

13/12/2023

- Calcolare l'area sottesa alla funzione

$$f(x) = \frac{e^{2x}}{4e^{2x} + 4e^x + 1} \quad \text{nell'intervallo } [0, 2]$$

$$A = \int_0^2 |f(x)| dx$$

$f(x) > 0 \quad \forall x \in \mathbb{R}$, a maggior ragione su $[0, 2]$

$$|f(x)| = f(x)$$

$$A = \int_0^2 f(x) dx = \int_0^2 \frac{e^{2x} e^x \cdot e^x dx}{4e^{2x} + 4e^x + 1} =$$

$$y = \varphi(x) = e^x \\ dy = \varphi'(x) dx = e^x dx$$

$$e^{2x} = (e^x)^2$$

$$e^{2x} = e^x \cdot e^x$$

$$\text{se } x=0 \Rightarrow y = e^0 = 1$$

$$\text{se } x=2 \Rightarrow y = e^2$$

$$= \int_1^{e^2} \frac{y}{4y^2 + 4y + 1} dy$$

f. integranda fatta con grado num < grado den.

$$4y^2 + 4y + 1 = (2y + 1)^2$$

riscrivo il num in modo da evidenziare la derivata del denominatore con l'obiettivo di ottenere

$$\int \frac{\psi'(y)}{\psi(y)} dy \quad \psi \text{ psi}$$

$$D(4y^2 + 4y + 1) = 8y + 4$$

lavoro con l'integrale indefinito, cioè
cerco una primitiva di $\frac{y}{4y^2+4y+1}$

$$\int \frac{y}{4y^2+4y+1} dy = \frac{1}{8} \int \frac{8y}{4y^2+4y+1} dy = \frac{1}{8} \int \frac{\overbrace{8y+4}^{\downarrow} - 4}{4y^2+4y+1} dy$$
$$\text{lin.} = \underbrace{\frac{1}{8} \int \frac{8y+4}{4y^2+4y+1} dy}_{I_1} + \underbrace{\frac{1}{8} \int \frac{-4}{4y^2+4y+1} dy}_{I_2}$$

$$\underline{I_1} = \frac{1}{8} \int \frac{8y+4}{4y^2+4y+1} dy = \frac{1}{8} \int \frac{dt}{t} = \frac{1}{8} \log |t| =$$
$$t = 4y^2+4y+1 \quad = \frac{1}{8} \log |4y^2+4y+1|$$
$$dt = (8y+4) dy$$

$$\underline{I_2} = -\frac{1}{2} \int \frac{1}{4y^2+4y+1} dy = -\frac{1}{2} \int \frac{1 \cdot 2}{(2y+1)^2} dy =$$

$$t = 2y+1$$
$$dt = 2 \cdot dy$$

$$= -\frac{1}{4} \int \frac{dt}{t^2} = -\frac{1}{4} \int t^{-2} dt = -\frac{1}{4} \cdot \frac{1}{-2+1} t^{-2+1} = \frac{1}{4} t^{-1} =$$

$$= \frac{1}{4} \cdot (2y+1)^{-1} = \underline{\underline{\frac{1}{4(2y+1)}}}$$

$$\int \frac{y}{4y^2+4y+1} dy = I_1 + I_2 = \frac{1}{8} \log |4y^2+4y+1| + \frac{1}{4(2y+1)}$$

$$\Rightarrow A = \int_1^{e^2} \dots dy = \left[\frac{1}{8} \log |4y^2+4y+1| + \frac{1}{4(2y+1)} \right]_1^{e^2}$$

$$= \frac{1}{8} \log (4e^4 + 4e^2 + 1) + \frac{1}{4(2e^2+1)} - \left(\frac{1}{8} \log |4+4+1| + \frac{1}{4 \cdot 3} \right)$$

$$= \frac{1}{8} \log (4e^4 + 4e^2 + 1) + \frac{1}{4(2e^2+1)} - \frac{1}{8} \log 9 - \frac{1}{12}$$

$$\circ \int \frac{1}{x(\log^2 x + 4 \log x + 5)} dx =$$

$$y = \log x$$

$$dy = \frac{1}{x} \cdot dx$$

$$f(x) = \frac{1}{x(\log^2 x + 4 \log x + 5)} = \frac{1}{x} \cdot \frac{1}{(\log^2 x + 4 \log x + 5)}$$

$$= \int \frac{1}{y^2 + 4y + 5} dy =$$

$$y^2 + 4y + 5 = 0$$

$$y_{1,2} = -2 \pm \sqrt{4 - 5} \notin \mathbb{R}$$

non posso scomporre il trinomio

$$y^2 + 4y + 5 = (y^2 + 4y + 4) + 1 = (y+2)^2 + 1$$

facio la sostituzione $t = y+2$

$$dt = dy$$

$$= \int \frac{1}{(y+2)^2 + 1} dy = \int \frac{dt}{t^2 + 1} = \operatorname{arctg} t =$$

$$= \operatorname{arctg}(y+2) = \operatorname{arctg}(-\log x + 2)$$

$$I = \operatorname{arctg}(\log x + 2) + C$$

$$\bullet \int \frac{x-1}{\sqrt{x}-3} dx =$$

$$y = \sqrt{x}$$

$$dy = \frac{1}{2\sqrt{x}} dx$$

nelle \int integrate $\frac{1}{\sqrt{x}}$ non c'è

però ho:

$$2 \underbrace{\sqrt{x}}_y dy = dx$$

$$2 y dy = dx$$



$$\text{da } y = \sqrt{x} \implies y^2 = x$$

$$= \int \frac{y^2 - 1}{y - 3} \cdot 2y \, dy = \int \frac{2y^3 - 2y}{y - 3} \, dy$$

$$= 2 \int \frac{y^3 - y}{y - 3} \, dy = \textcircled{\times} \quad n = 3 > 1 = m$$

$y^3 + 0y^2 - y + 0$		$y^2 + 3y + 8$
$-y^3 + 3y^2$		<hr/>
$\parallel \quad 3y^2 - y + 0$		$y - 3$
$\quad -3y^2 + 9y$		divisore
$\parallel \quad 8y + 0$		
$\quad -8y + 24$		
$\parallel \quad 24 \text{ resto}$		

$$\frac{\text{dividendo}}{\text{divisore}} = \text{quoto} + \frac{\text{resto}}{\text{divisore}}$$

$$\frac{y^3 - y}{y - 3} = (y^2 + 3y + 8) + \frac{24}{y - 3}$$

$$\begin{aligned}
 \textcircled{*} &= 2 \left[\int y^2 + 3y + 8 \, dy + \int \frac{24}{y-3} \, dy \right] \\
 &= 2 \left[\frac{1}{3} y^3 + \frac{3}{2} y^2 + 8y + 24 \log |y-3| \right] \\
 &= 2 \left(\frac{1}{3} x^{3/2} + \frac{3}{2} x + 8 x^{1/2} + 24 \log |x^{1/2} - 3| \right) + C
 \end{aligned}$$

$y = \sqrt{x} = x^{1/2}$

$$\int \frac{x^2 - 4}{x^3 + 2x} \, dx \qquad m=2 < m=3$$

$$x^3 + 2x = x \cdot (x^2 + 2)$$

uscivo
$$\frac{x^2 - 4}{x^3 + 2x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2} =$$

dove A, B e C sono da determinare imponendo l'uguaglianza con la fraz di partenze

(Al numeratore va un pol di un grado inferiore rispetto a quello del denominatore)

$$= \frac{Ax^2 + 2A + Bx^2 + Cx}{x(x^2 + 2)} = \frac{(A+B)x^2 + Cx + 2A}{x(x^2 + 2)}$$

A, B, C dovranno essere tali che

$$(A+B)x^2 + Cx + 2A = x^2 - 4$$

$$\left\{ \begin{array}{l} A+B=1 \\ C=0 \\ 2A=-4 \end{array} \right. \quad \left\{ \begin{array}{l} B=1-A=3 \\ C=0 \\ A=-2 \end{array} \right.$$

$$\frac{x^2-4}{x^3+2x} = \frac{A}{x} + \frac{Bx+C}{x^2+2} = \frac{-2}{x} + \frac{3x}{x^2+2}$$

$$\int \frac{x^2-4}{x^3+2x} dx = \underbrace{-2 \int \frac{1}{x} dx}_{I_1} + 3 \underbrace{\int \frac{x}{x^2+2} dx}_{I_2}$$

$$I_1 = -2 \log |x|$$

$$I_2 = \frac{3}{2} \int \frac{2x}{x^2+2} dx = \frac{3}{2} \int \frac{1}{y} dy = \frac{3}{2} \log |y| =$$

$$\begin{aligned} y &= x^2+2 \\ dy &= 2x dx \end{aligned} \quad = \frac{3}{2} \log(x^2+2)$$

$$\int \frac{x^2-4}{x^3+2x} dx = -2 \log |x| + \frac{3}{2} \log(x^2+2) + C$$

- $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$ (dalle regole di derivazione)

- $\int \frac{1}{\sqrt{x^2-1}} dx =$
 $y = \sqrt{x^2-1} + x = \varphi(x)$

$$dy = \varphi'(x) dx$$

$$= \left(\frac{x}{\sqrt{x^2-1}} + 1 \right) dx$$

$$dy = \underbrace{\frac{x + \sqrt{x^2-1}}{\sqrt{x^2-1}}}_{\varphi'(x)} dx$$

$$= \int \frac{1}{\sqrt{x^2-1}} \cdot \frac{x + \sqrt{x^2-1}}{x + \sqrt{x^2-1}} dx \quad dy =$$

$$= \int \frac{1}{y} dy = \log|y| = \log|x + \sqrt{x^2-1}| + C$$

- $I = \int \sqrt{x^2-1} dx =$
 $x = \sinh t$ uso the
 stesso

$$\int \sqrt{1-x^2} dx \quad x = \sin t$$

perché risulterebbe radicando < 0

$$= \int \underbrace{1}_{f'} \cdot \underbrace{\sqrt{x^2-1}}_g dx = \quad \text{per parti}$$
$$\int f'g = fg - \int fg'$$

$$f'(x) = 1 \quad g(x) = \sqrt{x^2-1}$$

$$f(x) = x \quad g'(x) = \frac{2x}{2\sqrt{x^2-1}}$$

$$= x\sqrt{x^2-1} - \underbrace{\int \frac{x^2}{\sqrt{x^2-1}} dx}_{I_1}$$

$$I_1 = \int \frac{x^2 - 1 + 1}{\sqrt{x^2-1}} dx = \int \frac{x^2-1}{\sqrt{x^2-1}} dx + \underbrace{\int \frac{1}{\sqrt{x^2-1}} dx}_{\text{vedi } \int \text{ prec}}$$

$$= \int \sqrt{x^2-1} dx + \log |x + \sqrt{x^2-1}|$$

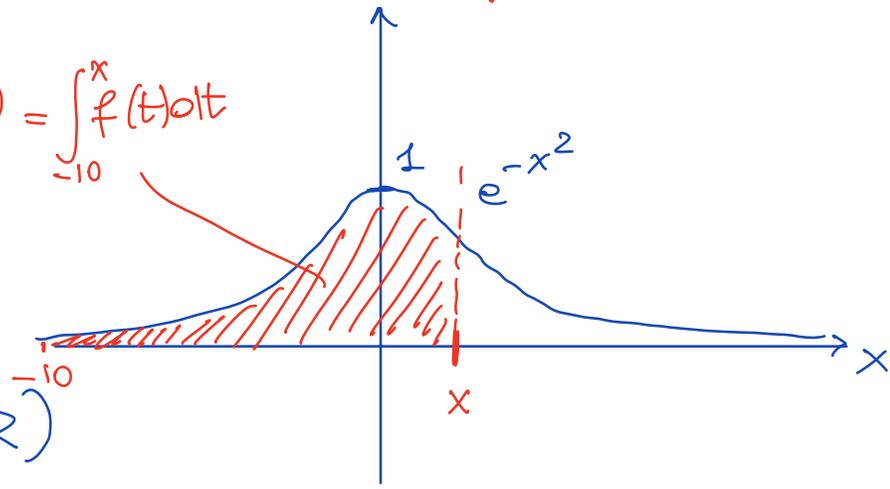
$$I = x\sqrt{x^2-1} - \underbrace{\int \sqrt{x^2-1} dx}_I = \log |x + \sqrt{x^2-1}|$$

$$2I = x\sqrt{x^2-1} - \log |x + \sqrt{x^2-1}|$$

$$I = \frac{1}{2} \left(x\sqrt{x^2-1} - \log |x + \sqrt{x^2-1}| \right) + C$$

- $f(x) = e^{-x^2}$ funzione gaussiana

$$F(x) = \int_{-10}^x f(t) dt$$



$$f \in C^\infty(\mathbb{R})$$

è integrabile secondo Riemann

∄ una espressione elementare di una sua primitiva $F(x) = \int e^{-x^2} dx$

f non è integrabile elementarmente

Costruisco la funzione integrale di f su un certo intervallo

Funzione integrale di $f(x)$ a partire da x_0 è

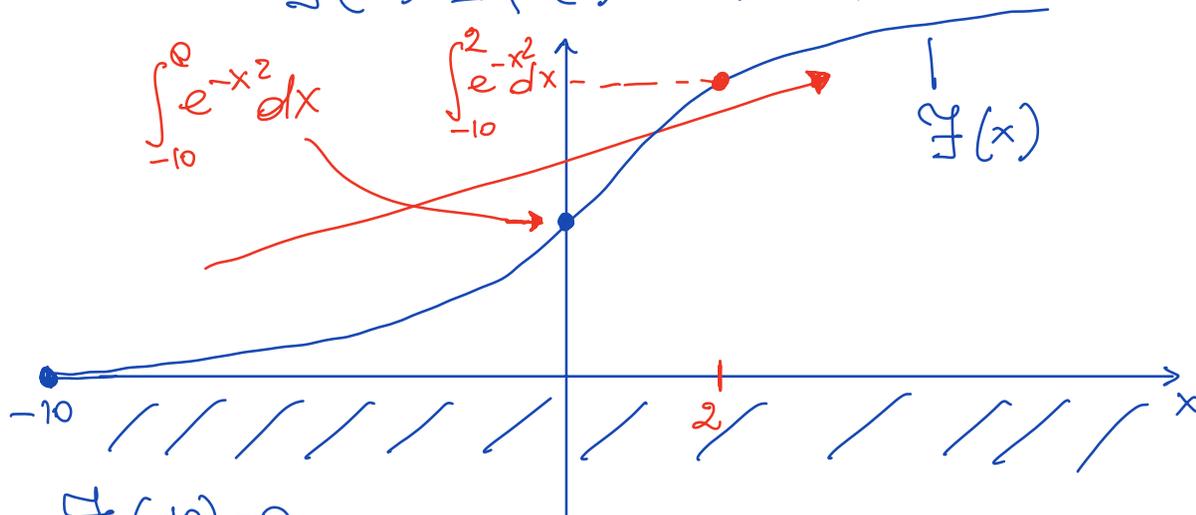
$$F_{x_0}(x) = \int_{x_0}^x f(t) dt$$

$$x_0 = -10 \quad F_{-10}(x) = \int_{-10}^x e^{-t^2} dt$$

voglio di scrivere $F_{-10}(x)$ - $f(x) = F_{-10}(x)$

Dal 1° fine fond del calcolo ho che

$$F'(x) = f(x) \quad \forall x \in \mathbb{R}$$



$$F(-10) = 0$$

? $F(x) \geq 0$ per quali x per ora non so rispondere

$$? F'(x) \geq 0 \quad F'(x) = f(x) = e^{-x^2} > 0$$

$\Rightarrow F$ è crescente $\forall x \geq -10$

$$\Rightarrow F(x) \geq 0 \quad \forall x \geq -10$$

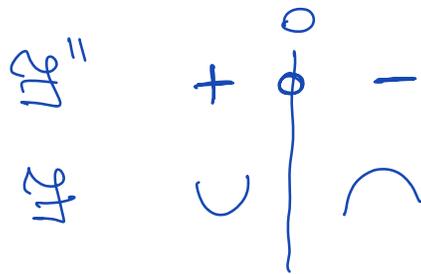
? F è concava o convessa

$$? F''(x) \geq 0 \quad \text{se } F'(x) = f(x)$$

$$\Rightarrow F''(x) = f'(x)$$

$$\text{calcolo } f'(x) = e^{-x^2} \cdot (-2x) = F''(x)$$

$$f''(x) = -2x \cdot e^{-x^2} \geq 0 \quad \text{se} \quad x \leq 0$$



$x=0$ punto di flesso

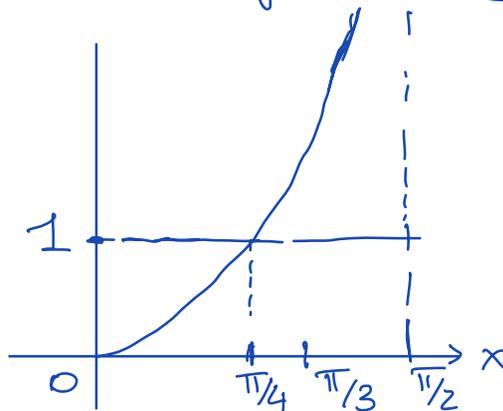
? Area sottesa al grafico di $f(x) = \frac{\sqrt{\tan x} - 1}{\cos^2 x}$ su $[0, \frac{\pi}{3}]$

$$A = \int_0^{\pi/3} |f(x)| dx$$

OSS: se $x \in [0, \frac{\pi}{3}] \Rightarrow \tan x \geq 0$ e f è ben definita

$$f(x) = \frac{\sqrt{\tan x} - 1}{\cos^2 x} \geq 0 \quad \text{su} \quad [0, \frac{\pi}{3}]$$

$$N = \sqrt{\tan x} - 1 \geq 0 \Leftrightarrow \sqrt{\tan x} \geq 1 \Leftrightarrow \tan x \geq 1$$



So che $\operatorname{tg} x = 1$ per $x = \frac{\pi}{4}$

$\Rightarrow \operatorname{tg} x \geq 1$ se $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$

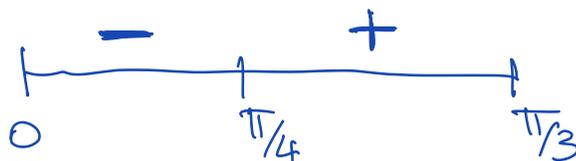
$N \geq 0 \Leftrightarrow \frac{\pi}{4} \leq x \leq \frac{\pi}{3}$

$D > 0 \Leftrightarrow \cos^2 x > 0$ vero $\forall x \in [0, \frac{\pi}{3}]$

\Rightarrow

$f(x) \geq 0$ se $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$

$$A = \int_0^{\pi/4} -f(x) dx + \int_{\pi/4}^{\pi/3} f(x) dx =$$



$$I = \int f(x) dx = \int \frac{\sqrt{\operatorname{tg} x} - 1}{\cos^2 x} dx \stackrel{(*)}{=} \text{integrale indef.}$$

$$\begin{aligned} y = \operatorname{tg} x = \varphi(x) & \quad \varphi'(x) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} \\ & = \frac{\sin x}{\cos x} & = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \\ & = 1/\cos^2 x & = 1 + (\operatorname{tg} x)^2 \end{aligned}$$

$$dy = \varphi'(x) dx = \frac{1}{\cos^2 x} \cdot dx$$

$$\textcircled{*} = \int (\sqrt{y} - 1) \cdot dy = \int y^{1/2} dy - \int dy =$$

$$= \frac{1}{\frac{1}{2} + 1} \cdot y^{\frac{1}{2} + 1} - y = \frac{2}{3} y^{3/2} - y =$$

$$= \frac{2}{3} (\operatorname{tg} x)^{3/2} - \operatorname{tg} x = G(x)$$

$$A = \int_0^{\pi/4} -f(x) dx + \int_{\pi/4}^{\pi/3} f(x) dx =$$

$$= - \left[\frac{2}{3} (\operatorname{tg} x)^{3/2} - \operatorname{tg} x \right]_0^{\pi/4} + \left[\frac{2}{3} (\operatorname{tg} x)^{3/2} - \operatorname{tg} x \right]_{\pi/4}^{\pi/3} =$$

$$= - \left(\frac{2}{3} \underbrace{(\operatorname{tg} \frac{\pi}{4})^{3/2}}_1 - 1 - (0 - 0) \right) +$$

$$+ \left[\frac{2}{3} (\operatorname{tg} \frac{\pi}{3})^{3/2} - \operatorname{tg} \frac{\pi}{3} - \left(\frac{2}{3} \cdot 1 - 1 \right) \right] =$$

$$\left[\operatorname{tg} \frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \right]$$

$$= -\left(\frac{2}{3}-1\right) + \left[\frac{2}{3} \cdot 3^{1/2 \cdot 3/2} - 3^{1/2} - \left(-\frac{1}{3}\right)\right]$$

$$= \frac{1}{3} + \frac{2}{3} \sqrt[4]{3^3} - \sqrt{3} + \frac{1}{3}$$