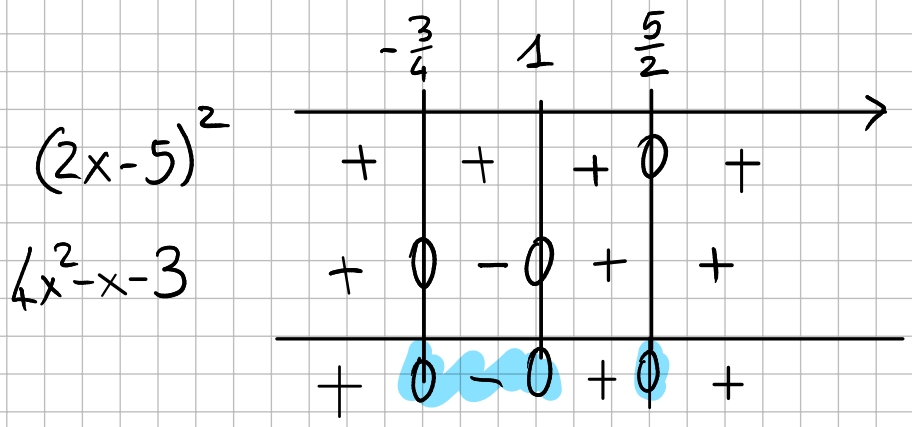
$y = 4x^2 - 3x + 1$



$\nexists x \in \mathbb{R} \mid 4x^2 - 3x + 1 < 0$

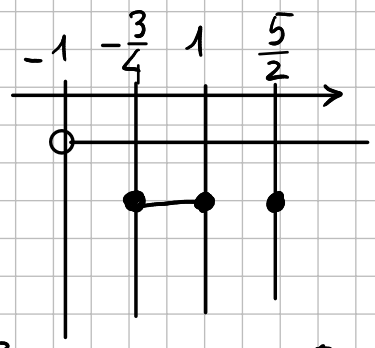
<u>Oss</u>. La disequazione $\dfrac{(2x-5)^2(4x^2-x-3)}{[(x^3+1)^4+\sqrt{7}](3x-1-4x^2)} \geq 0$

è equivalente a $(2x-5)^2(4x^2-x-3) \leq 0$



$$(2x-5)^2(4x^2-x-3) \leq 0 \iff -\frac{3}{4} \leq x \leq 1 \vee x = \frac{5}{2}$$

$$\begin{cases} x > -1 \\ -\frac{3}{4} \leq x \leq 1 \vee x = \frac{5}{2} \end{cases}$$



$$-\frac{3}{4} \leq x \leq 1 \vee x = \frac{5}{2}$$

$$\text{dom}(f) = \left[-\frac{3}{4}, 1\right] \cup \left\{\frac{5}{2}\right\}$$

Maggioranti di dom(f) $\quad x \geq \frac{5}{2}$

$\sup(\text{dom}(f)) = \frac{5}{2}$ $\qquad$ $\max(\text{dom}(f)) = \frac{5}{2}$

perchè $\frac{5}{2} \in \text{dom}(f)$ e $\frac{5}{2}$ è sup

Minoranti di dom(f) $\quad x \leq -\frac{3}{4}$

$\inf(\text{dom}(f)) = -\frac{3}{4}$ $\qquad$ $\min(\text{dom}(f)) = -\frac{3}{4}$

$$f(x) = \begin{cases} \sqrt{7-x^2} & \text{se } 0 \leqslant x < 2 \\ \dfrac{1}{\sqrt{x+3}+\sqrt{5}-2} & \text{se } -3 \leqslant x < 0 \\ \sqrt{\dfrac{|3-2x|+1}{|x^2-3|-2}} & \text{se } x < -3 \end{cases}$$

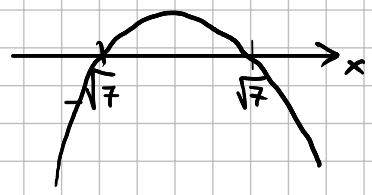
$\sqrt{7-x^2}$ è definita se e solo se $7-x^2 \geq 0$

Equaz. associata $\quad 7-x^2=0$

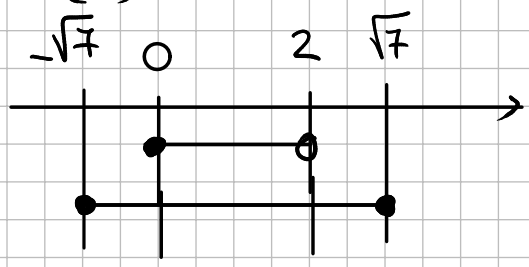
$x^2=7$

$x=-\sqrt{7} \vee x=\sqrt{7}$

$y=7-x^2$



$7-x^2 \geq 0 \iff -\sqrt{7} \leq x \leq \sqrt{7}$



Se $x \in [0,2)$ allora è anche vero che $x \in [-\sqrt{7}, \sqrt{7}]$, quindi $f(x)$ è definita se $0 \leq x < 2$

$\dfrac{1}{\sqrt{x+3}+\sqrt{5}-2}$ è definita se e solo se

$$\begin{cases} x+3 \geq 0 \\ \underbrace{\sqrt{x+3}}_{\geq 0} + \underbrace{\sqrt{5}-2}_{>0} \neq 0 \end{cases} \iff x \geq -3$$

quando è definita

perchè $\sqrt{x+3}+\sqrt{5}-2>0$ se la radice quadrata è definita

$f(x)$ è definita per $-3 \leq x < 0$

$$\sqrt{\frac{|3-2x|+1}{|x^2-3|-2}}$$ è definita se e solo se

$$\begin{cases} \dfrac{|3-2x|+1}{|x^2-3|-2} \geq 0 \\[2ex] |x^2-3|-2 \neq 0 \end{cases}$$

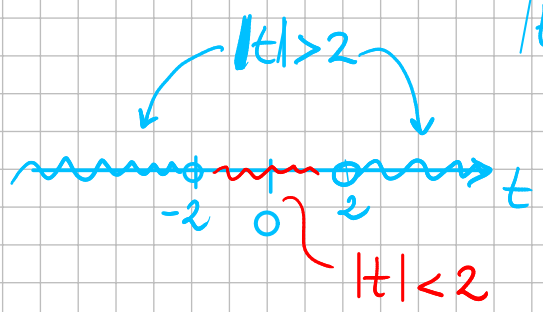
$$\frac{|3-2x|+1}{|x^2-3|-2} \geq 0$$

$$\underbrace{|3-2x|}_{\geq 0 \ \forall x \in \mathbb{R}} + \overbrace{1}^{>0} > 0 \quad \forall x \in \mathbb{R}$$

$$|x^2-3|-2>0$$

$$|x^2-3|>2$$

$|t|>2 \iff t<-2 \vee t>2$

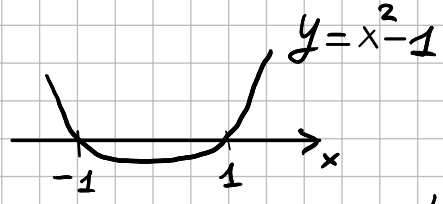


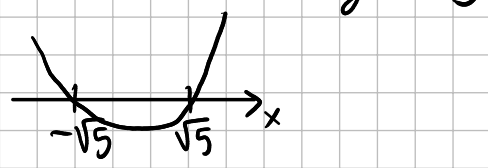
$$x^2-3<-2 \ \vee \ x^2-3>2$$

$$x^2-1<0 \ \vee \ x^2-5>0$$

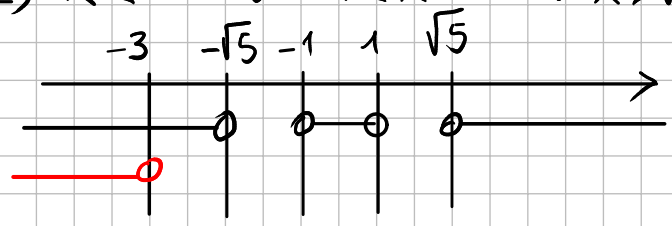
$$-1<x<1 \ \vee \ x<-\sqrt{5} \ \vee \ x>\sqrt{5}$$

$$x<-\sqrt{5} \ \vee \ -1<x<1 \ \vee \ x>\sqrt{5}$$





$$\frac{|3-2x|+1}{|x^2-3|-2} \geq 0 \iff x<-\sqrt{5} \ \vee \ -1<x<1 \ \vee \ x>\sqrt{5}$$



Se $x<-3$, allora $x<-\sqrt{5}$, quindi $f(x)$ è definita se $x<-3$

$\text{dom}(f) = (-\infty, 2)$

Maggioranti di $\text{dom}(f)$ $\quad x \geq 2$

$\sup(\text{dom}(f)) = 2$ $\quad 2 \notin \text{dom}(f)$, quindi $\text{dom}(f)$ non ha massimo

Minoranti di $\text{dom}(f)$ nessun numero reale

$(-\infty, 2)$ è inferiormente illimitato

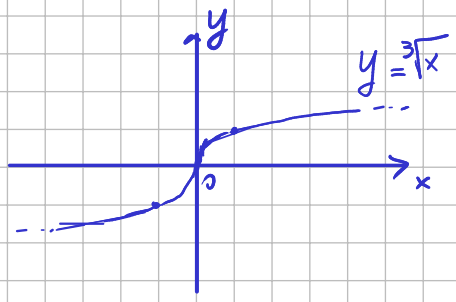
$\text{dom}(f)$ non ha minimo

$$f(x) = \frac{x^4}{[x]} - \frac{\sqrt[3]{x}}{2}\left(\sqrt{x-1} + x - 3\right)^{-\frac{1}{4}}$$

$x \longmapsto [x]$ ha dominio $\mathbb{R}$

$[x] = \max\{k \in \mathbb{Z} \mid k \leq x\}$

$x \longmapsto \sqrt[3]{x}$ ha dominio $\mathbb{R}$

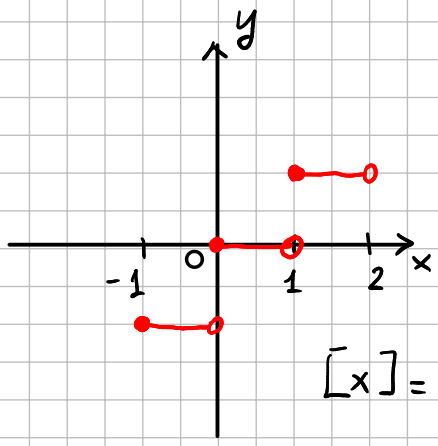


$x \longmapsto x^{-\frac{1}{4}}$ ha dominio $(0, +\infty)$

$$x^{-\frac{1}{4}} = \frac{1}{x^{1/4}} = \frac{1}{\sqrt[4]{x}} \quad \forall x > 0$$

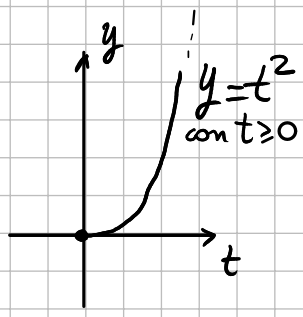
$$\begin{cases} [x] \neq 0 \\ \sqrt{x-1} + x - 3 > 0 \\ x - 1 \geq 0 \end{cases}$$

$[x] \neq 0 \iff x < 0 \vee x \geq 1$



$[x] = 0 \iff 0 \leq x < 1$

$$\begin{cases} \sqrt{x-1} + x - 3 > 0 \\ x - 1 \geq 0 \end{cases} \iff \begin{cases} \sqrt{x-1} > 3 - x \\ x \geq 1 \end{cases}$$

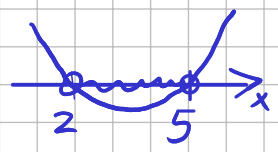


$$\iff \begin{cases} 3 - x < 0 \\ x \geq 1 \end{cases} \vee \begin{cases} 3 - x \geq 0 \\ x - 1 > (3-x)^2 \\ x \geq 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x > 3 \\ x \geq 1 \end{cases} \vee \begin{cases} x \leq 3 \\ x \geq 1 \\ x - 1 > 9 + x^2 - 6x \end{cases}$$

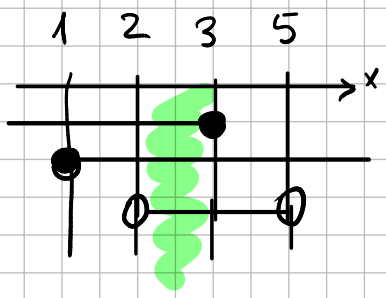
$x^2 - 7x + 10 < 0$

$(x-2)(x-5) < 0$



$2 < x < 5$

$$\Leftrightarrow \begin{cases} x > 3 \\ x \geq 1 \end{cases} \vee \begin{cases} x \leq 3 \\ x \geq 1 \\ 2 < x < 5 \end{cases}$$



$\Leftrightarrow x > 3 \vee 2 < x \leq 3$

$\Leftrightarrow x > 2$

$$\begin{cases} x < 0 \vee x \geq 1 \\ x > 2 \end{cases} \Leftrightarrow x > 2$$

$\mathrm{dom}(f) = (2, +\infty)$

Minoranti di $\mathrm{dom}(f)$ $\quad x \leq 2$

$\inf(\mathrm{dom}(f)) = 2$ $\quad 2 \notin \mathrm{dom}(f)$, quindi $\mathrm{dom}(f)$ non ha minimo

$\mathrm{dom}(f)$ è superiormente illimitato, quindi non ha massimo

$$f(x) = \sqrt{e^{2x} - 4e^x + 3} \cdot (\log^2 x - 1)^{-\frac{1}{2}}$$

$$\begin{cases} e^{2x} - 4e^x + 3 \geq 0 \\ \log^2 x - 1 > 0 \\ x > 0 \end{cases}$$

$$\log^2 x = (\log x)^2$$

$$(\log^2 x - 1)^{-\frac{1}{2}} = \frac{1}{(\log^2 x - 1)^{1/2}} = \frac{1}{\sqrt{\log^2 x - 1}}$$

$$e^{2x} - 4e^x + 3 \geq 0$$

$$(e^x)^2 - 4e^x + 3 \geq 0 \qquad e^x = t$$

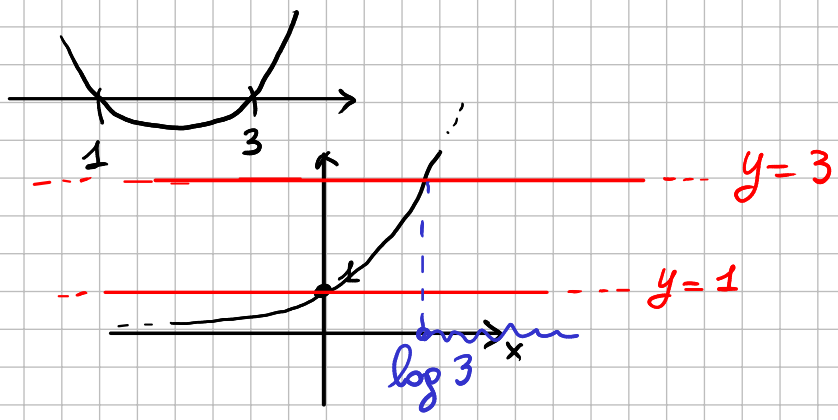
$$t^2 - 4t + 3 \geq 0$$

$$(t-1)(t-3) \geq 0$$

$$t \leq 1 \vee t \geq 3$$

$$e^x \leq 1 \vee e^x \geq 3$$

$$x \leq 0 \vee x \geq \log 3$$



$$\log^2 x - 1 > 0 \qquad t = \log x$$

$$t^2 - 1 > 0$$

Da completare per esercizio